Probabilistic Invariant Sets for Closed-Loop Re-Identification

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Abstract— **Recently, a Model Predictive Control (MPC) suitable for closed-loop re-identification was proposed, which solves the potential conflict between the persistent excitation of the system and the stabilization of the closed-loop by extending the equilibrium-point-stability to the invariant-set-stability. The proposed objective set, however, derives in large regions that contain conservatively the excited system evolution. In this work, based on the concept of probabilistic invariant sets, the controller target sets are substantially reduced ensuring the invariance with a sufficiently large probability (instead of deterministically), giving the resulting MPC controller the necessary flexibility to be applied in a wide range of systems.**

Keywords— **Model predictive control, closed-loop identification, probabilistic invariant set**

1. INTRODUCTION

In most Model predictive control (MPC) applications a periodic updating of the system model are desired to reach meaningful performances. In this context, a re-identification procedure should be developed in a closed-loop fashion, since the process cannot be stopped each time an update is needed. As it is known, the main problem of a closed-loop identification is that the dynamic control objectives are incompatible with the identification objectives. In fact, to perform a suitable identification, a persistent excitation of the system modes is needed, while the controller takes this excitation as disturbance that it tries to reject from the output to stabilize the system. On the other hand, the identification objective is to excite the system in a open-loop fashion to obtain uncorrelated input-output (and input-output noise) data that permits to obtain a suitable model.

Several strategies were developed to perform closed-loop re-identification under MPC controllers: [1] proposed a controller named Model Predictive Control and Identification (MPCI) where a persistent excitation condition is added by means of an additional constraints in the optimization problem. This strategy turns the MPC optimization problem non-convex, and so, most of the well-known properties of the MPC formulation cannot be established. In [2] a two-step controller approach is presented: the first stage is devoted to optimize the control trajectory - as usual in MPC, while the second stage is devoted to generate the persistent excitation input signal by maximizing the minimal eigenvalue of the information matrix (a matrix describing the input variability). The second optimization problem, however, is nonlinear and difficult to solve. In [3], a study of several MPC re-identification methods is made, focusing on the so-called MPC Relevant Identification (MRI), which is an identification method that not only takes into account the identified model accuracy but also the model aptitude for predictions, i.e., the model aptitude for the controller point of view.

In general, none of these reports have shown results regarding the stability of the MPC while it is re-identifying the system. Recently, [4] have proposed a novel MPC suitable for re-identification that ensures stability and performs a safe closed-loop re-identification. The main idea is this paper is to extend the concept of equilibrium-point-stability to the invariant-set-stability, and proposed an MPC that: steers the system to that invariant set, when outside, and persistently excites the system, when inside. The MPC problem formulation is based on the concept of generalized distance from a point (the state and input trajectory) to a set (target invariant set and input excitation set). So, it guaranties stability of the target invariant set and also the persistent excitation of the system, since both tasks are developed separately in the state space.

The method proposed in [4], however, derives in large regions that contain conservatively the excited system evolution. This way, the region where the controller leaves the system in open-loop (given that the entire invariant set is considered as a generalized equilibrium and no control action are injected to the system when the system is in it) is a large region in the state space, which is not safety enough for many control systems.

In this work, based on the concept of probabilistic invariant sets presented in [5], the controller target sets are substantially reduced ensuring the invariance with a sufficiently large probability (instead of deterministically), giving the resulting MPC controller the necessary flexibility to be applied in a wide range of systems. Several closed-loop re-identification scenarios are simulated to clearly show the proposed controller benefits and limitations.

1.1. Problem Statement

We shall consider a system described by a linear time-invariant discrete-time model

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0 \qquad (1)$$

where $x(k) \in \mathcal{X} \subset \mathbb{R}^n$ is the system state at the current time instant k, x_0 is the initial state, and $u(k) \in \mathcal{U} \subset \mathbb{R}^m$ is the current control input. All along this work we assume that matrix $A \in \mathbb{R}^{n\times n}$ has all its eigenvalues strictly inside the unit circle, the pair (A, B) is controllable, the set $\mathcal{X}$ is convex and closed, the set $\mathcal{U}$ is convex and compact and both contain the origin in their interior.

Our goal is to develop a MPC strategy that account for the closed-loop re-identification of such a system.

2. BACKGROUND

In this section, we recall definitions and properties that will be used in the next section to derive the main results of the work.

