A Stochastic Multi-stage Fixed Charge Transportation Problem: Worst-Case Analysis of the Rolling Horizon Approach

Luca Bertazzi[a], Francesca Maggioni[b]

[a] *Corresponding author: L. Bertazzi Department of Economics and Management, University of Brescia, Contrada S. Chiara 50, 25122, Brescia, ITALY*
e-mail: luca.bertazzi@unibs.it
[b] *F. Maggioni Department of Management, Economics and Quantitative Methods, Bergamo University, Via dei Caniana 2, 24127, Bergamo ITALY*
e-mail:francesca.maggioni@unibg.it

Abstract

We introduce a stochastic multi-stage fixed charge transportation problem, in which a producer has to satisfy an uncertain demand within a deadline. At each time period, a fixed transportation cost can be paid to buy a transportation capacity. If the transportation capacity is used, the supplier also pays an uncertain unit transportation cost. A unit inventory cost is charged for the unsatisfied demand. The aim is to determine the transportation capacities to buy and the quantity to send at each time period in order to minimize the expected total cost. We prove that this problem is NP-hard, we propose a multi-stage stochastic optimization model formulation, and we determine optimal policies for particular cases, with deterministic unit transportation costs or demand and zero fixed costs. Furthermore, we provide the worst–case analysis of the *rolling horizon* approach, a classical heuristic approach for solving multi-stage stochastic programming models, applied to this NP-hard problem and to polynomially solvable particular cases. Worst–case results show that the rolling horizon approach can be very suboptimal. We also provide experimental results.

Keywords: Logistics, fixed charge transportation problem, Multi-stage stochastic programming, Rolling horizon, Worst-case analysis.

1. Introduction

The *fixed charge transportation problem* is a generalization of the classical *transportation problem*, where the transportation cost function from each source to each sink is composed of a fixed value and a variable value proportional to the quantity of the shipment. It has been studied in the past and in the last years (see [17], [22], [1], [32], [2] and [37] for exact algorithms, [20], [36] and [8] for heuristic algorithms, see [29], [3] and [41] for cases with uncertain variables, see [11] for an extension of this problem with piecewise linear transportation costs and [33] for an application of this problem to service network design). It has several practical applications, such as distribution, transportation, scheduling, location. Specifically, allocation of launch vehicles to space missions, solid-waste management, process selection, teacher assignment. Moreover, it is applicable for e-commerce companies such as Amazon, Alibaba, eBay, Rakuten, Zalando, Groupon etc, which over the past years, have transformed how people buy and sell online. The Internet provides a fast and easy way for people to purchase things without visiting an actual store. In Europe, the percentage of turnover on e-sales in 2014 rose 17% of the total turnover of enterprises with 10 or more employees, and this percentage increases with the size of the company. At the same time, new challenges arise in the e-commerce supply chain management due to demand variations in time and higher requirements in delivery services. As a result, a new optimization problem has to be addressed. This problem is by nature stochastic and dynamic like the one addressed in this paper, in which a producer has to ship an uncertain demand to a customer within a deadline. We assume that transportation is outsourced, as in [7], [12], [13], [35] and [43]. Combinatorial auctions are typically used in transportation procurement to determine the best offer of transportation services (see [40]). We assume that, for each time period, the transportation company offering the best fixed transportation cost for a given transportation capacity, is known. If the transportation capacity is used, the supplier also pays an uncertain unit transportation cost with given probability distribution. A unit inventory cost is charged for the quantity that remains to be sent at the end of the time period. The aim is to determine the transportation capacities to buy and the quantities to ship at each time period in order to minimize the expected total cost. The deterministic counterpart of this problem can be viewed as the *Single-sink fixed charge transportation problem* studied in [19], [16], [10]. In fact, each time period can be viewed as a different source from which a quantity can be sent to the sink. However, at each time period, we have to pay a unit backordering cost for the quantity not already sent. A typical application for this problem is in long-haul transportation, where containers or vehicles are rent to send products from one supplier to a customer located in a different country.

The main contributions and research questions of the paper are as follows:

- to present a new problem, namely a stochastic multi-stage fixed charged transportation problem;
- to formulate a new multi-stage mixed-integer stochastic optimization model for this problem;
- to prove the computational complexity analysis of this problem;
- to prove worst-case performance bounds for the classical rolling horizon approach by considering a sequence of multi-stage stochastic subproblems with reduced time horizon;
- to provide extensive numerical experiments with the aim of understanding:
    1 the maximum dimension of the multi-stage stochastic programming models, in terms of stages, that can be solved by a state-of-the-art solver;
    2 how sensitive the optimal policies and the optimal total cost are with respect to the reduced time horizon;
    3 the average performance of the rolling horizon approach in a given set of instances, compared to the worst-case performance bounds.

More specifically, we first formulate a multi-stage mixed-integer stochastic programming model (see [5] and [14] for a brief review of history and achievements of stochastic programming and for selected modeling issues concerning applications of multi-stage stochastic programs). Multi-stage stochastic mixed integer linear programs are among the most challenging optimization problems combining stochastic programs and discrete optimization problems (see [23, 38, 39, 34, 21, 31] for some major results in this area). Exact solution methods are in general based on branch and bound type algorithms or branch and price methods, see [26]. Bounds and approximations for such a class of problems are provided in [27, 28, 30].

After proving that this problem is NP-hard, we design exact polynomial time algorithms for the solution of two particular cases, having deterministic unit transportation cost or demand. Our main aim is to provide the worst–case analysis of the classical *rolling horizon approach*, a heuristic approach frequently used to solve multi-stage stochastic programming models. In this approach, a policy is computed by optimally solving a sequence of stochastic programming subproblems having a reduced time horizon. At each iteration, only the value of the first-stage variables is captured (we refer to [25], [24], [18], [42], for applications of this approach to different problems, to [9] for a classified bibliography of the literature and to [4] for the choice of the time horizon, stages, methods for generating scenario trees). Since in most of the cases the optimal policy of a multi-stage stochastic program cannot be computed, an evaluation of the performance of the rolling horizon approach with respect to the optimal policy is missing in the literature.

Worst-case analysis (see [15]) is a useful tool to give this comparison in the worst case and, to the best of our knowledge, it has never been applied to the rolling horizon approach. This analysis computes the value of the ratio between the total cost of the heuristic approach and the optimal total cost, in the worst case. Upper bounds on the total cost provided by the heuristic approach and lower bounds on the optimal total cost are used to prove the worst-case performance bound, that holds for any instance of the problem. Then, a worst-case instance, or a sequence of worst-case instances, are provided to show that the bound is tight, i.e. it is not overestimated. More formally, a heuristic approach $\mathcal{H}$, which gives a solution whose cost is $z^{\mathcal{H}}(I)$ on an instance I for which the optimal cost is $z^*(I)$, has a worst–case performance bound δ if $\frac{z^{\mathcal{H}}(I)}{z^*(I)} \leq \delta$, for any instance I. The ratio δ is tight if, for any $\delta' < \delta$, an instance I' exists for which $\frac{z^{\mathcal{H}}(I')}{z^*(I')} > \delta'$. Finally, we provide a systematic computational experiment that allows us to show the maximum dimension of the instances (in terms of number of stages) that can be solved by using a state-of-the-art solver, the sensitivity of the optimal total cost of the stochastic programming models solved at each iteration of the rolling horizon approach with respect to increasing values of the reduced time horizon, and finally, the average performance of the rolling horizon approach in a given set of problem instances.

The paper is organized as follows. In Section 2 the stochastic fixed charge transportation problem we study is formally described. In Section 3 a multi-stage stochastic programming model is formulated. In Section 4 we prove that this problem is NP-hard and provide exact polynomial time algorithms for the solution of the two particular cases with deterministic demand or unit transportation costs. In Section 5, the worst-case analysis of the rolling horizon approach is provided. In Section 6 the computational results are shown. Finally, in Section 7, we conclude the paper.

2. Problem Description

A producer has to ship an uncertain demand to a customer within a deadline H. The demand is composed of an uncertain number of units L, that is a random variable having discrete probability distribution $\mathcal{L}$ defined over the support $\mathscr{U}_1 = \{L_{\min}, \ldots, L_{\max}\}$, where $0 < L_{\min} \leq L_{\max}$. A shipment can be performed at any of the discrete time periods $t \in T = \{0, 1, \ldots, H-1\}$, paying a fixed transportation cost Q_t to buy a transportation capacity K_t and an uncertain unit transportation cost P_t. The fixed transportation cost Q_t is given. The unit transportation cost is described by a discrete random variable having probability distributions $\mathcal{P}_t$ defined over the support $\mathscr{U}_2 = \{m_2, \ldots, M_2\}$, where $0 < m_2 \leq M_2$. We assume that the probability distributions $\mathcal{P}_t$, $t \in T$, and $\mathcal{L}$ are stage independent and mutually independent. The realization of the random variables in each time period is available at the end of the time period. A unit inventory cost h is paid for the quantity that remains to be sent at the end of time t. The aim is to determine, for each time period $t \in T$, whether to buy the transportation service or not and the quantity of the shipment, in order to minimize the expected total cost.

3. A Multi-stage Stochastic Programming Formulation

In this section, we present a multi-stage stochastic programming formulation of the problem (see [6]). If we assume that the transportation cost $\boldsymbol{P}_{H-1} := (P_0, \ldots, P_t, \ldots, P_{H-1})$ and demand L are random parameters evolving as discrete-time stochastic processes with finite and discrete support, then the information structure can be described in the form of a scenario tree. At each stage t, there is a discrete number of atoms (nodes), where a specific realization of the uncertain cost P_t and of the demand L takes place. We assume that the information on the uncertain demand will be available after the first period. There are $H+1$ levels (stages) in the tree, that correspond to specific times. Each node, except the root, is connected to a unique node at stage $t-1$, called the ancestor node, and to nodes at stage $t+1$, called

the successors ones. A scenario ω is a path through nodes from the root node to a leave node. We indicate with Ω the set of scenarios and with $p(\omega)$ the probability of scenario ω. Let $\mathbf{P}_t(\omega)$ be the history of the ω-realization, $\omega \in \Omega$, of the transportation costs up to stage t. We denote with $L(\omega)$, the possible realizations of the demand to be shipped within the deadline H in scenario $\omega \in \Omega$.
Let us now define the following notation:

Sets:

$\Omega = \{\omega\}$: set of scenarios

$T' = \{1, \ldots, H-1\}$: subset of discrete times

Deterministic Parameters:

K_t: transportation capacity offered at time $t \in T$

Q_t: fixed transportation cost to buy the capacity K_t at time $t \in T$

h: unit inventory cost

Stochastic Parameters:

$P_t(\omega)$: realization of the unit transportation cost at time $t \in T$ in scenario $\omega \in \Omega$

$\mathbf{P}_t(\omega)$: the history of the ω-realization of the unit transportation cost up to stage t

$L(\omega)$: demand to be shipped within the deadline H in scenario $\omega \in \Omega$

$p(\omega)$: probability of scenario $\omega \in \Omega$

Decision Variables:

$y_0 \geq 0$: quantity shipped at stage $t = 0$ at the uncertain cost $P_0(\omega)$

$x_0 \in \{0, 1\}$: 1 if capacity K_0 is used at stage 0, 0 otherwise

$y_t(\omega) \geq 0$: quantity shipped at stage $t \in T'$ in scenario ω

$x_t(\omega) \in \{0, 1\}$: 1 if capacity K_t is used at stage $t \in T'$ in scenario ω, 0 otherwise.

