Bounding Multistage Stochastic Programs: a Scenario Tree Based Approach

Francesca Maggioni and Elisabetta Allevi

Abstract Multistage mixed-integer stochastic programs are among the most challenging optimization problems combining stochastic programs and discrete optimization problems. Approximation techniques which provide lower and upper bounds to the optimal value are very useful in practice. In this paper we present a critic summary of the results in [6] and in [7] where we consider bounds based on the assumption that a sufficiently large discretized scenario tree describing the problem uncertainty is given but is unsolvable. Bounds based on group subproblems, quality of the deterministic solution and rolling-horizon approximation will be then discussed and compared.

Key words: Multistage stochastic programs - bounds - group subproblems

1 Introduction

In general the uncertainty of multistage stochastic programs is defined by means of a scenario process which may take uncountable infinite values. In order to solve it, is possible to consider a sufficiently large discretized scenario tree describing the uncertainty and considering it as benchmark. However in most of the real cases this problem is unsolvable requiring the inclusion of a large number of samples. Bounding its solution is then of practical interests.

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The authors would like to dedicate this work to the memory of Marida Bertocchi, exceptional colleague and friend sadly passed away on November 16, 2016.

The aim of the paper is to present a brief and critic summary of bounds in multistage mixed-integer stochastic programs introduced in [6] and [7] based on the assumption that a sufficiently large discretized scenario tree describing the problem uncertainty is given but is unsolvable. Chain of lower bounds less complex than the original problem are solved by solving sets of group subproblems made by fixed and free scenarios, and taking an expectation across scenario groups. Monotonicity results are provided. Other approximations of the optimal stochastic solution have been quantified by the introduction of measures of the quality of the deterministic solution and rolling horizon measures which update the estimation and add more information at each stage. The general idea behind construction of the proposed bounds, is that for every optimization problem of minimization type, lower bounds to the optimal solution can be found by relaxation of constraints and upper bound by inserting feasible solutions. Bounds for multistage convex problems with concave risk functionals based on scenario tree approaches are also provided in [8]. Other approaches bounding the infinite problem are presented in [9].

The paper is organized as follows: Section 2 introduces the notation and basic definitions. Lower bounds based on solving group subproblems are in Section 3 and upper bounds for the optimal multistage objective value are in Section 4. Section 5 concludes the paper.

2 Preliminaries

We consider the following scenario formulation of a *multistage mixed-integer stochastic program* (see [11]):

$$
\begin{aligned}
RP &= \min_{\mathbf{x}} E_{\boldsymbol{\xi}^{H-1}} z(\mathbf{x}, \boldsymbol{\xi}^{H-1}) \\
&= \min_{x^1,\dots,x^H} c^1 x^1 + \sum_{s=1}^{S} \pi_s \left(c^2\left(\xi_s^1\right) x^2(\xi_s) + \cdots + c^H\left(\xi_s^{H-1}\right) x^H(\xi_s)\right) \\
&\quad \text{s.t. } A x^1 = h^1 \ , \\
&\qquad T^1(\xi_s^1) x^1(\xi_s) + W^2(\xi_s^1) x^2(\xi_s) = h^2(\xi_s^1) \ , \quad s = 1,\dots,S \ , \\
&\qquad \vdots \\
&\qquad T^{H-1}(\xi_s^{H-1}) x^{H-1}(\xi_s) + W^H(\xi_s^{H-1}) x^H(\xi_s) = h^H(\xi_s^{H-1}) \ , \ s = 1,\dots,S \ , \\
&\qquad x^t(\xi_{j'}) = x^t(\xi_{j''}), \forall j', j'' \in \{1,\dots,S\} \text{ for which } \xi_{j'}^t = \xi_{j''}^t, \ t = 2,\dots,H \ ,
\end{aligned}
\tag{1}
$$

where $c^1 \in \mathbb{R}^{n_1}$ and $h^1 \in \mathbb{R}^{m_1}$ are known vectors, $A \in \mathbb{R}^{m_1 \times n_1}$ is a known matrix, $h^t \in \mathbb{R}^{m_t}$, $c^t \in \mathbb{R}^{n_t}$, $T^{t-1} \in \mathbb{R}^{m_t \times n_{t-1}}$, $W^t \in \mathbb{R}^{m_t \times n_t}$, $t = 2,\dots,H$ are the uncertain parameter vectors and matrices. The random process ξ^t, $t = 1,\dots,H-1$, is revealed gradually over time in H periods and $\boldsymbol{\xi}^t := (\xi^1,\dots,\xi^t)$, $t = 1,\dots,H-1$ denotes the history of the process up to time t. ξ^t is defined on a probability space $(\Xi^t, \mathscr{A}^t, p)$ with support $\Xi^t \in \mathbb{R}^{n_t}$ and given probability distribution p on the $\sigma-$algebra $\mathscr{A}^t$