2.1. Invariant sets and control

Definition 1. ***(γ-Control Invariant Set, γ-CIS)*** *Given $\gamma \in [0,1]$, a set $\Omega \subseteq \mathcal{X}$ is γ-control invariant for system (1) associated with the set $\mathcal{U}$, if $x(k) \in \Omega$ implies $x(k+1) \in \gamma\Omega$, for some $u(k) \in \mathcal{U}$.*

Definition 2. ***(Reachability Set)*** *Given the set $\Omega \subset \mathcal{X}$, the reachability set $\mathcal{R}(\Omega)$ from Ω in one step, associated to the input set $\mathcal{U}$, is the set of all $z \in \mathcal{X}$ for which there exists $x \in \Omega$ and $u \in \mathcal{U}$ such that $Ax + Bu = z$. $\mathcal{R}(\Omega) =$*

$$= \{z \in \mathcal{X} : \exists x \in \Omega, \exists u \in \mathcal{U} \text{ such that } z = Ax + Bu\}$$

Definition 3. ***(Controllability Set)*** *Given the set $\Omega \subset \mathcal{X}$, the controllability set $\mathcal{Q}(\Omega)$ to Ω in one step, associated to the input set $\mathcal{U}$, is the set of all $x \in \mathcal{X}$ for which there exists $u \in \mathcal{U}$ such that $Ax + Bu \in \Omega$.*

$$\mathcal{Q}(\Omega) = \{x \in \mathcal{X} : \exists u \in \mathcal{U} \text{ such that } Ax + Bu \in \Omega\}$$

2.2. Probabilistic Invariants

In this work, the concept of probabilistic invariant set is associated to the excitation requirements necessary to perform suitable identifications. So we first define:

Definition 4. ***Bounded Persistent Excitation*** *Given a compact non empty set $V \subset \mathbb{R}^m$, we say that a stationary stochastic Markovian process[1] $v : \mathbb{N} \to V$ is a persistent excitation bounded by V if $\mathrm{E}[v(k)] = 0$ and $\mathrm{cov}[v(k)] > 0$ for all $k \in \mathbb{N}$.*

Note that this definition is directly related to the usual one used to define persistent excitation inputs in the identification system literature ([6]). Now, associated to the persistent excitation, we define:

[1] i.e., $v(k)$ takes random values on set V so that $v(k)$ is uncorrelated with $v(j)$ for $k \neq j$.

Definition 5. ***γ–Probabilistic Invariant Set (γ–PIS)*** *Let $p \in (0,1]$ and $\gamma \in (0,1]$. A set $S \subseteq \mathcal{X}$ is a γ–Probabilistic Invariant Set with probability p of system (1) with $u(k)$ being a persistent excitation bounded by $V \subset \mathcal{U}$, if and only if $x(k) \in S \Rightarrow \Pr[x(k+j) \in \gamma S] \geq p$, for any $j > 0$.*

Notice that when $\gamma = 1$, a γ-PIS is simply a PIS. On the other hand, when $p = 1$, a γ-PIS is a γ-ISI set, as the one defined in [4].

3. MAIN RESULTS

3.1. One Step Probabilistic Invariant Sets

In [4], the γ-Invariant Set for Identification (γ-ISI) was defined as a γ-PIS with probability 1. This set is used as target set when the system is outside, and it is used as an identification set when it is inside. Given that in this approach the stochastic nature of the Bounded Persistent Excitation is not used, the resulting sets are too large, and contain conservatively the excited system evolution.
We will introduce below the concept of *one step probabilistic invariant sets*, which will be then used for identification purposes. The next, is the set we will use to formulate the MPC for re-identification.

Definition 6. ***γ–One Step Probabilistic Invariant Set (γ–OSPIS)*** *Let $p \in (0,1]$ and $\gamma \in (0,1]$. A set $S \subseteq \mathcal{X}$ is a γ–One Step Probabilistic Invariant Set with probability p of system (1) with $u(k)$ being a persistent excitation bounded by $V \subset \mathcal{U}$, if and only if $x(k) \in S \Rightarrow \Pr[x(k+1) \in \gamma S] \geq p$.*

Again, when $\gamma = 1$ a γ–OSPIS is simply an OSPIS. Furthermore, when $p = 1$ a γ-OSPIS is a γ-ISI set, as the one defined in [4].

Remark 1. *Notice that by definition, a γ-PIS with probability $p \in (0,1]$ is also γ-OSPIS with the same probability, for the same system. Although the opposite is not necessarily true.*

The following lemma establishes that a γ–OSPIS is also a γ–CIS. This property will play a fundamental role to prove the convergence of the MPC scheme we shall propose.