The risk-neutral mixed integer linear multi-stage stochastic programming model is formulated as follows:

$$\begin{aligned} \min \quad & Q_0 \cdot x_0 + \sum_{\omega \in \Omega} p(\omega) \Bigg\{ P_0(\omega) \cdot y_0 + \\ & + \sum_{t \in T'} \left[P_t(\omega) \cdot y_t(\omega) + Q_t \cdot x_t(\omega) + h \left(L(\omega) - y_0 - \sum_{k=1}^{t-1} y_k(\omega) \right) \right] \Bigg\} \end{aligned} \tag{1}$$

$$\begin{aligned} \text{s.t.} \quad & y_{0+\sum_{t \in T'}} y_t(\omega) = L(\omega), \qquad \omega \in \Omega, \\ & y_0 \le K_0 \cdot x_0, \\ & y_t(\omega) \le K_t \cdot x_t(\omega), \quad t \in T', \ \omega \in \Omega, \\ & y_t(\omega') = y_t(\omega''), \forall \omega', \omega'' \mid (\mathbf{P}_t(\omega'), L(\omega')) = (\mathbf{P}_t(\omega''), L(\omega'')), \ t \in T', \\ & y_0 \in \mathbb{R}^+, \\ & y_t(\omega) \in \mathbb{R}^+, \ t \in T', \ \omega \in \Omega, \\ & x_t(\omega') = x_t(\omega''), \forall \omega', \omega'' \mid (\mathbf{P}_t(\omega'), L(\omega')) = (\mathbf{P}_t(\omega''), L(\omega'')), \ t \in T', \\ & x_0 \in \{0, 1\}, \\ & x_t(\omega) \in \{0, 1\}, \qquad t \in T', \ \omega \in \Omega. \end{aligned}$$

The first term in the objective function denotes the fixed transportation cost paid at stage 0, while the second term the expected total cost of paying the unit transportation cost $P_t(\omega)$, the fixed transportation cost Q_t and the inventory cost h in each scenario ω at each stage $t \in T'$. The first constraint guarantees that, for each scenario ω, the total quantity shipped to the customer within the deadline H is equal to $L(\omega)$. The second constraint guarantees that the quantity y_0 that can be sent at stage 0 is not greater than the transportation capacity K_0 when it has been bought, while it is equal to 0 otherwise. The same is guaranteed for each stage $t \in T'$ by the third constraint. The fourth and seventh constraints represent the so-called non-anticipativity constraints on the decision variables y_t and x_t. All the other constraints define the decision variables of the problem. We denote the optimal total cost of model (1), referred to as Case 1), with z^*.

4. Computational Complexity and Polynomially Solvable Cases

In this section, we prove the computational complexity of model (1) and provide two polynomially solvable cases.

Theorem 1. *Model (1) is NP–hard.*

Proof Consider the set of instances such that the unit transportation cost is deterministic and equal to 0, the unit inventory cost $h = 0$ and the demand is deterministic, say L. Then, model (1) becomes

$$\begin{aligned}
\min \quad & \sum_{t\in T} Q_t \cdot x_t \\
\text{s.t.} \quad & \sum_{t\in T} y_t = L, \\
& y_t \leq K_t \cdot x_t, \quad t \in T, \\
& y_t \geq 0, \quad t \in T, \\
& x_t \in \{0,1\}, \qquad t \in T,
\end{aligned}$$

which is equivalent to the following *Min Cost 0–1 Knapsack Problem*:

$$\begin{aligned}
\min \quad & \sum_{t\in T} Q_t \cdot x_t \\
\text{s.t.} \quad & \sum_{t\in T} K_t \cdot x_t \geq L, \\
& x_t \in \{0,1\}, \qquad t \in T,
\end{aligned}$$

which is known to be NP-hard. □

We now provide two polynomially solvable special cases, under the assumption that the transportation capacity $K_t = \infty$ and the fixed cost $Q_t = 0$. Similar results can be derived also for cases 2) and 3) with capacity constraints ($K_t < \infty$, $t \in T$) under the assumption that no fixed cost is paid.

4.1. Case 2) Deterministic demand (uncapacitated)

If we assume that the demand is deterministic, i.e. $L(\omega) = L$, $\omega \in \Omega$, there are no constraints on the capacities, i.e. $K_t = \infty$, and the fixed costs $Q_t = 0$, $t \in T$, then model (1) reduces to the following model:

$$\min \sum_{\omega\in\Omega} p(\omega) \left\{ P_0(\omega) y_0 + \sum_{t\in T'} \left[P_t(\omega) \cdot y_t(\omega) + h \left(L - y_0 - \sum_{k=1}^{t-1} y_k(\omega) \right) \right] \right\} \quad (2)$$

$$
\begin{aligned}
\text{s.t.} \quad & y_0 + \sum_{t \in T'} y_t(\omega) = L, \qquad \omega \in \Omega, \\
& y_t(\omega') = y_t(\omega''), \qquad \forall\, \omega', \omega'' \mid \mathbf{P}_t(\omega') = \mathbf{P}_t(\omega''),\ t \in T', \\
& y_0 \in \mathbb{R}^+, \\
& y_t(\omega) \in \mathbb{R}^+, \qquad t \in T',\ \omega \in \Omega.
\end{aligned}
$$

This case would arise at the operational level, when a supplier has to ship a known demand to a customer by using external transportation companies, having to pay an unknown transportation cost.

Theorem 2. *The optimal total cost of model (2) is*

$$
z^* = \min_{t \in T} \{\mathbb{E}(P_t) + ht\} L.
$$

Therefore, an optimal policy can be computed in $O(H)$ time.

Proof We first note that in the two-stage case ($H = 1$), since the unique feasible solution is to send the overall quantity L at time 0, the optimal total cost z^* of model (2) is $\mathbb{E}(P_0)L$ and the thesis is verified.

We now prove the theorem by induction on the deadline H.

- (Base Case) We consider the case of a three-stage problem ($H = 2$). The model (2) reduces to the following model:

$$
\begin{aligned}
\min \quad & \sum_{\omega \in \Omega} p(\omega) \{P_0(\omega) y_0 + P_1(\omega) \cdot y_1(\omega) + h(L - y_0)\} \\
\text{s.t.} \quad & y_0 + y_1(\omega) = L, \qquad \omega \in \Omega, \\
& y_1(\omega') = y_1(\omega''), \forall\, \omega', \omega'' \mid \mathbf{P}_1(\omega') = \mathbf{P}_1(\omega''), \\
& y_0 \in \mathbb{R}^+, \\
& y_1(\omega) \in \mathbb{R}^+,\ \omega \in \Omega.
\end{aligned}
$$

  From the first type of constraints, we have

$$
y_1(\omega) = L - y_0 \qquad \omega \in \Omega,
$$

  meaning that $y_1(\omega)$ is constant, say y_1, over all scenarios $\omega \in \Omega$. Therefore, the non-anticipativity constraints are unnecessary. Substituting

$y_1 = L - y_0$ in the objective function and taking into account that $y_1 \geq 0$, the model reduces to the following model:

$$\begin{aligned} \min \quad & \sum_{\omega \in \Omega} p(\omega) \left\{ P_0(\omega) y_0 + (L - y_0)(P_1(\omega) + h) \right\} \\ \text{s.t.} \quad & y_0 \leq L, \\ & y_0 \in \mathbb{R}^+, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \min \quad & (\mathbb{E}(P_1) + h)L + (\mathbb{E}(P_0) - \mathbb{E}(P_1) - h)y_0 \\ \text{s.t.} \quad & y_0 \leq L, \\ & y_0 \in \mathbb{R}^+. \end{aligned}$$

If $\min_{t \in \{0,1\}} \{\mathbb{E}(P_t) + ht\} = \mathbb{E}(P_0)$ then $y_0^* = L$ and the corresponding total cost $z^* = \mathbb{E}(P_0)L$; otherwise $y_0^* = 0$, with total cost $z^* = (\mathbb{E}(P_1) + h)L$ and the thesis is verified. Notice that in this case the stochastic model corresponds to the (deterministic) expected value model.

(Inductive step) We assume now, as induction hypothesis, that the thesis is verified for a model with deadline H. We need to prove that the thesis is also verified for a model with deadline $H + 1$. If $\min_{t \in \{0,1,\dots,H\}} \{\mathbb{E}(P_t) + ht\} = \min_{t \in \{0,1,\dots,H-1\}} \{\mathbb{E}(P_t) + ht\}$, the thesis follows by the induction hypothesis. Otherwise, if $\min_{t \in \{0,1,\dots,H\}} \{\mathbb{E}(P_t) + ht\} = \mathbb{E}(P_H) + hH$ then $y_H^*(\omega) = L$, $\forall \omega \in \Omega$, and the corresponding total cost $z^* = (\mathbb{E}(P_H) + hH)L$, which again satisfies the thesis. $\square$

4.2. Case 3) Deterministic unit transportation cost (uncapacitated)

If we assume that the unit transportation cost is deterministic, i.e. $P_t(\omega) = P_t$, $\omega \in \Omega$, there are no constraints on the capacities, i.e. $K_t = \infty$, and the fixed costs $Q_t = 0$, $t \in T$, then the model (1) reduces to the following model:

$$\begin{aligned} \min \quad & \sum_{\omega \in \Omega} p(\omega) \left\{ P_0 y_0 + \sum_{t \in T'} \left[P_t \cdot y_t(\omega) + h \left(L(\omega) - y_0 - \sum_{k=1}^{t-1} y_k(\omega) \right) \right] \right\} \quad (3) \\ \text{s.t.} \quad & y_0 + \sum_{t \in T'} y_t(\omega) = L(\omega), \qquad \omega \in \Omega, \\ & y_t(\omega') = y_t(\omega''), \qquad \forall\, \omega', \omega'' \mid L(\omega') = L(\omega''),\ t \in T', \\ & y_0 \in \mathbb{R}^+, \\ & y_t(\omega) \in \mathbb{R}^+, \qquad t \in T',\ \omega \in \Omega. \end{aligned}$$

This case would arise at a tactical level, when a supplier owns the vehicles to perform the shipping to the customer, while the demand is unknown.

Theorem 3. *The optimal total cost of model (3) is*

$$z^* = \begin{cases} P_0 \cdot L_{min} + \min_{t\in T'}\{P_t + ht\}[\mathbb{E}(L) - L_{min}] & \text{if } P_0 = \min_{t\in T}\{P_t + ht\} \\ \mathbb{E}(L)\min_{t\in T'}\{P_t + ht\} & \text{otherwise.} \end{cases}$$

Therefore, an optimal policy can be computed in $O(H)$ time.

Proof We first note that the two-stage case ($H = 1$) is infeasible, since the unique possibility would be to send the stochastic quantity $L(\omega)$ at stage 0 for all scenarios $\omega \in \Omega$, clearly impossible due to the non-anticipativity constraints which force the solution y_0 to be constant for all scenarios. For this reason we assume $H \geq 2$.

As before, we prove the theorem by induction on the deadline H.

- (Base Case) We consider the case of a three stage problem ($H = 2$). The model (3) reduces to the following model:

$$\begin{aligned} \min \quad & \sum_{\omega\in\Omega} p(\omega)\left\{P_0 y_0 + P_1 y_1(\omega) + h\left(L(\omega) - y_0\right)\right\} \\ \text{s.t.} \quad & y_0 + y_1(\omega) = L(\omega), \qquad \omega \in \Omega, \\ & y_1(\omega') = y_1(\omega''), \forall\, \omega', \omega'' \mid L(\omega') = L(\omega''), \\ & y_0 \in \mathbb{R}^+ \\ & y_1(\omega) \in \mathbb{R}^+, \quad \omega \in \Omega. \end{aligned}$$

  From the first type of constraints, we have

$$y_1(\omega) = L(\omega) - y_0 \qquad \omega \in \Omega.$$

  Substituting in the objective function and taking into account that $y_1(\omega) \geq 0$, the model reduces to the following:

$$\begin{aligned} \min \quad & \sum_{\omega\in\Omega} p(\omega)\left\{P_0 y_0 + (P_1 + h)(L(\omega) - y_0)\right\}, \\ \text{s.t.} \quad & y_0 \leq L(\omega), \qquad \omega \in \Omega, \\ & y_0 \in \mathbb{R}^+, \end{aligned}$$

which is equivalent to

$$\begin{array}{ll} \min & (P_1 + h)\mathbb{E}(L) + (P_0 - P_1 - h)y_0 \\ \text{s.t.} & y_0 \leq L_{\min}, \\ & y_0 \in \mathbb{R}^+. \end{array}$$

If $\min_{t\in\{0,1\}}\{P_t + ht\} = P_0$ then $y_0^* = L_{min}$, $y_1^*(\omega) = L(\omega) - L_{min}$, $\omega \in \Omega$, and the corresponding cost $z^* = P_0 L_{min} + (P_1+h)[\mathbb{E}(L) - L_{min}]$; otherwise $y_0^* = 0$ and $y_1^*(\omega) = L(\omega)$, with cost $z^* = \mathbb{E}(L)(P_1 + h)$ and the thesis is verified.