(with $\mathscr{A}^t \subseteq \mathscr{A}^{t+1}$) and E_{ξ^t} denotes the expectation with respect to ξ^t. Let $\xi_1, \ldots, \xi_S$, be the possible realizations (or scenarios) of $\boldsymbol{\xi}^{H-1}$, Ξ the finite support of possible scenarios and ξ_s^t the history of the s-realization, $s = 1, \ldots, S$, up to stage t, $t = 1, \ldots, H-1$. Let π_s the probability of scenario s, $s = 1, \ldots, S$. See Figure 1 for a scenario tree visualization of the scenario process with approximate distribution. The decision vector $\mathbf{x} := (x^1, x^2, \ldots, x^H)$, with $x^t \in \mathbb{R}_+^{n_t - d_t} \times \mathbb{N}^{d_t}$, $t = 1, \ldots, H$, is *nonanticipative* which means it depends on the information up to time t. The last set of constraints enforce this condition. In the following, for a simpler presentation, the feasibility condition on x^t will be omitted even if assumed to be satisfied.

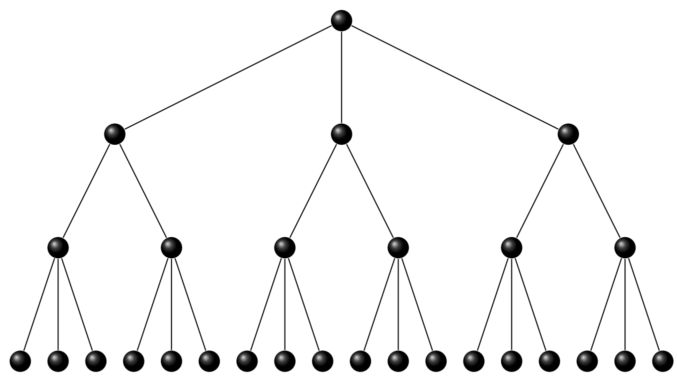

Fig. 1 Scenario tree representation of the random process $\boldsymbol{\xi}^{H-1}$ with approximate distribution.

The main principle to obtain lower bounds of problem (1) is given by the relaxation of some constraints. This is the case of the *multistage wait-and-see* problem (WS), where the *nonanticipativity* constraints are relaxed. WS is then obtained by averaging the total costs of the S deterministic programs:

$$\begin{aligned} WS = \sum_{s=1}^{S} \pi_s \; \min\nolimits_{x^1(\xi_s), \ldots, x^H(\xi_s)} \; & c^1 x^1(\xi_s) + c^2(\xi_s^1) x^1(\xi_s) + \ldots + c^H(\xi_s^{H-1}) x^H(\xi_s) \\ \text{s.t.} \quad & A x^1(\xi_s) = h^1 \; , \\ & T^1(\xi_s^1) x^1(\xi_s) + W^2(\xi_s^1) x^2(\xi_s) = h^2(\xi_s^1) \; , \\ & \vdots \\ & T^{H-1}(\xi_s^{H-1}) x^{H-1}(\xi_s) + W^H(\xi_s^{H-1}) x^H(\xi_s) = h^H(\xi_s^{H-1}) \; . \end{aligned} \tag{2}$$

The *Expected Value problem EV* is obtained by replacing all random parameters by their expected values and solving the deterministic program, with $\bar{\boldsymbol{\xi}} := (\bar{\xi}^1, \bar{\xi}^2, \ldots, \bar{\xi}^{H-1}) = (E\xi^1, E\xi^2, \ldots, E\xi^{H-1})$:

$$EV := \min_{\mathbf{x}} z(\mathbf{x}, \bar{\boldsymbol{\xi}}) \; . \tag{3}$$

2.1 Basic Bounds

The following relations between *RP*, *WS* and *EV* have been proved in [3].