Lemma 1. ***(OSPIS $\Rightarrow$ CIS)*** *Let $p \in (0,1]$ and $\gamma \in (0,1]$. Let S be a γ–OSPIS with probability p of System (1) with $u(k)$ being a persistent excitation bounded by $V \subset \mathcal{U}$. Then S is a γ–CIS for the same system, associated with the set V.*

Demostración. Let $x(k) \in S$. Consider $x(k+1) = Ax(k) + Bu(k)$, being $u(k)$ a persistent excitation bounded by $V \subset \mathcal{U}$. Then $Pr[x(k+1) \in \gamma S] \geq p$.
Since $p > 0$, and $u(k)$ takes values on $V \subset \mathcal{U} \implies$ there is $\hat{u}(k) \in V$ such that $\hat{x}(k+1) = Ax(k) + B\hat{u}(k) \in \gamma S$. Therefore, S is a γ-CIS of system (1), association with the set $V \subset \mathcal{U}$. $\square$

3.2. Application to MPC

When the bounded persistent excitation inputs is considered for the computation of the γ–OSPIS in the context of identification, it must belong to a compact set smaller than $\mathcal{U}$, but large enough to sufficiently excite the system. Formally:

Definition 7. ***(Excitation input set, EIS).*** *An input proper C-set* [2] $\mathcal{U}^t \subset \mathcal{U}$*, with enough size to excite the system will be denoted as excitation input set.*

The target set used to implement the MPC for re-identification is defined as a proper C-set $\mathcal{S}^t$ that is a γ-OSPIS with probability $p \in (0,1]$, and $\gamma \in (0,1)$, for system (1), associated to a persistent excitation input, $u(k)$, bounded by $\mathcal{U}^t \subset \mathcal{U}$.

Remark 2. *Notice that a set S^t could be an OSPIS with probability $p \in (0,1]$ for some stochastic process and an OSPIS with probability $q \neq p$ for some other stochastic process.*

In this context, the main idea of the MPC formulation consists in penalize the distance from the predicted state to a target set, for which it is necessary the following definition:

Definition 8. ***Distance from a point to a set.*** *Let $\mathcal{V}$ be a proper C-set on $\mathbb{R}^p$, the distance from $x \in \mathbb{R}^p$ to $\mathcal{V}$ is defined as*

$$d_{\mathcal{V}}(x) \triangleq \inf\{(x-\hat{x})^T M (x-\hat{x}) : \hat{x} \in \mathcal{V}\}$$

with $M \in \mathbb{R}^{pxp}$ positive definite.

Notice that $d_{\mathcal{V}}(x)$ is a convex and continuous function, and $d_{\mathcal{V}}(x) \geq 0$ for all $x \in \mathbb{R}^p$, and $d_{\mathcal{V}}(x) = 0$ if and only if $x \in \mathcal{V}$.

The proposed controller cost function is based on the aforementioned distance to an OSPIS $\mathcal{S}^t$ with probability $p \in (0,1]$, and is given by:

$$V_N^{OSPIS}(x, \mathcal{S}^t; \mathbf{u}) = \sum_{j=0}^{N-1} [\alpha d_{\mathcal{S}^t}(x(j)) + \beta d_{\mathcal{U}^t}(u(j))] \quad (2)$$

where α and β are positive real numbers, $N \in \mathbb{N}$ is the prediction horizon, and $\mathcal{S}^t$ is the *objective set*, where the system needs to be placed to start an identification procedure.

For any current state in the set of states that can be feasibly steered to $\mathcal{S}^t$ in N steps (the N-step controllable set to $\mathcal{S}^t$), $x \in \mathcal{X}_N$, the optimization problem $P_N^{OSPIS}(x, \mathcal{S}^t)$ to be solved is given by:

[2] A proper C-set is a convex and compact set that contains the origin as an interior point.

Problem $P_N^{OSPIS}(x, \mathcal{S}^t)$

$$\begin{aligned}
\min_{\mathbf{u}} \quad & V_N^{OSPIS}(x, \mathcal{S}^t; \mathbf{u}) \\
s.t. \quad & \\
& x(0) = x, \\
& x(j+1) = Ax(j) + Bu(j), \quad j = 0, ..., N-1. \\
& x(j) \in \mathcal{X},\ u(j) \in \mathcal{U}, \qquad j = 0, ..., N-1. \\
& x(N) \in \mathcal{S}^t.
\end{aligned}$$