(Inductive step) We assume now, as induction hypothesis, that the thesis is verified for a model with deadline H. We need to prove that the thesis is also verified for a model with deadline $H + 1$. If $\min_{t\in\{0,1,\ldots,H\}}\{P_t + ht\} = \min_{t\in\{0,1,\ldots,H-1\}}\{P_t + ht\}$, the thesis follows by the induction hypothesis. Otherwise, if $\min_{t\in\{0,1,\ldots,H\}}\{P_t + ht\} = P_H + hH$ then $y_H^*(\omega) = L(\omega)$, $\forall\omega \in \Omega$, and the corresponding total cost $z^* = \mathbb{E}(L)(P_H + hH)$, which again satisfies the thesis. □

5. Worst-Case Analysis of the Rolling Horizon Approach

In this section, we evaluate the worst-case performance of the classical rolling horizon approach in solving multi-stage stochastic programs with finite time horizon. In this heuristic approach, a policy is computed by optimal solving a sequence of subproblems with less number of consecutive periods. In particular, in the first step, the $(W + 1)$-stage stochastic programming model defined on $t = 0, 1, \ldots, W < H$ is optimally solved and only the values of the first-stage decision variables x_0 and y_0 are captured as the decision $x_0^{(W+1)S}$ to buy or not the capacity and the quantity $y_0^{(W+1)S}$ to send at stage 0 in the rolling horizon policy. In the second step, the $(W + 1)$-stage stochastic programming model defined on $t = 1, 2, \ldots, W + 1$ is optimally solved by setting the demand equal to the residual quantity to be sent, given by $S(\omega) = L(\omega) - y_0$. Only the values of the new first-stage variables x_1 and y_1 are captured as the decision $x_1^{(W+1)S}(\omega)$ to buy or not the capacity and the quantity $y_1^{(W+1)S}(\omega)$ to send on each scenario $\omega \in \Omega$ at stage 1 in the rolling horizon policy. This process is repeated until stage $t = H - W$. After solving the last $(W + 1)$-stage stochastic programming model, a W-stage stochastic programming model on $t = H - W + 1, H - W + 2, \ldots, H$, is solved. Then, a $(W-1)$-stage stochastic programming model on $t = H-W+2, H-W+3, \ldots, H$ is solved. The process is repeated until the 2-stage stochastic programming model defined on $t = H-1, H$ is solved (see Figure 1). In the following, we denote the total cost of the rolling horizon approach by $z^{(W+1)S}$.

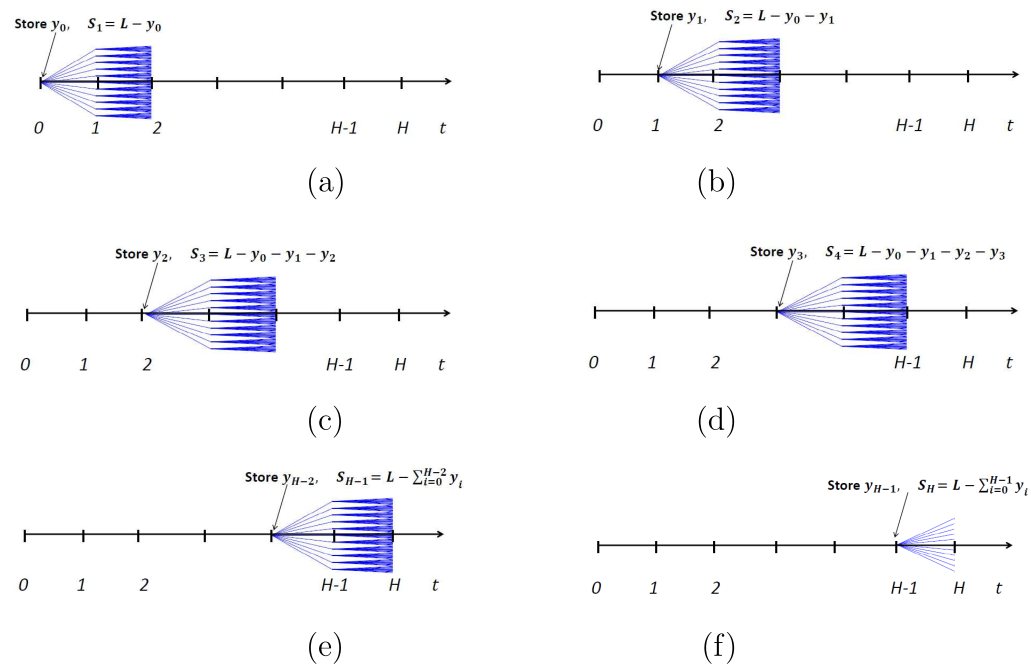

Figure 1: The rolling horizon approach with $W = 2$ over $H+1$ stages. Follow the sequence (a)-(b)-(c)-(d)-(e)-(f).

We now prove the worst-case performance bound of the rolling horizon approach. Similar results hold true also for cases 2) and 3) with capacity constraints ($K_t < \infty$, $t \in T$) and for reduced time horizon $W > 2$.

Case 1): stochastic demand and unit transportation cost.

Remember that in this case any 2-stage stochastic programming model is infeasible. Therefore, we study the worst-case performance of the rolling horizon approach with $W = 2$, based on the optimal solution of a sequence of 3-stage stochastic programming models. The following theorem holds.

Theorem 4. *In Case 1), there exists an instance such that* $\frac{z^{3S}}{z^*} \longrightarrow \infty$.

Proof Consider the following instance: Deterministic demand $L = 1$; deadline $H = 4$; deterministic unit transportation costs $P_t = 0$, $t \in T$; transportation capacities $K_0 = K_1 = K_2 = \frac{L}{2}$ and $K_3 = L$; fixed transportation costs $Q_0 = Q_1 = Q_2 = 1$ and $Q_3 = \epsilon << 1$; unit inventory cost $h = \epsilon$.

Let us apply the rolling horizon approach with $W = 2$. In the first step, the first three-stage stochastic programming model is solved. Since $Q_0 + h\frac{L}{2} < Q_1 + hL$, $x_0^* = 1$ and $y_0^* = \frac{L}{2}$. Therefore, in the rolling horizon policy, we have $x_0^{3S} = 1$ and $y_0^{3S} = \frac{L}{2}$. In the second step, since $Q_1 < Q_2 + h\frac{L}{2}$, $x_1^* = 1$ and $y_1^* = \frac{L}{2}$. Therefore, in the rolling horizon policy, we have $x_1^{3S} = 1$ and $y_1^{3S} = \frac{L}{2}$. All remaining variables are equal to 0. Therefore, the total cost is $z^{3S} = 2 + h\frac{L}{2} = 2 + \frac{\epsilon}{2}$.

The optimal total cost z^* is not greater than the cost of the following solution: $x_0 = x_1(\omega) = x_2(\omega) = 0$, $x_3(\omega) = 1$ and $y_0 = y_1(\omega) = y_2(\omega) = 0$ and $y_3(\omega) = L$, having total cost $Q_3 + 3hL = 4\epsilon$.

Therefore, in this instance

$$\frac{z^{3S}}{z^*} \geq \frac{2 + \frac{\epsilon}{2}}{4\epsilon} \to \infty \qquad \text{for } \epsilon \to 0.$$

□

According to the previous theorem, a finite worst-case bound does not exist. Consequently, a guarantee on the performance of the rolling horizon approach is not at our disposal, implying that the rolling horizon approach can be infinitely suboptimal.

Case 2) deterministic demand (uncapacitated).

In this case, we start studying the rolling horizon approach with $W = 1$, where a two-stage stochastic programming model is solved at each iteration, and then we study the case with $W = 2$. The following theorem holds.

Theorem 5. *In Case 2),* $\frac{z^{2S}}{z^*} \leq \frac{M_2}{m_2}$ *and the bound is tight.*

Proof Since the optimal solution of the two-stage stochastic programming model solved at stage 0 is to send the overall quantity L at stage 0, the policy provided by the rolling horizon approach with $W = 1$ is $y_0^{2S} = L$ and $y_t^{2S}(\omega) = 0$, $t \in T'$, $\omega \in \Omega$. The corresponding total cost z^{2S} is $\mathbb{E}(P_0)L$, which is not greater than $M_2 L$.

Let us now compute the worst-case performance bound. If the optimal policy is to send L at stage 0, then $z^* = \mathbb{E}(P_0)L$ and therefore the worst-case performance bound is equal to 1. Otherwise, a lower bound on the optimal total cost z^* is $m_2 L$. Therefore,

$$\frac{z^{2S}}{z^*} \leq \frac{M_2}{m_2}.$$

We now prove that the bound is tight. Consider the following instance: demand $L = 1$; deadline $H = 2$; probability distribution at stage 0: M_2 with probability 1; probability distribution at time 1: m_2 with probability 1; unit inventory cost $h = 0$.

The total cost of the rolling horizon approach with $W = 1$ is $z^{2S} = M_2$, while the optimal total cost z^* is m_2. Therefore, in this instance

$$\frac{z^{2S}}{z^*} = \frac{M_2}{m_2}.$$

□

Consider now the rolling horizon approach with $W = 2$, where a three-stage stochastic programming model is solved at each iteration. Let L_t be the residual quantity to ship at stage t and Ω_t^W be the set of scenarios obtained by considering the probability distributions of the unit transportation costs at time t and $t+1$. Then, the three-stage stochastic programming model to solve at each time $t = 0, 1, \ldots, H-2$ is:

$$\begin{aligned}
\min \quad & \sum_{\omega \in \Omega_t^W} p(\omega) \left\{ P_t(\omega) y_t + P_{t+1}(\omega) y_{t+1}(\omega) \right\} + h\left(L_t - y_t\right) \\
\text{s.t.} \quad & y_t + y_{t+1}(\omega) = L_t, \qquad \omega \in \Omega_t^W, \\
& y_t \in \mathbb{R}^+, \\
& y_{t+1}(\omega) \in \mathbb{R}^+, \qquad \omega \in \Omega_t^W.
\end{aligned}$$

Note that the non–anticipativity constraints are not needed due to the first type of constraints, as $y_{t+1}(\omega) = L_t - y_t$ is constant for all $\omega \in \Omega_t^W$. Replacing in the objective function and taking into account that $y_{t+1}(\omega) \geq 0$, this model can be written as follows:

$$\begin{aligned}
\min \quad & \sum_{\omega \in \Omega_t^W} p(\omega) \left\{ P_t(\omega) y_t + (P_{t+1}(\omega) + h)(L_t - y_t) \right\} \\
\text{s.t.} \quad & y_t \leq L_t, \\
& y_t \in \mathbb{R}^+,
\end{aligned}$$

which is equivalent to

$$\begin{array}{ll} \min & [\mathbb{E}(P_{t+1}) + h]L_t + [\mathbb{E}(P_t) - \mathbb{E}(P_{t+1}) - h]y_t \\ \text{s.t.} & y_t \leq L_t, \\ & y_t \in \mathbb{R}^+. \end{array}$$

Therefore, if $\mathbb{E}(P_t) \leq \mathbb{E}(P_{t+1}) + h$, then $y_t^* = L_t$, otherwise $y_t^* = 0$. This is the quantity $y_t^{3S}(\omega)$ to send at time t on each scenario $\omega \in \Omega$ in the policy provided by the rolling horizon approach with $W = 2$. At time $H-1$ a two-stage stochastic programming model is solved. Since a shipment can be performed only at time $H-1$, the optimal solution is simply $y_{H-1}^* = L_{H-1}$. This is the quantity $y_{H-1}^{3S}(\omega)$ to send at time $H-1$ on each scenario $\omega \in \Omega$. Note that the overall quantity L is shipped in just one time $t \in T$. The following theorem holds.