Theorem 1. *For multistage stochastic mixed-integer programs of the form (1), the following inequalities hold true*

$$WS \leq RP \leq EEV \ , \tag{4}$$

where EEV denotes the solution value of the RP model, having the first stage decision variables fixed at the optimal values obtained by using the expected value of coefficients.

Similarly EEV^t, $t = 1,\ldots,H-1$ (see [2] and [6]), is defined by fixing the decision variables up to stage t of *RP* at the optimal values obtained by using the problem *EV*. The *Value of the Stochastic Solution at stage t*, VSS^t is then defined as $VSS^t := EEV^t - RP, \quad t = 1,\ldots,H-1.$

However, in several problems of practical interest the difference between EEV^t and *WS* is quite large. In the next sections we will discuss how to solve simpler problems for finding lower and upper bounds and proceed to find tighter and tighter bounds to *RP*.

3 Lower Bounds by Group Subproblems

In order to obtain lower bounds on *RP* problem which improve the left-hand side inequality in (4), one can solve smaller problems than the original one. The proposed approach (see [7]) divides a given problem into independent subproblems. We suppose to fix a number $1 \leq R < S$ of *reference scenarios* among the possible S scenarios. Let $\mathscr{R} = \{1,\ldots,R\}$ be the index set of fixed scenarios. Without loss of generality we suppose they are the first R scenarios among the available S ones. We choose among the $K = S - R$ scenarios (ξ_i, $i = R+1,\ldots S$) a subgroup of cardinality $k = 1,\ldots,K$. Let $\mathscr{K} = \{R+1,\ldots,S\}$ be the index set of scenarios excluding those belonging to the fixed scenario set $\mathscr{R}$. Let $\mathscr{P}(\mathscr{K})$ the power set of $\mathscr{K}$ excluding the empty set. Let $\mathscr{P}_k(\mathscr{K})$ the set of all subset of $\mathscr{P}(\mathscr{K})$ with cardinality k. For any subset $\Psi_k \in \mathscr{P}_k(\mathscr{K})$, let $\pi(\Psi_k) = \sum_{i \in \Psi_k} \pi_i$ be the probability assigned to scenarios group Ψ_k.

Definition 1. For any given scenario group Ψ_k, the group subproblem $MGR(\Psi_k, R)$ is defined as $\min z^R(\Psi_k) :=$

$$\min_{x^1,\ldots,x^H} \left(c^1x^1 + \sum_{r=1}^{R} \left(\pi_r \sum_{t=2}^{H} c^t(\xi_r^{t-1})x^t(\xi_r) \right) + (1 - \sum_{r=1}^{R} \pi_r) \sum_{i\in\Psi_k} \frac{\pi_i}{\pi(\Psi_k)} \sum_{t=2}^{H} c^t(\xi_i^{t-1})x^t(\xi_i) \right)$$
$$\begin{aligned} \text{s.t.}\quad & Ax^1 = h^1\ , \\ & T^{t-1}(\xi_r^{t-1})x^{t-1}(\xi_r) + W^t(\xi_r^{t-1})x^t(\xi_r) = h^t(\xi_r^{t-1})\ ,\ r\in\mathscr{R}, \quad t=2,\ldots,H \\ & T^{t-1}(\xi_i^{t-1})x^{t-1}(\xi_i) + W^t(\xi_i^{t-1})x^t(\xi_i) = h^t(\xi_i^{t-1})\ ,\ i\in\Psi_k, \quad t=2,\ldots,H \\ & x^t(\xi_{j'}) = x^t(\xi_{j''}), \forall j',j''\in\mathscr{R}\cup\Psi_k \text{ for which } \xi_{j'}^t = \xi_{j''}^t \quad t=2,\ldots,H. \end{aligned} \tag{5}$$

Remark 1. $MGR(\Psi_1,1)$ reduces to the definition of *PAIRS* subproblem (see [6]).