In this MPC formulation $\mathbf{u} = \{u(0), \cdots, u(N-1)\}$ is the optimization variable, while the initial state x and the target set $\mathcal{S}^t$ are the optimization parameters. The last constraint is a terminal constraint that forces the the terminal state (the state at the end of the control horizon) to be in $\mathcal{S}^t$, avoiding this way the use of a terminal penalization in the cost. The control law derived from the application of the receding horizon policy is given by $\kappa_N(x, \mathcal{S}^t) = u^o(0; x)$, where $u^o(0; x)$ is the first element of the (optimal) solution sequence $\mathbf{u}^o(x)$. This way, the closed-loop system under the MPC law is described as $x(j) = \phi_{\kappa_N}(j; x, \mathcal{S}^t) = A^j x + \sum_{i=0}^{j-1} A^{j-i-1} B \kappa_N(x, \mathcal{S}^t)$. Now, the following Theorems can be established

Theorem 1. *Consider an OSPIS $\mathcal{S}^t \subseteq \mathcal{X}$, with probability $p \in (0,1]$, for system (1) associated to the persistent excitation $u(k)$, which is bounded by $\mathcal{U}^t$. Then, $\mathcal{S}^t$ is an Invariant Set for the closed-loop system $x(j) = \phi_{\kappa_N}(j; x, \mathcal{S}^t)$, $x(0) = x$, $j \in \mathbb{N}$.*

Demostración. Consider a state $x \in \mathcal{S}^t$. Then, given that an OSPIS is also a CIS associated with the EIS $\mathcal{U}^t$ (by Lemma 1), an input sequence $\hat{\mathbf{u}} = \{u(0), \ldots, u(N-1)\}$, exists with $u(j) \in \mathcal{U}^t$, for $j = 0, ..., N-1$, that produces a sequence of states that remain in $\mathcal{S}^t$. So, considering the definition of the generalized distance function, the optimal solution of $P_N^{OSPIS}(x, \mathcal{S}^t)$ will have the properties of $\hat{\mathbf{u}}$, producing a cost function $V_N^{OSPIS}(x, \mathcal{S}^t; \hat{\mathbf{u}}) = 0$. On the other hand, any input sequence $\hat{\mathbf{u}}$ with $u(j) \notin \mathcal{U}^t$, for some $j = 0, ..., N-1$, produces a cost $V_N^{OSPIS}(x, \mathcal{S}^t; \hat{\mathbf{u}}) > 0$. This means that necessarily $u^o(0; x) \in \mathcal{U}^t$. This proves that the MPC cost $V_N^{OSPIS}(x, \mathcal{S}^t; \mathbf{u})$ is null along every trajectory starting in an initial state inside $\mathcal{S}^t$, and furthermore, $u^o(0; x)$ is a control input inside $\mathcal{U}^t$. From this fact, it directly follows that $\mathcal{S}^t$ is an IS set for the MPC closed-loop system. □

Theorem 2. *Consider an OSPIS $\mathcal{S}^t \subseteq \mathcal{X}$, with probability $p \in (0,1]$, for system (1) associated to the persistent excitation $u(k)$, which is bounded by $\mathcal{U}^t$. Then, $\mathcal{S}^t$ is locally attractive for the closed-loop system $x(j) = \phi_{\kappa_N}(j; x, \mathcal{S}^t)$, with $x(0) = x \in \mathcal{X}_N$ and $j \in \mathbb{N}$.*

Demostración. Since $\mathcal{S}^t$ is a CIS associated with the EIS $\mathcal{U}^t$ (by Lemma 1), the proof can be followed from here on [4]. □

3.3. Including the exciting mode

Theorems 1 and 2 suggest that an extra requirement to the input, such as a persistent excitation requirement, could be included in the proposed cost function. In fact, under a persistent excitation context, the state trajectory will be inside $\mathcal{S}^t$ most of the time (i.e., with probability p), and when the state trajectory leaves $\mathcal{S}^t$, the MPC controller could resume the control action to bring it back to $\mathcal{S}^t$.

To precise this idea, consider the following cost function which depends on the current time k (k denotes the time instant at which the MPC optimization Problem is solved):

$$
\begin{aligned}
&V_N^{EXC}(x, \mathcal{S}^t, u_{pe}(k); \mathbf{u}) = \\
&(1-\rho(x))V_N^{OSPIS}(x, \mathcal{S}^t; \mathbf{u}) + \rho(x)\|u(0) - u_{pe}(k)\|,
\end{aligned}
$$

where $\rho(x) = 1$ if $x \in \mathcal{S}^t$, and $\rho(x) = 0$ otherwise. Here, $u_{pe}(k)$ is a persistent excitation input bounded by $\mathcal{U}^t$ (the time dependence is necessary to make explicit that we have one different persistent excitation input for each time k).