Theorem 6. *In Case 2),* $\frac{z^{3S}}{z^*} \leq \frac{M_2}{m_2}$ *and the bound is tight.*

Proof We first compute an upper bound on the cost z^{3S} of the rolling horizon approach with $W = 2$. Let τ be the time having $y_\tau^* = L$ when the three-stage stochastic programing model on $\tau, \tau+1, \tau+2$ is solved (i.e. for $0 \leq \tau < H-1$) or when the two-stage stochastic programming model on $\tau, \tau+1$ is solved (i.e. for $\tau = H-1$). In order to have this solution, we need to have $\mathbb{E}(P_0) > \mathbb{E}(P_1) + h$, $\mathbb{E}(P_1) > \mathbb{E}(P_2) + h, \ldots$, $\mathbb{E}(P_{\tau-1}) > \mathbb{E}(P_\tau) + h$. The cost z^{3S} is $\mathbb{E}(P_\tau)L + h\tau L$, which is not greater than $(M_2 + h\tau)L$.

Since the optimal cost is equal to $\min_{t \in T} \{\mathbb{E}(P_t) + ht\} L$ (see Theorem 2), we have just two cases. In the first one, the policy is able to find the optimal cost and therefore the worst-case performance bound is equal to 1. In the second case, since $\mathbb{E}(P_0) > \mathbb{E}(P_1) + h$, $\mathbb{E}(P_1) > \mathbb{E}(P_2) + h, \ldots$, $\mathbb{E}(P_{\tau-1}) > \mathbb{E}(P_\tau) + h$, a lower bound on the optimal cost z^* is $(m_2 + h(\tau+1))L$, meaning that L is shipped at minimum cost m_2 at time $\tau + 1$. Therefore,

$$\frac{z^{3S}}{z^*} \leq \frac{(M_2 + h\tau)L}{(m_2 + h(\tau+1))L} \leq \frac{M_2}{m_2}.$$

We now prove that the bound is tight. Consider the following instance: demand $L = 1$; deadline $H = 3$; probability distribution at time 0 and at time 1: M_2 with probability 1; probability distribution at time 2: m_2 with probability 1; unit inventory cost $h = \epsilon << 1$.

Let us apply the rolling horizon approach with $W = 2$. In the first step, the first three-stage stochastic programming model is solved. Since $\mathbb{E}(P_0) < \mathbb{E}(P_1) + h$, $y_0^* = L$. Therefore, $y_0^{3S} = L$ and $z^{3S} = M_2$. The optimal total cost z^* is not greater than the cost of the following solution: $y_0 = 0, y_1(\omega) = 0, y_2(\omega) = L$ having total cost $m_2 + 2\epsilon$. Therefore, in this instance

$$\frac{z^{3S}}{z^*} \geq \frac{M_2}{m_2 + 2\epsilon} \rightarrow \frac{M_2}{m_2} \qquad \text{for } \epsilon \rightarrow 0.$$

□

Case 3) deterministic unit transportation cost (uncapacitated).

Remember that in this case any 2-stage stochastic programming model is infeasible. Therefore, we study the worst-case performance of the rolling horizon approach with $W = 2$, based on the optimal solution of a sequence of 3-stage stochastic programming models. The following theorem holds.

Theorem 7. *In Case 3),* $\frac{z^{3S}}{z^*} \leq \max\left\{\frac{M_2}{m_2}, H-1\right\}$ *and the bound is tight.*

Proof Let us apply the rolling horizon approach with $W = 2$. In the first step, the first three-stage stochastic program is solved. If $P_0 \leq P_1 + h$, then $y_0^* = L_{\min}$, otherwise, $y_0^* = 0$ (see Theorem 3). In the rolling horizon policy, we capture only the value of the first stage variable y_0^{3S}. Therefore, we have $y_0^{3S} = L_{\min}$ in the former case and $y_0^{3S} = 0$ in the latter case. Then, at each time $1 \leq t < H-2$, the corresponding three-stage stochastic model is solved starting from time t. We have two cases:

1. Total quantity sent up to time $t-1$ equal to 0, i.e. $\sum_{\tau=0}^{t-1} y_\tau^{3S} = 0$: if $P_t \leq P_{t+1} + h$ then $y_t^* = L_{\min}$, otherwise $y_t^* = 0$. Since in the rolling horizon policy we just capture the value of the first-stage variable y_t^{3S}, we have $y_t^{3S} = L_{\min}$ in the former case and $y_t^{3S} = 0$ in the latter case.
2. Total quantity sent up to time $t-1$ equal to $L_{\min}$, i.e. $\sum_{\tau=0}^{t-1} y_\tau^{3S} = L_{\min}$: since $\min_\omega \{L(\omega) - L_{\min}\} = 0$, then $y_t^* = 0$ and therefore $y_t^{3S} = 0$.

At time $H-2$, where the last three-stage stochastic model is solved, if $\sum_{\tau=0}^{H-3} y_\tau^{3S} = L_{\min}$, then $L(\omega) - L_{\min}$ is shipped at time $H-1$ in each scenario ω. If $\sum_{\tau=0}^{H-3} y_\tau^{3S} = 0$, then $L(\omega)$ is shipped at time $H-1$ in each scenario ω. Therefore, $y_{H-1}^{3S} = L(\omega) - L_{\min}$ and $z^{3S} \leq L_{\min} M_2 + [M_2 + (H-1)h][\mathbb{E}(L) - L_{\min}] = M_2 \mathbb{E}(L) + (H-1)h[\mathbb{E}(L) - L_{\min}]$ in the former case, while $y_{H-1}^{3S} = L(\omega)$ and $z^{3S} \leq [M_2 + (H-1)h]\mathbb{E}(L)$. in the latter one.

Let us now compute a lower bound on the optimal total cost z^*. If $P_0 = \min_{t\in T}\{P_t + ht\}$, then $z^* \geq m_2 L_{\min} + (m_2+h)[\mathbb{E}(L) - L_{\min}] = m_2\mathbb{E}(L) + h[\mathbb{E}(L) - L_{\min}]$. Otherwise, $z^* \geq \mathbb{E}(L)(m_2 + h)$ (see Theorem 3).

Therefore, if $P_0 = \min_{t\in T}\{P_t + ht\}$, then

$$\frac{z^{3S}}{z^*} \leq \frac{M_2\mathbb{E}(L) + (H-1)h[\mathbb{E}(L) - L_{\min}]}{m_2\mathbb{E}(L) + h[\mathbb{E}(L) - L_{\min}]} \leq \max\left\{\frac{M_2}{m_2}, H-1\right\}.$$

Otherwise,

$$\frac{z^{3S}}{z^*} \leq \frac{[M_2 + (H-1)h]\mathbb{E}(L)}{(m_2+h)\mathbb{E}(L)} \leq \max\left\{\frac{M_2}{m_2}, H-1\right\}.$$

In order to prove that the bound is tight, consider the following two instances.

Instance 1: Unit transportation cost $P_t = \frac{1}{h}$ for all $t \in T$, where h is the unit inventory cost assumed to be greater than 0. Let us apply the rolling horizon approach with $W = 2$. Since $P_0 < P_1 + h$, $y_0^* = L_{\min}$. Therefore, $y_0^{3S} = L_{\min}$ and $y_t^{3S}(\omega) = 0$, for $t = 1, 2, \ldots, H-2$, while $y_{H-1}^{3S}(\omega) = L(\omega) - L_{\min}$. Therefore, $z^{3S} = \frac{1}{h}\mathbb{E}(L) + (H-1)h[\mathbb{E}(L) - L_{\min}]$. The optimal total cost z^* is not greater than the total cost of the solution in which $y_0 = L_{\min}$ and $y_1(\omega) = L(\omega) - L_{\min}$, that is $z^* \leq \frac{1}{h}\mathbb{E}(L) + h[\mathbb{E}(L) - L_{\min}]$. Therefore, in this instance:

$$\frac{z^{3S}}{z^*} \geq \frac{\frac{1}{h}\mathbb{E}(L) + (H-1)h[\mathbb{E}(L) - L_{\min}]}{\frac{1}{h}\mathbb{E}(L) + h[\mathbb{E}(L) - L_{\min}]} \to H-1 \qquad \text{for } h \to \infty.$$

Instance 2: Deadline $H = 3$, unit inventory cost $h << 1$, $L_{\min} = h$, unit transportation costs $P_0 = P_1 = m_2$ and $P_2 = M_2$. Let us apply the rolling horizon approach with $W = 2$. Since $P_0 < P_1 + h$, $y_0^* = L_{\min}$. Therefore, $y_0^{3S} = L_{\min}$ and $y_1^{3S} = 0$, while $y_2^{3S}(\omega) = L(\omega) - L_{\min}$. Therefore, $z^{3S} = m_2 L_{\min} + (M_2 + 2h)[\mathbb{E}(L) - L_{\min}]$. The optimal total cost z^* is not greater than the total cost of the solution in which $y_0 = 0$ and $y_1(\omega) = L(\omega)$, that is $z^* \leq (m_2 + h)\mathbb{E}(L)$. Therefore, in this instance:

$$\frac{z^{3S}}{z^*} \geq \frac{m_2 h + (M_2 + 2h)[\mathbb{E}(L) - h]}{(m_2+h)\mathbb{E}(L)} \to \frac{M_2}{m_2} \qquad \text{for } h \to 0.$$

□

6. Numerical Results

In this section, our aim is three-fold. First, we aim at understanding the maximum dimension of the multi-stage stochastic programming models in term of stages that can be solved by a state-of-the-art solver. Second, we aim at understanding how sensitive are the optimal solutions and the optimal total cost of the $(W+1)$-stage stochastic programming model with respect to the reduced time horizon W. Third, we aim at comparing the average performance of the rolling horizon approach in a given set of instances with respect to the worst-case performance bounds provided in the previous section.

We use AMPL environment along CPLEX 12.5.1.0 solver to solve the stochastic programming models (see [44]). All the computations were run on a 64-bit machine with 12 GB of RAM and a 2.90 GHz processor. Since the problem is new, benchmark instances are not available in the literature. Therefore, we generate a set of instances inspired by a real case problem provided by an Italian Logistics company named *Gamba Logistica srl.* In the problem the supply of an uncertain number of pallets, is performed by tracks with limited and different capacities in time due to groupage transportation. The instances are built as follows:

1. Deadline H: up to 8 time periods.
2. Demand L: In the cases 1) and 3), having stochastic demand, the support of the probability distribution is the set of integer numbers in the interval $[L_{\min}, L_{\max}]$, with $L_{\min} = 8$ pallets and $L_{\max} = 12$ pallets. The probability distribution $\mathcal{L}$ is given by a *Beta distribution* $B(\alpha, \beta)$, with $\alpha = 9$ and $\beta = 15$, having average demand $\mathbb{E}(L) = 10.00875104$ pallets. In the Case 2), we assume deterministic demand, $L = 10$ pallets.
3. Transportation capacities K_t (units of pallets): $K_0 = 6$, $K_1 = 7$, $K_2 = 4$, $K_3 = 6$, $K_4 = 9$. These values are such that $K_t < L$, $\forall t \in T$, and $\sum_{t \in T} K_t > L$. We note that the capacity K_t corresponds to the available capacity at time t and not to the capacity of the full track.
4. Unit transportation costs P_t (in Euros): in cases 1) and 2), having stochastic unit transportation costs, the support of the probability distribution at each time $t \in T$ is the set of integer numbers in the interval $[m_2, M_2]$, with $m_2 = 90$ and $M_2 = 100$. The probability distribution $\mathcal{P}_t$ at time $t \in T$ is given by a *Beta Distribution* $B(\alpha_t, \beta_t)$. The values of α_t and β_t are shown in Table 1. They are selected in such a way that

the expected values $\mathbb{E}(P_t)$ of the probability distributions are decreasing over time, as shown in Table 2. In Case 3), having deterministic costs, the unit transportation cost P_t at each time $t \in T$ is equal to $\mathbb{E}(P_t)$. Notice that the considered unit price corresponds to the shipping of one good pallet with 100-200 Kg weight on a distance up to 500 Km.