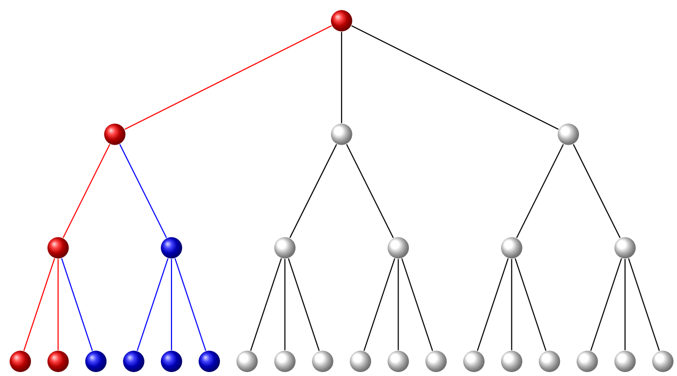

Fig. 2 Representation of the Group Subproblem $MGR(\Psi_4,2)$ with $R=2$ reference scenarios (in red) and one subset Ψ_4 (in blue).

A representation of $MGR(\Psi_4,2)$ is shown in Figure 2 with $R=2$ reference scenarios (in red) and one subset Ψ_4 (in blue).

Definition 2. Given an integer $k\in\{1,\ldots,K\}$, and R fixed scenarios, the *Multistage Expected value of the Group Subproblem Objective* function with k scenarios in each group and R fixed scenarios, *MEGSO(k,R)* is defined as

$$MEGSO(k,R) := \frac{1}{\binom{K-1}{k-1}(1-\sum_{r=1}^{R}\pi_r)} \left[\sum_{\Psi_k\in\mathscr{P}_k(\mathscr{K})} \pi(\Psi_k) \min z^R(\Psi_k) \right]. \tag{6}$$

Remark 2. The *Multistage Sum of Pairs Expected Values*, *MSPEV* [6] reduces to $MEGSO(1,1)$ as follows

$$MSPEV = MEGSO(1,1) = \frac{1}{1-\pi_a} \sum_{\Psi_1\in\mathscr{P}_1(\mathscr{K})} \pi(\Psi_1) \min z^P(\Psi_1)\ . \tag{7}$$

Theorem 2. *Given an integer R,* $1\le R<S$ *and an integer k,* $1\le k\le K$ *the following chains of inequalities hold true*

$$WS \leq MEGSO(1,R) \leq MEGSO(2,R) \leq \ldots \leq MEGSO(K,R) = RP \ , \quad (8)$$

$$MEGSO(k,1) \leq MEGSO(k,2) \leq \cdots \leq MEGSO(k,S-k) = RP \ . \quad (9)$$

Theorem 2 says that $MEGSO$ is monotonically nondecrasing in the cardinality k of scenarios of the subsets Ψ_k with R fixed and monotonically nondecrasing in the number of reference scenarios R with k fixed. The proof can be found in [7].

4 Upper Bounds

In this section we discuss and compare some types of upper bounds for multistage mixed-integer programs. We focus on bounds based on solving group subproblems, quality measures of the deterministic solution and on rolling horizon measures.

4.1 Upper Bounds from Multistage Group Subproblems

In this section we recall upper bounds for multistage stochastic programs based on solving group subproblems (see [10] for the two-stage case and [7] for the multistage one). Given $\check{\mathbf{x}}_R^1$ the optimal first stage solution of the stochastic problem $\min_{\mathbf{x}} z(\mathbf{x}, \xi_1, \ldots, \xi_R)$, based only on the R reference scenarios, then a possible upper bound of RP is:

$$MEVRS^{1,R} := E_{\boldsymbol{\xi}^{H-1}} \min_{\mathbf{x}^{(2,H)}} z(\check{\mathbf{x}}_R^1, \mathbf{x}^{(2,H)}, \boldsymbol{\xi}^{H-1}) \ . \quad (10)$$

A tighter upper bound to RP is the *Multistage Expectation of Group Subproblems* $MEGS(k,R)$, which represents the minimum optimal value among those obtained by solving the original stochastic program (1), using the optimal first stage solution $\hat{x}^1_{\Psi_k,R}$ of each group subproblem (5). This can be expressed as follows:

$$MEGS(k,R) := \min_{\Psi_k \in \mathscr{P}_k(\mathscr{K}) \cup \mathscr{R}} (E_{\boldsymbol{\xi}^{H-1}} \min_{\mathbf{x}^{(2,H)}} z(\hat{x}^1_{\Psi_k,R}, \mathbf{x}^{(2,H)}, \boldsymbol{\xi}^{H-1})) \ . \quad (11)$$

The following inequality holds (see [7]).