For any initial state x in $\mathcal{X}_N$, at a given time step k, the optimization problem $P_N^{EXC}(x, \mathcal{S}^t, u_{pe}(k), k)$, to be solved at each time instant k, is given by:

Problem $P_N^{EXC}(x, \mathcal{S}^t, u_{pe}(k), k)$

$$
\begin{aligned}
\min_{\mathbf{u}} \quad & V_N^{EXC}(x, \mathcal{S}^t, u_{pe}(k); \mathbf{u}) \\
s.t. \quad & \\
& x(0) = x, \\
& x(j+1) = Ax(j) + Bu(j), \quad j = 0, ..., N-1. \\
& x(j) \in \mathcal{X},\ u(j) \in \mathcal{U}, \qquad j = 0, ..., N-1. \\
& x(N) \in \mathcal{S}^t
\end{aligned}
$$

Notice that the function $\rho(x)$ is a discontinuous function necessary to cancel the persistent excitation in case the state leaves $\mathcal{S}^t$. This could occur by the presence of an external disturbance or even, with a small probability $(1-p)$ by the persistent excitation itself.

Assume that $T_{id} \in \mathbb{N}$ is the length of the data necessary to perform a suitable identification of (1), so we must excite the system at least T_{id} times to complete the identification process.

The OSPIS with probability p, ensures that only the first step of any trajectory starting inside the set remains there with probability greater than p. Accordingly, to ensure the excitation of the system as long as needed, the following theorem, subject to the assumption below, is introduced to formalize the properties of the proposed MPC controller.

Assumption 1. *Let $\mathcal{S}^t$ be an OSPIS with probability $p \in (0, 1]$, of system* (1) *with $u(k)$ being a persistent excitation bounded by $\mathcal{U}^t$. If $\mathcal{R}(\mathcal{S}^t)$ is the reachability set from $\mathcal{S}^t$ in one step, associated to the EIS set $\mathcal{U}^t$, and $\mathcal{Q}(\mathcal{S}^t)$ is the controllability set to $\mathcal{S}^t$ in one step, associated to the input set $\mathcal{U}$, then we assume that $\mathcal{R}(\mathcal{S}^t) \subseteq \mathcal{Q}(\mathcal{S}^t)$.*

Theorem 3. *Let Assumption 1 holds. Consider an OSPIS $\mathcal{S}^t \subseteq \mathcal{X}$ with probability $p \in (0, 1]$ of system* (1) *with $u(k)$ being a persistent excitation bounded by $\mathcal{U}^t$. Then, $\mathcal{S}^t$ is a PIS with probability p, for the closed-loop system $x(j) = \phi_{\kappa_N}(j; x, \mathcal{S}^t)$, $x(0) = x$, and $j \in \mathbb{N}$.*

Demostración. Suppose $x \in \mathcal{S}^t$ at time instant k. Then $\rho(x) = 1$, and the cost of the Problem $P_N^{EXC}(x, \mathcal{S}^t, u_{pe}(k), k)$ will be

$$
V_N^{EXC}(x, \mathcal{S}^t, u_{pe}(k); \mathbf{u}) = \|u(0) - u_{pe}(k)\|
$$

then, $u^o(0; x) = u_{pe}(k)$, where $u_{pe}(k)$ is a bounded persistent excitation. Given that $\mathcal{S}^t$ is an OSPIS with probability p, this means that

$$
\begin{aligned}
Pr[Ax + Bu^o(0; x) \in \mathcal{S}^t] &= Pr[Ax + Bu_{pe}(k) \in \mathcal{S}^t] \\
&\geq p
\end{aligned}
$$

This occurs whenever the system state remains in $\mathcal{S}^t$. Otherwise, if some state $\tilde{x}$ is brought outside $\mathcal{S}^t$, we must show that $Pr[A\tilde{x} + B\kappa_N(\tilde{x}) \in \mathcal{S}^t] \geq p$. In fact, suppose that $\tilde{x}$ leaves $\mathcal{S}^t$ at time $k+j$, then $\tilde{x}$ belongs to $\mathcal{R}(\mathcal{S}^t)$, the reachability set from $\mathcal{S}^t$ in one step, associated to the EIS set $\mathcal{U}^t$, since that $u_{pe}(k+j-1) \in \mathcal{U}^t$. But, $\mathcal{R}(\mathcal{S}^t) \subseteq \mathcal{Q}(\mathcal{S}^t)$ by Assumption 1, where $\mathcal{Q}(\mathcal{S}^t)$ is associated to the input set $\mathcal{U} \supseteq \mathcal{U}^t$.