5. Fixed transportation costs Q_t (in Euros) to buy the full capacity K_t on the track: they are generated to maintain a predefined ratio θ between the total variable cost $\mathbb{E}(P_t) \cdot K_t$ and the fixed cost Q_t. The instances are grouped into 2 classes characterized by $\theta = 0.2$ and 0.5. Similar results were obtained for $\theta = 0.1, 0.3, 0.4, 0.6, 0.7, 0.8, 0.9, 1$. The idea of a predefined ratio between the total variable cost and the fixed cost has been inspired by [37].
6. Unit inventory cost h (in Euros): $0, 1, \ldots, 10$ in Case 1) and specific intervals, provided in the following, in cases 2) and 3). From a practical point of view, the considered unit inventory costs approximatively correspond to the 5% of the value of a pallet of 100 Kg.

t	0	1	2	3	4	5	6	7	8	9
α_t	9	15	6	5	9	8	3	5	1	1
β_t	10	20	10	10	20	20	10	20	10	20

Table 1: Values of α_t and β_t in the Beta distribution $B(\alpha_t, \beta_t)$

t	0	1	2	3	4	5	6	7	8	9
$\mathbb{E}(P_t)$	95.23	94.78	94.25	93.83	93.60	93.35	92.80	92.50	91.49	91.13

Table 2: Expected value of the unit transportation costs P_t

Notice that, since the support of the uncertain demand and costs are discrete, in our approach we solve the full stochastic programming problem with the complete scenario tree structure. No scenario reduction techniques are adopted.

6.1. Solving the multi-stage stochastic programming models

In this subsection, we provide statistics concerning the performance of a state-of-the-art solver (CPLEX) to find an optimal solution of the multi-stage stochastic programming models formulated in Sections 3 and 4 for the cases 1), 2) and 3). In particular, Tables 3-4-5 show the number of simplex iterations and the CPU time (in seconds) in the cases 1), 2) and 3), respectively, required by CPLEX to find an optimal solution of the corresponding multi-stage stochastic programming model, when the number of stages increases.

	three-stage	four-stage
simplex iterations	0	1373
CPU time (s)	0.23	80.40

Table 3: Case 1) Summary statistics

	two-stage	three-stage	four-stage	five-stage	six-stage
simplex iterations	0	0	65	488	3033
CPU time (s)	0.01	0.01	0.09	0.65	36.56

Table 4: Case 2) Summary statistics

	three-stage	four-stage	five-stage	six-stage	seven-stage	eight-stage	nine-stage
simplex it.	0	16	52	217	888	2943	9695
CPU time (s)	0.01	0.03	0.06	0.26	1.91	23.68	221.56

Table 5: Case 3) Summary statistics

These results show that in Case 1) an optimal solution of the model (2), which is NP-hard, can be obtained just up to the four-stage stochastic programming model, while CPLEX runs out of memory starting from the five-stage case. Therefore, heuristic algorithms, like the rolling horizon approach, are required. More interesting, even the polynomially solvable cases 2) and 3) can be solved just up to the six-stage and the nine-stage stochastic programming models, respectively. This gives additional value to the optimal policies provided in Section 4, that are able to solve any H-stage stochastic model in cases 2) and 3) in $O(H)$ time.

6.2. Analysis of the $(W + 1)$-stage stochastic programming model

Since the previous model, with a reduced time horizon $W < H$, is embedded in the rolling horizon approach, we now show a sensitivity analysis of the optimal solution and of the total cost of this model with respect to the reduced time horizon W. Our aim is to understand for which values of the unit inventory cost h the optimal value of the first-stage variable and the total cost of this model are significantly affected by the value of the reduced time horizon W.

Case 1): stochastic demand and unit transportation costs.

Table 6 shows the optimal value of the first-stage variable y_0^* and the total cost in Case 1) for different values of the unit inventory cost $h = 0, \ldots, 10$, when the predefined ratio θ between the expected total variable cost $\mathbb{E}(P_t) \cdot K_t$ and the fixed cost Q_t is 0.2 and 0.5. We do not show the values of the variables $y_t(\omega)$ for $t > 0$ because they are different in different scenarios $\omega \in \Omega$. We just consider the cases with $W = 2$ (three-stage) and $W = 3$ (four-stage), as the optimal solution cannot be computed for larger numbers of stages (as previously shown in Table 3).

W	h	y_0^*	Total cost ($\theta = 0.2$)	Total cost ($\theta = 0.5$)
2	0	5	7166.28	3426.52
2	1	6	7171.15	3431.39
2	2	6	7175.24	3435.48
2	3	6	7179.32	3439.56
2	4	6	7183.41	3443.65
2	5	6	7187.49	3447.73
2	6	6	7191.58	3451.82
2	7	6	7195.66	3455.90
2	8	6	7199.75	3459.99
2	9	6	7203.83	3464.07
2	10	6	7207.92	3468.16
3	0	6	5971.76	2946.92
3	1	6	5979.11	2954.28
3	2	6	5986.46	2961.63
3	3	6	5993.82	2968.98
3	4	6	6001.17	2976.33
3	5	6	6008.52	2983.69
3	6	6	6015.87	2991.04
3	7	6	6023.23	2998.39
3	8	6	6030.58	3005.74
3	9	6	6037.93	3013.10
3	10	6	6045.28	3020.45

Table 6: Case 1) Optimal value of the first-stage variable y_0^* and of the total cost with $\theta = 0.2$ and 0.5

The results show that, in all cases, the value of the optimal first-stage variable y_0^* is to send a quantity equal to the capacity K_0, with exception of the case with $W = 2$ and $h = 0$ only, in which a lower quantity is sent at time 0. Moreover, the three-stage ($W = 2$) stochastic programming model is significantly more costly than the four-stage one ($W = 3$). In particular, the average percent increase of the total cost of the model with $W = 2$ with respect to the model with $W = 3$ is about 19% and 15% for $\theta = 0.2$ and 0.5, respectively. As a managerial insight, this implies that a manager should consider models with larger time horizon, as they allow to significantly reduce the costs. These effects are evident especially for low values of the ratio θ.

Case 2) deterministic demand (uncapacitated).

Table 7 and Figure 2 show the optimal values of the variables y_0 and $y_t(\omega)$, $t > 0$, and the total cost in Case 2), having deterministic demand $L = 10$, for different values of the inventory cost h, when the reduced time horizon W increases from 1 to 5.

W	h	$y_0,\, y_t(\omega) \neq 0,\, t > 0$	Total cost
1	$[0, \infty)$	$y_0 = L,\ \omega = 1, \ldots, 11$	$\mathbb{E}(P_0) \cdot L = 952.36$
2	$[0, 0.4511)$	$y_1(\omega) = L,\ \omega = 1, \ldots, 121$	$(\mathbb{E}(P_1) + h) \cdot L$
2	$[0.4511, \infty)$	$y_0 = L,\ \omega = 1, \ldots, 121$	$\mathbb{E}(P_0) \cdot L = 952.368$
3	$[0, 0.4934)$	$y_2(\omega) = L,\ \omega = 1, \ldots, 1331$	$(\mathbb{E}(P_2) + 2h) \cdot L$
3	$[0.4934, \infty)$	$y_0 = L,\ \omega = 1, \ldots, 1331$	$\mathbb{E}(P_0) \cdot L = 952.36$
4	$[0, 0.4167)$	$y_3(\omega) = L\ \omega = 1, \ldots, 14641$	$(\mathbb{E}(P_3) + 3h) \cdot L$
4	$[0.4167, 0.4934)$	$y_2(\omega) = L\ \omega = 1, \ldots, 14641$	$(\mathbb{E}(P_2) + 2h) \cdot L$
4	$[0.4934, \infty)$	$y_0 = L\ \omega = 1, \ldots, 14641$	$\mathbb{E}(P_0) \cdot L = 952.36$
5	$[0, 0.2299)$	$y_4(\omega) = L\ \omega = 1, \ldots, 161051$	$(\mathbb{E}(P_4) + 4h) \cdot L$
5	$[0.0.2299, 0.4167)$	$y_3(\omega) = L\ \omega = 1, \ldots, 161051$	$(\mathbb{E}(P_3) + 3h) \cdot L$
5	$[0.4167, 0.4934)$	$y_2(\omega) = L\ \omega = 1, \ldots, 161051$	$(\mathbb{E}(P_2) + 2h) \cdot L$
5	$[0.4934, \infty)$	$y_0 = L\ \omega = 1, \ldots, 161051$	$\mathbb{E}(P_0) \cdot L = 952.36$

Table 7: Case 2) Optimal value of the variable y_0 and $y_t(\omega)$, $t > 0$, and of the total cost

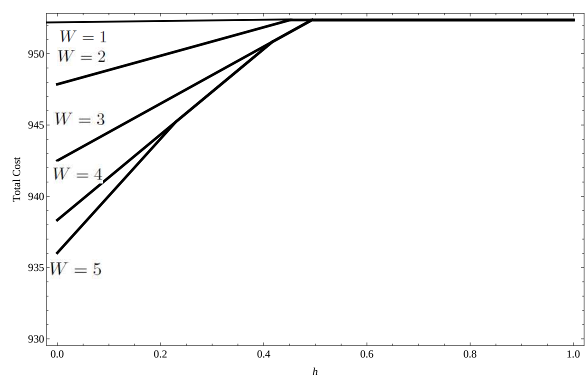

Figure 2: Case 2) Total cost against increasing values of h for different values of W

The results show that, in the simpler two-stage stochastic programming model ($W = 1$), the unique solution, irrespectively to the inventory cost values, is to ship the total quantity L at time 0, i.e. $y_0 = L$, with a total expected cost $\mathbb{E}(P_0) \cdot L = 952.368$, the highest one (see Table 7). For the three-stage stochastic programming model ($W = 2$), $y_0 = L$ only for $h \geq 0.4511$, while for $h < 0.4511$ it is more convenient to wait until the second stage. In the four-stage stochastic programming model ($W = 3$), the total quantity L is shipped at time 0, i.e. $y_0 = L$, if $\mathbb{E}(P_0) < \mathbb{E}(P_t) + ht, \forall\ t = 1, \ldots, H-1$, which is verified for $h \geq 0.4934$. On the contrary, when $h < 0.4934$, it is more convenient to wait until the third stage, with a total expected cost $(94.2500 + 2h)L$. The demand is never shipped at the second stage since this requires that $\mathbb{E}(P_1) + h < \mathbb{E}(P_2) + 2h$, which is satisfied for $h > 0.5357$, but in such a range the total quantity L is shipped at time 0.

Similarly, in the five-stage stochastic programming model ($W = 4$), the total quantity L is shipped at time 0, i.e. $y_0 = L$, for $h \geq 0.4934$. On the contrary, when $0.4167 \leq h < 0.4934$, it is more convenient to wait until the third stage, with a total expected cost of $L(94.2500 + 2h)$ and, when $h < 0.4167$, it is more convenient to wait until the fourth stage, with a total expected cost $10(93.8333 + 3h)$.

Similar arguments can be applied to the six-stage stochastic programming model ($W = 5$).

In conclusion, this analysis, based on different values of the reduced time horizon W, allows us to deduce that the same total cost is obtained for $h \geq 0.4934$. On the other hand, for $h < 0.4934$, the higher is the reduced time horizon W, the lower is the total cost to be paid. This gives a measure of the value of having more stages to ship the demand. As a managerial insight, this implies that for the manager, in case of low values of the ratio θ, it is more convenient to solve models with larger time horizon, as they allow to significantly reduce the costs.

Note that all the results confirm the optimal policy derived in Theorem 2.

Case 3) deterministic unit transportation cost (uncapacitated).

Table 8 and Figure 3 show the optimal values of the variables y_0 and $y_t(\omega)$, $t > 0$, and the total cost in Case 3), having deterministic unit transportation costs, for different values of the inventory cost h, when the reduced time horizon W increases from 2 to 5. The case with $W = 1$ is infeasible because it does not allow to satisfy the different demands in different scenarios.