Proposition 1. *For a fixed number R of reference scenarios and any* $1 \leq k \leq K$ *we have*

$$RP \leq MEGS(k,R) \leq MEVRS^{1,R} \ . \quad (12)$$

4.2 Upper Bounds based on the Expected Value Skeleton Solution

Measures of the structure (skeleton) of the deterministic solution such as the *Multistage Loss Using the Skeleton Solution MLUSS* has been introduced in [6], in relation to the standard VSS (see in [5] the definition in the two-stage case). The aim of these measures is to identify meaningful information, which can be extracted from the solution of the deterministic problem, in order to reduce the size of the stochastic one. $MLUSS^t$ are computed as the difference between the optimal values of the stochastic problem RP and its reduced version $MESSV^t$ obtained by fixing the out-of-basis variables up to stage in the expected value solution. Having a $MLUSS^t$ close to zero suggests that the out-of-basis variables chosen by the expected value model until stage t are correct also in a stochastic environment. The following relations hold true (see [6]):

Proposition 2.

$$MLUSS^{t+1} \geq MLUSS^t, \quad t = 1, \ldots, H-2, \tag{13}$$

$$RP \leq MESSV^t \leq EEV^t, \quad t = 1, \ldots, H-1. \tag{14}$$

In these lines, other measures of the goodness of deterministic solutions based on reduced costs of the deterministic solution are proposed in [1].

4.3 Upper bounds based on Rolling Horizon approaches

Multistage problems such as EEV^t are often infeasible since they require to fix many variables to their value obtained via the expected value model. An alternative approach to consider is the *rolling time horizon* procedure taking into account the arrival of new information at each stage. This is obtained by solving a sequence of H scenario trees with random parameters in periods $t, \ldots, H-1$ replaced with their expected value and solve the associated model with fixing the solutions obtained in the previous steps (see Figure 3). The *Rolling Horizon Value of the Expected Value Solution* is then given by the difference with respect to RP. In similar way other rolling horizon measures are defined in [6].

5 Conclusions

In this paper lower and upper bounds for mixed-integer multistage stochastic programs have been discussed and compared. The bounds are based on the assumption that a sufficiently large scenario-tree process is given as approximation of the general infinite problem and it is considered as a benchmark. The lower and upper bounds proposed are based on groups subproblems, quality of deterministic solu-

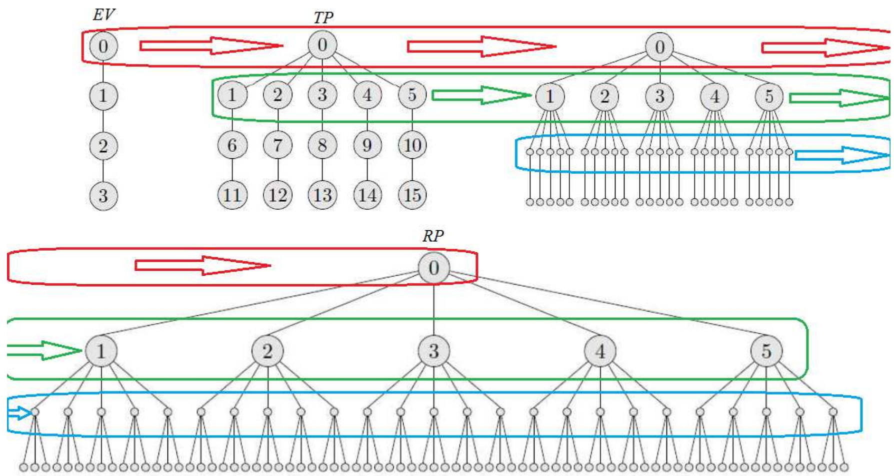

Fig. 3 Procedure to compute the Rolling Horizon Value of the Expected Value Solution.

tion and rolling horizon approaches. The approach discussed is both of theoretical and practical importance arising when solving problems of large instances where it is fundamental to have approximations of the optimal solution.

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