Given that the system is outside $\mathcal{S}^t$, it will be $\rho(\tilde{x}) = 0$, and so the MPC controller will implement the control action $u^o(0; \tilde{x})$, which will steer the state back to $\mathcal{S}^t$ in one step. In other words, given that it is possible to do that (since $\tilde{x} \in \mathcal{Q}(\mathcal{S}^t)$), and this will be the optimal control action (since it produces a null cost), then $Pr[A\tilde{x} + Bu^o(0; \tilde{x}) \in \mathcal{S}^t)] = 1 \geq p$.

On the other hand, if we assume that $x \notin \mathcal{S}^t$ at time instant k, the reasoning is the same.

Therefore, $Pr[\phi_{\kappa_N}(j; x, \mathcal{S}^t) \in \mathcal{S}^t] \geq p$, for all $j \in \mathbb{N}$, which concludes the proof. □

This later result is important since it shows that with just an OSPIS we obtain a PIS in the closed loop. This way we can estimate how often the system will be properly excited under the MPC control law.

Theorem 4. *Let $\mathcal{S}^t$ be an OSPIS with probability $p \in (0, 1]$ of system* (1) *with $u(k)$ being a persistent excitation bounded by $\mathcal{U}^t$. Then, for any initial state $x \in \mathcal{S}^t$, the system controlled by the receding horizon MPC control law $\kappa_N(x, \mathcal{S}^t) = u^o(0; x)$, will be persistently excited inside $\mathcal{S}^t$ with probability p, i.e., $Pr[x(j) = \phi_{\kappa_N}(j; x, \mathcal{S}^t, \mathbf{u}_{pe})] \geq p$, for $j \in \mathbb{N}$, and whenever $x(j) \in \mathcal{S}^t$, the persistent input u_{pe} will be injected to the system. Furthermore, for any initial state $x \in \mathcal{X}_N$, the closed-loop converges to $\mathcal{S}^t$.*

Demostración. i) Let $x = x(0) \in \mathcal{S}^t$. According to Theorem 3, the closed loop system will remain in $\mathcal{S}^t$ with probability greater than p, and in this case, the optimal control action to be applied to the system is given by the persistent excitation input u_{pe}. ii)

Let $x = x(0) \in \mathcal{X}_N \backslash \mathcal{S}^t$. Then, $\rho(x) = 0$, and so, Problem $P_N^{EXC}(x, \mathcal{S}^t, \mathbf{u}_{pe}, k)$ is equivalent to Problem $P_N^{OSPIS}(x, \mathcal{S}^t)$. Then, for Theorem 2 we have the result. □

A corollary of the later results is that, in absence of disturbances, the system under the MPC control law will be persistently excited p % of the time steps, in expectation.

3.4. Robust Analysis

It should be noticed that the OSPIS $\mathcal{S}^t$, which is a parameter of the proposed MPC optimization cost, depends on the model. Since the excitation scenario is precisely given when we suspect that the current model is no longer accurate, a guarantee that this PISI set is robust to some kind of model mismatch is necessary. The following parametric uncertainty description is selected here (although other could be equally considered):

$$x(k+1) = A(w)x(k) + B(w)u(k), \quad w \in \mathcal{W} \subseteq \mathbb{R}, \quad (3)$$

where $A(w)$ and $B(w)$ are affine functions of w, i.e., $A(w) = A + w\bar{A}$, $B(w) = B + w\bar{B}$ with w belonging to a proper C-set $\mathcal{W} \subset \mathbb{R}$.

Assume that the Nominal model is given by $x(k+1) = Ax(k) + Bu(k)$ ($w = 0$), and the unknown Real model, is given by $x(k+1) = A(w_R)x(k) + B(w_R)u(k)$, for some $w_R \in \mathcal{W}$. The following Theorem holds:

Theorem 5. *Consider a γ-OSPIS $\mathcal{S}^t$ with probability p, and $\gamma \in [0, 1)$, of system (1) with $u(k)$ being a persistent excitation bounded to $\mathcal{U}^t$. Then, there exists a proper C-set $\mathcal{W} \subset \mathbb{R}$ for which $\mathcal{S}^t$ is an OSPIS with probability p, of system (3) with $u(k)$ being a persistent excitation bounded to $\mathcal{U}^t$, and for all $w \in \mathcal{W}$.*

Demostración. Let $x(k) \in \mathcal{S}^t$. If $u(k)$ is a persistent excitation bounded to $\mathcal{U}^t$, then $Pr[x(k+1) \in \gamma\mathcal{S}^t] \geq p$.