The results show that, in the case with $W = 2$, $L_{\min}$ is sent at time $t = 0$ and the remaining $L(\omega) - L_{\min}$ at time 1 if $P_0 \leq P_1 + h$, which is satisfied for $h \geq 0.4511$. Otherwise, the stochastic demand $L(\omega)$ is sent at time 1. Analogous arguments apply for the cases with larger reduced time horizons $W = 3, \ldots, 5$. The results show that, for $h \geq 0.5357$, the optimal total costs are the same, irrespectively of the number of stages considered. On the other hand, for $h < 0.5357$, the larger is the reduced time horizon W, the lower is

W	h	$y_0,\, y_t(\omega) \neq 0,\, t>0$	Total cost
2	$[0, 0.4511)$	$y_1(\omega) = L(\omega),\ \omega = 1,\dots,5$	$\mathbb{E}(L)(P_1 + h)$
2	$[0.4511, \infty)$	$y_0 = L_{\min}, y_1 = L(\omega) - L_{\min}$	$P_0 \cdot L_{\min} + (P_1 + h)(\mathbb{E}(L) - y_0)$
3	$[0, 0.4934)$	$y_2(\omega) = L(\omega),\ \omega = 1,\dots,5$	$\mathbb{E}(L)(P_2 + 2h)$
3	$[0.4934, 0.5357)$	$y_0 = L_{\min},\ y_2 = L(\omega) - L_{\min},\ \omega = 1,\dots,5$	$P_0 \cdot L_{\min} + (P_2 + 2h)(\mathbb{E}(L) - y_0)$
3	$[0.5357, \infty)$	$y_0 = L_{\min},\ y_1 = L(\omega) - L_{\min},\ \omega = 1,\dots,5$	$P_0 \cdot L_{\min} + (P_1 + h)(\mathbb{E}(L) - y_0)$
4	$[0, 0.4167)$	$y_3(\omega) = L(\omega),\ \omega = 1,\dots,5$	$\mathbb{E}(L)(P_3 + 3h)$
4	$[0.4167, 0.4934)$	$y_2(\omega) = L(\omega),\ \omega = 1,\dots,5$	$\mathbb{E}(L)(P_2 + 2h)$
4	$[0.4934, 0.5357)$	$y_0 = L_{\min},\ y_2 = L(\omega) - L_{\min},\ \omega = 1,\dots,5$	$P_0 \cdot L_{\min} + (P_2 + 2h)(\mathbb{E}(L) - y_0)$
4	$[0.5357, \infty)$	$y_0 = L_{\min},\ y_1 = L(\omega) - L_{\min},\ \omega = 1,\dots,5$	$P_0 \cdot L_{\min} + (P_1 + h)(\mathbb{E}(L) - y_0)$
5	$[0, 0.2299)$	$y_4(\omega) = L(\omega),\ \omega = 1,\dots,5$	$\mathbb{E}(L)(P_4 + 4h)$
5	$[0.2299, 0.4167)$	$y_3(\omega) = L(\omega),\ \omega = 1,\dots,5$	$\mathbb{E}(L)(P_3 + 3h)$
5	$[0.4167, 0.4934)$	$y_2(\omega) = L(\omega),\ \omega = 1,\dots,5$	$\mathbb{E}(L)(P_2 + 2h)$
5	$[0.4934, 0.5357)$	$y_0 = L_{\min},\ y_2 = L(\omega) - L_{\min},\ \omega = 1,\dots,5$	$P_0 \cdot L_{\min} + (P_2 + 2h)(\mathbb{E}(L) - y_0)$
5	$[0.5357, \infty)$	$y_0 = L_{\min},\ y_1 = L(\omega) - L_{\min},\ \omega = 1,\dots,5$	$P_0 \cdot L_{\min} + (P_1 + h)(\mathbb{E}(L) - y_0)$

Table 8: Case 3) Optimal value of the variables y_0 and $y_t(\omega)$, $t > 0$, and of the total cost

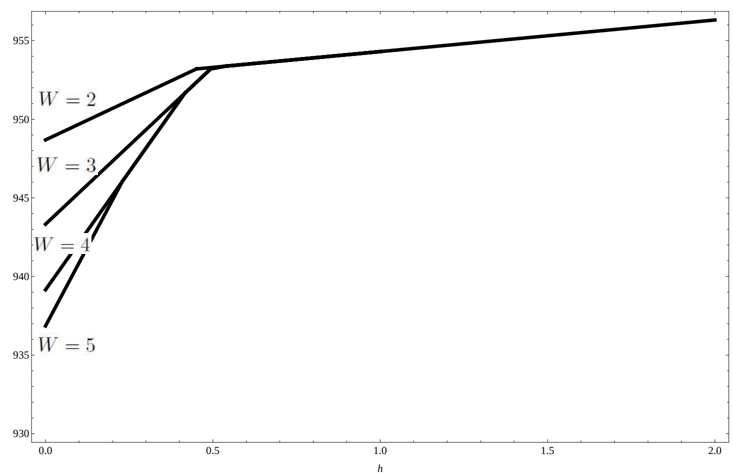

Figure 3: Case 3) Total cost against increasing values of h for different values of W

the total cost to be paid.

As a managerial insight, this implies that for the manager, in case of low values of the ratio θ, it is more convenient to solve models with larger time horizon, as they allow to significantly reduce the costs. On the contrary, for large values of ratio, the manager should choose a three-stage model, as it has the same cost of models with greater time horizon.

Note that all the computational results shown in Table 8 confirm the optimal policy provided in Theorem 3.

6.3. The rolling horizon approach

In this subsection, we evaluate the average performance of the rolling horizon approach in a given set of instances.

In order to do this, we solve the original multi-stage problem by building a series of models with reduced time horizon, which are easier to solve. This means that we solve a model with reduced time horizon, we store its first stage solution and step forward in time; then, we solve the problem starting from the next stage again and we store its first stage solution. The process is repeated until we reach the end of the original time horizon H. The strategy that we use is to reduce the time horizon from $[t, H]$ to $[t, t + W]$, where W is a suitable reduced time horizon and $t = 0, \ldots, H - W$. For our numerical analysis we create a multi-stage stochastic programming model with finite horizon, where uncertainties are captured using a scenario tree. Notice that, at each step of this procedure, we also need to make decisions over our planning horizon $t' = t, \ldots, t + W$, but the decisions we make at time $t' > t$ are purely for the purpose of making a better decision at time t.

Case 1) stochastic demand and unit transportation costs.

Table 9 and Figure 4 report numerical results of the rolling horizon approach in Case 1) for $W = 2$, in instances with deadline $H = 3$, in the cases with $\theta = 0.2$ and 0.5. The optimal policy and total cost of the four-stage problem ($H = 3$) are also reported for comparison.

The results show that the rolling horizon approach gives an average percent cost increase with respect to the optimal total cost of about 51% and 74% in the cases with $\theta = 0.2$ and 0.5, respectively. This is mainly due to the fact that, in the optimal policy, the quantity sent at time 1 is scenario dependent, while it is constant over all scenarios in the rolling horizon approach, as shown in the top part of Table 9. However, the percent cost increase of the rolling horizon is not very affected by the value of the inventory cost h. Comparing these results with the corresponding worst-case analysis provided in Section 5 (Theorem 4 where we showed that there is not a finite worst-case performance bound), we can conclude that in Case 1) the rolling horizon approach can be very suboptimal. As a managerial insight, these results imply that the manager should carefully evaluate the average performance of the rolling horizon approach in the typical set of instances considered by the company. This would avoid a very bad performance provided by a blind application of the heuristic approach.

Rolling horizon approach with $W=2$						
W	h	$y_0^{(W+1)S}$	$y_1^{(W+1)S}(\omega)$	$y_2^{(W+1)S}(\omega)$	Total cost ($\theta=0.2$)	Total cost ($\theta=0.5$)
2	0	5	3	$L(\omega)-8$	8985.51	5233.92
2	1	6	2	$L(\omega)-8$	9082.74	5148.40
2	2	6	2	$L(\omega)-8$	9084.72	5156.57
2	3	6	2	$L(\omega)-8$	9086.69	5164.74
2	4	6	2	$L(\omega)-8$	9088.66	5172.91
2	5	6	2	$L(\omega)-8$	9090.63	5181.08
2	6	6	2	$L(\omega)-8$	9092.60	5189.25
2	7	6	2	$L(\omega)-8$	9094.57	5197.42
2	8	6	2	$L(\omega)-8$	9096.55	5205.59
2	9	6	2	$L(\omega)-8$	9098.52	5213.76
2	10	6	2	$L(\omega)-8$	9100.49	5221.93
Optimal policy						
H	h	y_0^*	$y_1^*(\omega)$	$y_2^*(\omega)$	Total cost ($\theta=0.2$)	Total cost ($\theta=0.5$)
3	0	6	–	–	5971.76	2946.92
3	1	6	–	–	5979.11	2954.28
3	2	6	–	–	5986.46	2961.63
3	3	6	–	–	5993.82	2968.98
3	4	6	–	–	6001.17	2976.33
3	5	6	–	–	6008.52	2983.69
3	6	6	–	–	6015.87	2991.04
3	7	6	–	–	6023.23	2998.39
3	8	6	–	–	6030.58	3005.74
3	9	6	–	–	6037.93	3013.10
3	10	6	–	–	6045.28	3020.45

Table 9: Case 1) Policies and total costs against different values of the inventory cost h for the rolling horizon approach with $W=2$ and for the optimal policy, in instances with deadline $H=3$, in the cases with $\theta=0.2$ and 0.5

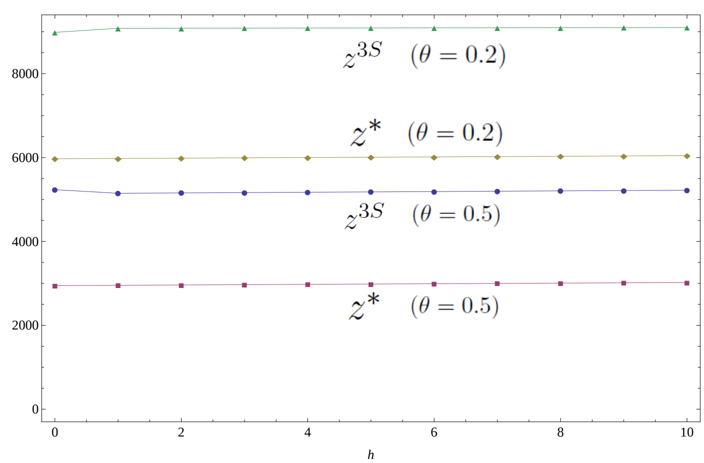

Figure 4: Case 1) Total cost against increasing value of the inventory cost h for the rolling horizon approach with $W=2$ and for the optimal policy in instances with deadline $H=3$, in the cases with $\theta=0.2$ and 0.5

Table 10 and Figure 5 report optimal policies and total costs in Case 1) for the rolling horizon approach with $W = 2$ and 3 in instances with deadline $H = 5$. Since the optimal policy is not available, we compare the results obtained by the rolling horizon approach for $\theta = 0.2$ and 0.5.