Notice that, if $x(k+1) \in \gamma\mathcal{S}^t$, sice $\gamma < 1$, there exists $r > 0$ such that $\mathcal{B}_r(x(k+1)) \subseteq \mathcal{S}^t$. [3]

Consider now the solution $\bar{x}(k+1)$ of system (3):

$$\begin{aligned} \bar{x}(k+1) &= A(w)x(k) + B(w)u(k) \\ &= (A + w\bar{A})x(k) + (B + w\bar{B})u(k) \\ &= Ax(k) + Bu(k) + w(\bar{A}x(k) + \bar{B}u(k)) \\ &= x(k+1) + w(\bar{A}x(k) + \bar{B}u(k)) \\ &= x(k+1) + w\theta \end{aligned}$$

with $w \in \mathcal{W}$, for some $\mathcal{W} \subseteq \mathbb{R}$.
But θ belongs to the bounded set $\Theta = \bar{A}\mathcal{X} \oplus \bar{B}\mathcal{U}$, so there is $\bar{w} > 0$ such that $\|w\theta\|_2 < r$ for all $w \in [-\bar{w}, \bar{w}]$. Then,

$$\|\bar{x}(k+1) - x(k+1)\|_2 = \|w\theta\|_2 < r, \quad w \in [-\bar{w}, \bar{w}]$$

[3] $\mathcal{B}_r(z) = \{x \in \mathbb{R}^n : \|z - x\|_2 < r\}$

which means that $\bar{x}(k+1) \in \mathcal{B}_r(x(k+1))$ for all $w \in \bar{\mathcal{W}} = [-\bar{w}, \bar{w}]$.

Furthermore, as we said before, if $x(k+1) \in \gamma\mathcal{S}^t \Rightarrow \mathcal{B}_r(x(k+1)) \subseteq \mathcal{S}^t$. Therefore, $\bar{x}(k+1) \in \mathcal{S}^t$, for all $w \in \bar{\mathcal{W}}$. Then

$$Pr[\bar{x}(k+1) \in \mathcal{S}^t] \geq Pr[x(k+1) \in \gamma\mathcal{S}^t] \geq p, \quad \forall\, w \in \bar{\mathcal{W}}$$

which concludes the proof. □

4. EXAMPLES

4.1. How to get the final OSPIS

It should be noted that in this work a method for computation of an OSPIS was not proposed. Instead, we will use the method proposed by [5], that compute polytopic PIS, that also are OSPIS (remark 1).

The goal of this section is to compare the OSPIS with the Invariant Set for Identification (the target set submitted on [4]), showing that we gain a much smaller target set.

A 2-state stable system of the form of (1) is used, with matrices:

$$\begin{aligned} A &= \begin{bmatrix} 0{,}42 & -0{,}28 \\ 0{,}02 & 0{,}6 \end{bmatrix}, \\ B &= \begin{bmatrix} 0{,}3 \\ -0{,}4 \end{bmatrix}, \end{aligned}$$

The constraints of the system are given by $\mathcal{X} = \left\{x \in \mathbb{R}^2 : \|x\|_\infty \leq 17\right\}$ and $\mathcal{U} = \{u \in \mathbb{R} : \|u\|_\infty \leq 1{,}5\}$. The EIS set has been selected to be $\mathcal{U}^t = \{u \in \mathbb{R} : \|u\|_\infty \leq 1{,}25\}$. The persistent excitation input $u_{pe}(k) \sim N(\mu, \sigma^2)$ has a normal distribution and lies within $\mathcal{U}^t$, with $\mu = 0$ and standard deviation $\sigma = 0{,}4$. Then, $u_{pe}(k)$ conditional on $u_{pe}(k) \in \mathcal{U}^t$ has a truncated normal distribution.

We compute an OSPIS with probability $p_1 = 0{,}9$, and the invariant set for identification (ISI) $\mathcal{X}^t$, proposed on [4], which actually is an OSPIS with probability $p_2 = 1$ (see figure 1 down).
By the property for the intersection of probabilistic invariant sets from [5], we can get a new OSPIS $\mathcal{S}^t = \mathcal{S}_1 \cap \mathcal{X}^t$ with probability $p = p_1 + p_2 - 1 = 0{,}9$, and the same probability $p = 0{,}9$ holds.

Remark 3. *Notice that by this way, we ensure that $\mathcal{S}^t \subseteq \mathcal{X}^t$. Since $\mathcal{X}^t$ is an ISI for $\mathcal{U}^t$, fulfills that $\mathcal{R}(\mathcal{S}^t) \subseteq \mathcal{X}^t$, with the reachability set $\mathcal{R}(\mathcal{S}^t)$ associated to the EIS set $\mathcal{U}^t$. So, if we show that $\mathcal{X}^t \subseteq \mathcal{Q}(\mathcal{S}^t)$ (see figure 1, down), with the controllability set $\mathcal{Q}(\mathcal{S}^t)$ associated to the input set $\mathcal{U}$, then Assumption 1 holds.*

Notice that, considering the probabilistic sets, the controller target set $\mathcal{S}^t$ is substantially reduced, compared with the deterministic set $\mathcal{X}^t$.

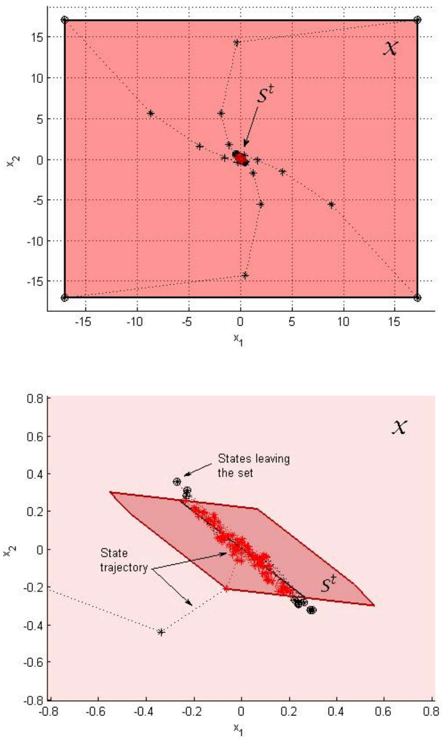

Figura 2: State evolution (left) outside and (right) inside $\mathcal{S}^t$.

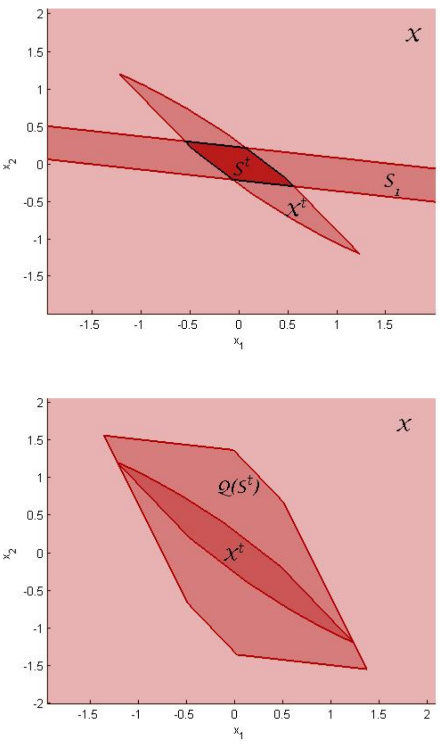

Figura 1: Intersection $\mathcal{S}^t = \mathcal{S}_1 \cap \mathcal{X}^t$ (up). Controllability set $\mathcal{Q}(\mathcal{S}^t)$ associated to the input set $\mathcal{U}$ (down).

4.2. Model including the exciting mode

In this section some simulations results are presented, to evaluate the state evolution with the proposed control strategy in the same model presented before. Figure 2 shows how the system is steered to the OSPIS $\mathcal{S}^t$, and once the system enters the target set, the exciting procedure is activated, until it comes out again. Once outside, the excitation procedure is stopped, aand the control is activated steering the system back to the OSPIS in one step. This is repeated until the identification process is finished.

Notice that, in figure 2, after the system enters S^t for the first time, it leaves that set nine times out of one hundred steps, which is consistent with the probability $p \geq 0{,}9$ of this OSPIS.

We are taking advantage of the probabilistic nature of the process to find the set. In this way, we can adjust the size of the OSPIS with its own invariance probabilistic character, and, at the same time, have control over the amount of excited states.

5. CONCLUSIONS

In this work a new MPC suitable for closed-loop reidentification is proposed. The main benefits consists in the use of a reduced target set, that is computed taking into account probabilistic invariance concepts. This way, the persistent excitation of the closed-loop system is ensured, and furthermore, state-input uncorrelated data can be obtained. In addition, from the control point of view, a less conservative formulation is obtained, which considerably improves the applicability of the proposed methodology.

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