W	h	$y_0^{(W+1)S}$	$y_1^{(W+1)S}(\omega)$	$y_2^{(W+1)S}(\omega)$	$y_3^{(W+1)S}(\omega)$	$y_4^{(W+1)S}(\omega)$	Total cost ($\theta = 0.2$)	Total cost ($\theta = 0.5$)
2	0	5	3	0	0	$L(\omega) - 8$	11467.95	3426.52
2	1	6	2	0	0	$L(\omega) - 8$	11470.32	3431.39
2	2	6	2	0	0	$L(\omega) - 8$	11471.89	3435.48
2	3	6	2	0	0	$L(\omega) - 8$	11473.46	3439.56
2	4	6	2	0	0	$L(\omega) - 8$	11475.03	3443.65
2	5	6	2	0	0	$L(\omega) - 8$	11476.60	3447.73
2	6	6	2	0	0	$L(\omega) - 8$	11478.17	3451.82
2	7	6	2	0	0	$L(\omega) - 8$	11479.74	3455.90
2	8	6	2	0	0	$L(\omega) - 8$	11481.31	3459.99
2	9	6	2	0	0	$L(\omega) - 8$	11482.88	3464.07
2	10	6	2	0	0	$L(\omega) - 8$	11484.45	3468.16
3	0	6	0	0	0	$L(\omega) - 6$	8155.05	2946.92
3	1	6	0	0	0	$L(\omega) - 6$	8158.62	2954.28
3	2	6	0	0	0	$L(\omega) - 6$	8162.20	2961.63
3	3	6	0	0	0	$L(\omega) - 6$	8165.77	2968.98
3	4	6	0	0	0	$L(\omega) - 6$	8169.34	2976.33
3	5	6	0	0	0	$L(\omega) - 6$	8172.91	2983.69
3	6	6	0	0	0	$L(\omega) - 6$	8176.48	2991.04
3	7	6	0	0	0	$L(\omega) - 6$	8180.05	2998.39
3	8	6	0	0	0	$L(\omega) - 6$	8183.62	3005.74
3	9	6	0	0	0	$L(\omega) - 6$	8187.19	3013.10
3	10	6	0	0	0	$L(\omega) - 6$	8190.76	3020.45

Table 10: Case 1) Policies and total costs against different values of the inventory cost h for the rolling horizon approach with $W = 2$ and 3 in instances with deadline $H = 5$, in the cases with $\theta = 0.2$ and 0.5

The results show the value of using a problem with a larger number of stages in the rolling horizon approach. In fact, the average percent cost increase in the total cost of the rolling horizon approach with $W = 2$ with respect to $W = 3$ is about 40% and 15% in the case with $\theta = 0.2$ and 0.5, respectively. However, the percent cost increase of the rolling horizon approach is not very affected by the value of the inventory cost h.

As a managerial insight, this implies that for the manager it is more convenient to solve models with larger time horizon, as they allow to significantly reduce the costs. These effects are evident especially for low values of the ratio θ.

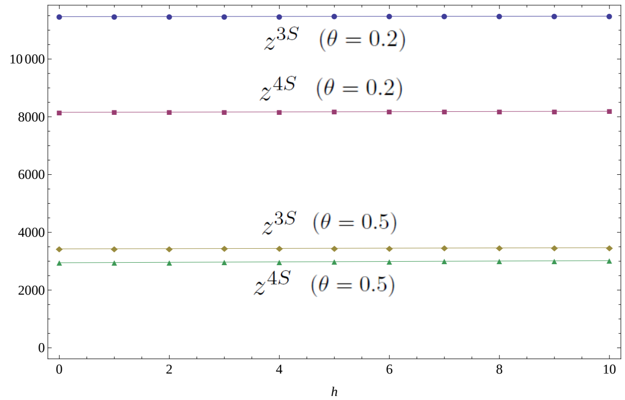

Figure 5: Case 1) Total cost against increasing values of the inventory cost h for the rolling horizon approach with $W = 2$ and 3 in instances with deadline $H = 5$, in the cases with with $\theta = 0.2$ and 0.5

Case 2) deterministic demand (uncapacitated).

Table 11 and Figure 6 show the numerical results obtained by applying the rolling horizon approach in Case 2) for increasing values of the inventory cost h in instances with deadline $H = 5$. The optimal policy and total cost of the six-stage problem ($H = 5$) are also reported for comparison.

Rolling horizon approach with $W = 1, 2, 3, 4$							
W	h	$y_0^{(W+1)S}$	$y_1^{(W+1)S}(\omega)$	$y_2^{(W+1)S}(\omega)$	$y_3^{(W+1)S}(\omega)$	$y_4^{(W+1)S}(\omega)$	Total cost
1	$[0,\infty)$	L	0	0	0	0	$\mathbb{E}(P_0) \cdot L$
2	[0, 0.2299)	0	0	0	0	L	$(\mathbb{E}(P_4) + 4h) \cdot L$
2	[0.2299, 0.4167)	0	0	0	L	0	$(\mathbb{E}(P_3) + 3h) \cdot L$
2	[0.4167,0.4511)	0	0	L	0	0	$(\mathbb{E}(P_2) + 2h) \cdot L$
2	$[0.4511,\infty)$	L	0	0	0	0	$\mathbb{E}(P_0) \cdot L$
3 /4	[0, 0.2299)	0	0	0	0	L	$(\mathbb{E}(P_4) + 4h) \cdot L$
3 /4	[0.2299, 0.4167)	0	0	0	L	0	$(\mathbb{E}(P_3) + 3h) \cdot L$
3 /4	[0.4167,0.4934)	0	0	L	0	0	$(\mathbb{E}(P_2) + 2h) \cdot L$
3 /4	$[0.4934,\infty)$	L	0	0	0	0	$\mathbb{E}(P_0) \cdot L$
Optimal policy							
H	h	y_0^*	$y_1^*(\omega)$	$y_2^*(\omega)$	$y_3^*(\omega)$	$y_4^*(\omega)$	Total cost
5	$[0, 0.2299)$	0	0	0	0	L	$(\mathbb{E}(P_4) + 4h) \cdot L$
5	$[0.2299, 0.4167)$	0	0	0	L	0	$(\mathbb{E}(P_3) + 3h) \cdot L$
5	$[0.4167, 0.4934)$	0	0	L	0	0	$(\mathbb{E}(P_2) + 2h) \cdot L$
5	$[0.4934, \infty)$	L	0	0	0	0	$\mathbb{E}(P_0) \cdot L$

Table 11: Case 2) Policies and total costs against different values of the inventory cost h for the rolling horizon approach with $W = 1, 2, 3, 4$ and for the optimal policy, in instances with deadline $H = 5$

The results show that in the simpler rolling horizon approach with $W =$ 1, the policy, irrespectively to the inventory cost values, is to ship the total

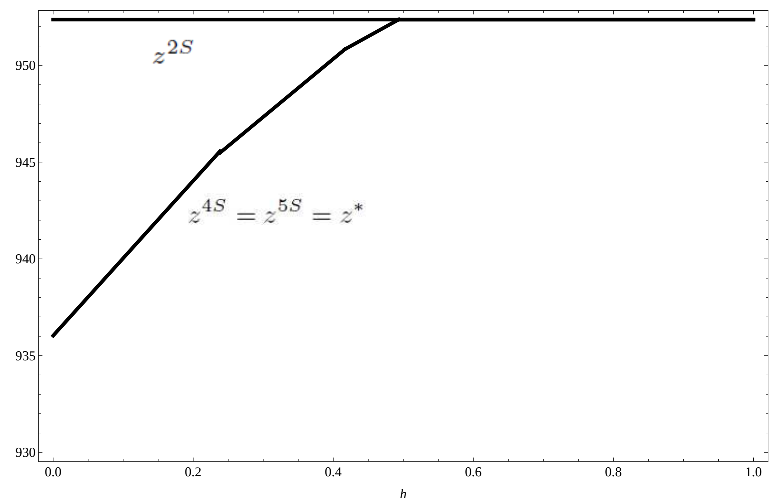

Figure 6: Case 2) Total cost against increasing value of the inventory cost h for the rolling horizon approach with $W = 1, 2, 3, 4$ and for the optimal policy in instances with deadline $H = 5$

quantity L at time 0, i.e. $y_0^{2S} = L$ with a total cost $\mathbb{E}(P_0)L$. In the rolling horizon approach with $W = 2$, L is sent at time 0 if $\mathbb{E}(P_0) \leq \mathbb{E}(P_1) + h$, i.e. $95.2368 \leq 94.7857 + h$, which is satisfied for $h \geq 0.4511$. Otherwise, if $h \leq 0.4511$, 0 units are sent at time 0. After storing the solution $y_0^{3S} = 0$ and solving the new three-stage stochastic programming model starting at time 1, L could be sent at time 1 if $\mathbb{E}(P_1) + h \leq \mathbb{E}(P_2) + 2h$, which is verified for $h \geq 0.535714$. However, this is not consistent with $h \leq 0.4511$. Consequently, $y_1^{3S}(\omega) = 0$. Then, a new three-stage stochastic programming model is solved: L is sent at time 2 if $\mathbb{E}(P_2)+2h \leq \mathbb{E}(P_3)+3h$, which is verified for $h \geq 0.4167$. Consequently, in the range $[0.4167, 0.4511)$, $y_2^{3S}(\omega) = L$. If $h \leq 0.4167$, $y_2^{3S}(\omega) = 0$ and a new three-stage stochastic programming model starting at time 3 is solved: L is sent at time 3 if $\mathbb{E}(P_3) + 3h \leq \mathbb{E}(P_4) + 4h$, which is verified for $h \geq 0.2299$. Consequently, in the range $[0.2299, 0.4167)$, $y_3^{3S}(\omega) = L$. Finally, if $h \leq 0.2299$ the demand L is sent at time 4. With similar arguments, the policies of the rolling horizon approach with $W = 3, 4$ can be explained. Notice that these policies are optimal.

Figure 6 clearly shows that the ratio between the cost of the rolling horizon approach and the optimal cost is always finite, as shown by the worst-case analysis (see Theorems 5 and 7), and that the maximum percent cost increase is obtained for $h = 0$, as in the worst-case analysis.

As a managerial insight, we can conclude that the manager should pay particular attention to the case of low inventory cost h, where the total cost can be kept low by applying a rolling horizon approach with at least 4 stages.

Case 3) deterministic unit transportation cost (uncapacitated).

Table 12 and Figure 7 report numerical results of the rolling horizon approach with $W = 2, 3, 4$ in the instances with deadline $H = 5$ for increasing values of the inventory cost h. The optimal policy and total cost are also reported for comparison. With similar arguments of the previous section, the policies of the rolling horizon approach can be explained.

Rolling horizon approach with $W = 2,3,4$							
W	h	$y_0^{(W+1)S}$	$y_1^{(W+1)S}(\omega)$	$y_2^{(W+1)S}(\omega)$	$y_3^{(W+1)S}(\omega)$	$y_4^{(W+1)S}(\omega)$	Total cost
2	$[0, 0.2299)$	0	0	0	0	$L(\omega)$	$(P_4{+}4h)\mathbb{E}(L)$
2	$[0.2299, 0.4167)$	0	0	0	$L_{\min}$	$L(\omega){-}L_{\min}$	$L_{\min}(P_3 + 3h){+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
2	$[0.4167, 0.4511)$	0	0	$L_{\min}$	0	$L(\omega){-}L_{\min}$	$L_{\min}(P_2 + 2h){+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
2	$[0.4511, \infty)$	$L_{\min}$	0	0	0	$L(\omega){-}L_{\min}$	$L_{\min}P_0{+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
3	$[0, 0.2299)$	0	0	0	0	$L(\omega)$	$(P_4{+}4h)\mathbb{E}(L)$
3	$[0.2299, 0.4167)$	0	0	0	$L_{\min}$	$L(\omega){-}L_{\min}$	$L_{\min}(P_3 + 3h){+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
3	$[0.4167, 0.4934)$	0	0	$L_{\min}$	0	$L(\omega){-}L_{\min}$	$L_{\min}(P_2 + 2h){+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
3	$[0.4934, \infty)$	$L_{\min}$	0	0	0	$L(\omega){-}L_{\min}$	$L_{\min}P_0{+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
4	$[0, 0.2299)$	0	0	0	0	$L(\omega)$	$(\mathbb{E}(P_4){+}4h)\mathbb{E}(L)$
4	$[0.2299, 0.4167)$	0	0	0	$L_{\min}$	$L(\omega){-}L_{\min}$	$L_{\min}(P_3 + 3h){+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
4	$[0.4167, 0.4934)$	0	0	$L_{\min}$	0	$L(\omega){-}L_{\min}$	$L_{\min}(P_2 + 2h){+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
4	$[0.4934, \infty)$	$L_{\min}$	0	0	0	$L(\omega){-}L_{\min}$	$L_{\min}P_0{+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$

Optimal policy							
H	h	y_0^*	$y_1^*(\omega)$	$y_{2^*}(\omega)$	$y_3^*(\omega)$	$y_4^*(\omega)$	Total cost
5	$[0, 0.2299)$	0	0	0	0	$L(\omega)$	$(P_4 + 4h)\mathbb{E}(L)$
5	$[0.2299, 0.4167)$	0	0	0	$L(\omega)$	0	$(P_3 + 3h)\mathbb{E}(L)$
5	$[0.4167, 0.4934)$	0	0	$L(\omega)$	0	0	$(P_2 + 2h)\mathbb{E}(L)$
5	$[0.4934, 0.5357)$	$L_{\min}$	0	$L(\omega) - L_{\min}$	0	0	$P_0 \cdot L_{\min}{+}(P_2 + 2h)(\mathbb{E}(L) - L_{\min})$
5	$[0.5357, \infty)$	$L_{\min}$	$L(\omega) - L_{\min}$	0	0	0	$P_0 \cdot L_{\min}{+}(P_1 + h)(\mathbb{E}(L) - L_{\min})$

Table 12: Case 3) Policies and total costs against different values of the inventory cost h for the rolling horizon approach with $W = 2, 3, 4$ and for the optimal policy, in instances with deadline $H = 5$

Figure 7 clearly shows that the ratio between the total cost of the rolling horizon approach and the optimal total cost increases with the inventory cost h. It can be easily checked that the ratio between these two costs tends to $H - 1 = 4$ for $h \to \infty$, as shown in the worst-case analysis.

As a managerial insight, we can conclude that the manager should pay particular attention to the case of high inventory cost h. Moreover, the worst-case results provided before give her a measure of the maximum extra-cost she could incur by applying the rolling horizon approach with respect to the optimal cost.

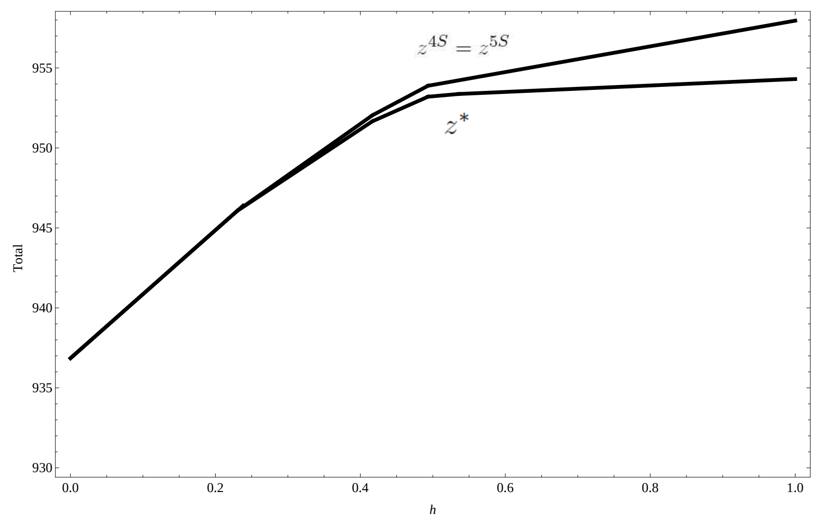

Figure 7: Case 3) Total cost against increasing value of the inventory cost h for the rolling horizon approach with $W = 2, 3, 4$ and for the optimal policy in instances with deadline $H = 5$

7. Conclusions

The paper presents a worst-case analysis of rolling horizon approach for the *Stochastic multi-stage fixed charge transportation problem*. Theoretical results showed that the rolling horizon approach can be very suboptimal in the worst case if it is used to solve the general case. Finite bounds exist for the polynomially solvable cases. Interesting results were also obtained by the computational experiment we carried out. First, we found that both the NP-hard problem and the polynomially solvable cases are very difficult to be solved by state-of-the-art solvers, when the complete scenario tree is used. In fact, we were able to solve the problems only up to 9 stages. Therefore, it is really important to design heuristic algorithms to solve the NP-hard problem and to be able to design an exact polynomial time algorithm to solve the particular cases. As a managerial insight, these results imply that the manager should carefully evaluate the average performance of the rolling horizon approach in the typical set of instances considered by the company. This would avoid a very bad performance provided by a blind application of the heuristic approach. Worst-case analysis of the rolling horizon approach on other multi-stage optimization problems under uncertainty will be addressed in future works.

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References

[1] Adlakha, V., Kowalski, K., Lev, B., 2010. A branching method for the fixed charge transportation problem. Omega 38 (5), 393–397.

[2] Agarwal, Y., Aneja, Y., 2012. Fixed-charge transportation problem: Facets of the projection polyhedron. Operations Research 60 (3), 638–654.

[3] Bertazzi, L., Maggioni, F., 2015. Solution approaches for the stochastic capacitated traveling salesmen location problem with recourse. Journal of Optimization Theory and Applications 166 (1), 321–342.

[4] Bertocchi, M., Moriggia, V., Dupačová, J., 2006. Horizon and stages in applications of stochastic programming in finance. Annals of Operations Research 142 (1), 63–78.

[5] Birge, J. R., 1997. State-of-the-art-survey-stochastic programming: Computation and applications. INFORMS Journal on Computing 9 (2), 111–133.

[6] Birge, J. R., Louveaux, F., 2011. Introduction to stochastic programming. Springer Science & Business Media.

[7] Bolduc, M.-C., Renaud, J., Boctor, F., Laporte, G., 2008. A perturbation metaheuristic for the vehicle routing problem with private fleet and common carriers. Journal of the Operational Research Society 59 (6), 776–787.

[8] Buson, E., Roberti, R., Toth, P., 2014. A reduced-cost iterated local search heuristic for the fixed-charge transportation problem. Operations Research 62 (5), 1095–1106.

[9] Chand, S., Hsu, V. N., Sethi, S., 2002. Forecast, solution, and rolling horizons in operations management problems: a classified bibliography. Manufacturing & Service Operations Management 4 (1), 25–43.

[10] Christensen, T. R., Andersen, K. A., Klose, A., 2013. Solving the single-sink, fixed-charge, multiple-choice transportation problem by dynamic programming. Transportation Science 47 (3), 428–438.

[11] Christensen, T. R., Labbé, M., 2015. A branch-cut-and-price algorithm for the piecewise linear transportation problem. European journal of operational research 245 (3), 645–655.

[12] Chu, C.-W., 2005. A heuristic algorithm for the truckload and less-than-truckload problem. European Journal of Operational Research 165 (3), 657–667.

[13] Côté, J.-F., Potvin, J.-Y., 2009. A tabu search heuristic for the vehicle routing problem with private fleet and common carrier. European Journal of Operational Research 198 (2), 464–469.

[14] Dupačová, J., 2002. Applications of stochastic programming: achievements and questions. European Journal of Operational Research 140 (2), 281–290.

[15] Gary, M. R., Johnson, D. S., 1979. Computers and Intractability: A Guide to the Theory of NP-completeness. WH Freeman and Company, New York.

[16] Görtz, S., Klose, A., 2009. Analysis of some greedy algorithms for the single-sink fixed-charge transportation problem. Journal of Heuristics 15 (4), 331–349.

[17] Gray, P., 1971. Technical note - exact solution of the fixed-charge transportation problem. Operations Research 19 (6), 1529–1538.

[18] Guigues, V., Sagastizábal, C., 2012. The value of rolling-horizon policies for risk-averse hydro-thermal planning. European Journal of Operational Research 217 (1), 129–140.

[19] Herer, Y. T., Rosenblatt, M., Hefter, I., 1996. Fast algorithms for single-sink fixed charge transportation problems with applications to manufacturing and transportation. Transportation Science 30 (4), 276–290.

[20] Jawahar, N., Balaji, A., 2009. A genetic algorithm for the two-stage supply chain distribution problem associated with a fixed charge. European Journal of Operational Research 194 (2), 496–537.

[21] Johnson, E. L., Nemhauser, G. L., Savelsbergh, M. W., 2000. Progress in linear programming-based algorithms for integer programming: An exposition. INFORMS Journal on Computing 12 (1), 2–23.

[22] Kennington, J., Unger, E., 1976. A new branch-and-bound algorithm for the fixed-charge transportation problem. Management Science 22 (10), 1116–1126.

[23] Klein Haneveld, W. K., van der Vlerk, M. H., 1999. Stochastic integer programming:general models and algorithms. Annals of Operations Research 85 (0), 39–57.

[24] Kouwenberg, R., 2001. Scenario generation and stochastic programming models for asset liability management. European Journal of Operational Research 134 (2), 279–292.

[25] Kusy, M. I., Ziemba, W. T., 1986. A bank asset and liability management model. Operations Research 34 (3), 356–376.

[26] Lulli, G., Sen, S., Jun. 2004. A branch-and-price algorithm for multistage stochastic integer programming with application to stochastic batch-sizing problems. Management Science 50 (6), 786–796.

[27] Maggioni, F., Allevi, E., Bertocchi, M., 2014. Bounds in multistage linear stochastic programming. Journal of Optimization Theory and Applications 163 (1), 200–229.

[28] Maggioni, F., Allevi, E., Bertocchi, M., 2016. Monotonic bounds in multistage mixed-integer stochastic programming. Computational Management Science 13 (3), 423–457.

[29] Maggioni, F., Kaut, M., Bertazzi, L., 2009. Stochastic optimization models for a single-sink transportation problem. Computational Management Science 6 (2), 251–267.

[30] Maggioni, F., Pflug, G. C., 2016. Bounds and approximations for multistage stochastic programs. SIAM Journal on Optimization 26 (1), 831–855.

[31] Pantuso, G., Fagerholt, K., Wallace, S. W., 2015. Solving hierarchical stochastic programs: Application to the maritime fleet renewal problem. INFORMS Journal on Computing 27 (1), 89–102.

[32] Papageorgiou, D. J., Toriello, A., Nemhauser, G. L., Savelsbergh, M. W., 2012. Fixed-charge transportation with product blending. Transportation Science 46 (2), 281–295.

[33] Paraskevopoulos, D., Bektaş, T., Crainic, T. G., Potts, C., 2016. A cycle-based evolutionary algorithm for the fixed-charge capacitated multi-commodity network design problem. European Journal of Operational Research 253 (2), 265–279.

[34] Parija, G. R., Ahmed, S., King, A. J., 2004. On bridging the gap between stochastic integer programming and mip solver technologies. INFORMS Journal on Computing 16 (1), 73–83.

[35] Potvin, J.-Y., Naud, M.-A., 2011. Tabu search with ejection chains for the vehicle routing problem with private fleet and common carrier. Journal of the Operational Research Society 62 (2), 326–336.

[36] Raj, K. A. A. D., Rajendran, C., 2012. A genetic algorithm for solving the fixed-charge transportation model: two-stage problem. Computers & Operations Research 39 (9), 2016–2032.

[37] Roberti, R., Bartolini, E., Mingozzi, A., 2014. The fixed charge transportation problem: An exact algorithm based on a new integer programming formulation. Management Science 61 (6), 1275–1291.

[38] Römisch, W., Schultz, R., 2001. Multistage stochastic integer programs: An introduction. In: Online optimization of large scale systems. Springer, pp. 581–600.

[39] Sen, S., 2005. Algorithms for stochastic mixed-integer programming models. In: K. Aardal, G. N., Weismantel, R. (Eds.), Discrete Optimization. Vol. 12 of Handbooks in Operations Research and Management Science. Elsevier, pp. 515 – 558.

[40] Sheffi, Y., 2004. Combinatorial auctions in the procurement of transportation services. Interfaces 34 (4), 245–252.

[41] Sheng, Y., Yao, K., 2012. Fixed charge transportation problem and its uncertain programming model. Industrial Engineering and Management Systems 11 (2), 183–187.

[42] Silventea, J., Kopanosb, G. M., Espuñaa, A., 2015. A rolling horizon stochastic programming framework for the energy supply and demand management in microgrids. Computer aided chemical engineering 37 (2), 2321–2326.

[43] Stenger, A., Vigo, D., Enz, S., Schwind, M., 2013. An adaptive variable neighborhood search algorithm for a vehicle routing problem arising in small package shipping. Transportation Science 47 (1), 64–80.

[44] Valente, C., Mitra, G., Sadki, M., Fourer, R., 2009. Extending algebraic modelling languages for stochastic programming. INFORMS Journal on Computing 21 (1), 107–122.