DISTRIBUTED MPC FOR TRACKING. APPLICATION TO A 4 TANKS PLANT.

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Abstract: The problem of controlling large scale systems, usually divided into subsystems controlled by different agents, is solved using cooperative distributed control schemes, where the agents share open-loop information in order to improve closed-loop performance. In a recent paper, a cooperative distributed linear model predictive control strategy applicable to any finite number of subsystems, has been presented. This controller is able to steer the system to any admissible setpoint in an admissible way. Feasibility is ensured under any changing of the target steady state. In this paper, this controller is applied to the plant proposed for the HD-MPC Benchmark: the four tanks plant situated in the lab of the University of Seville.

Key words: Model predictive control, distributed control, tracking non-zero setpoints.

1. INTRODUCTION

Large scale control systems usually consist of linked units of operations and can be divided into a number of subsystems controlled by different agents which may or may not share information. A first approach to this problem is decentralized control, in which interactions between the different subsystems are not considered (Sandell Jr. *et al.*, 1978). The main issue of this solution appears when the intersubsystem interactions become strong. Centralized control, a single agent controls the plantwide system, is another traditional solution that can cope with this control problem. The main problems of this solution are the computational burden and the coordination of subsystems and controller. Distributed control schemes, where agents share open-loop information in order to improve closed-loop performance, solve many of these problems (Rawlings and Mayne, 2009, Chapter 6).
The difference between the distributed control strategies is in the use of this open-loop information. In noncooperative distributed control each subsystem considers the other subsystems information as a known disturbance (Camponogara *et al.*, 2002; Dunbar, 2007). This strategy leads the whole system to converge to a Nash equilibrium.
In cooperative distributed control the agents share a common objective, which can be considered as the overall plant objective. This means that any player calculates its corresponding inputs by minimizing the same and unique cost function, by means of an iterative (and hence suboptimal) distributed optimization problem (Pannocchia *et al.*, 2009; Stewart *et al.*, 2010). This strategy is a form of suboptimal control: stability is deduced from suboptimal control theory (Scokaert *et al.*, 1999) and convergence to a Pareto optimum is ensured.
Another interesting approach to distributed control is dual decomposition (Rantzer, 2009; Negenborn *et al.*, 2009). This method is based on the use of Lagrange multipliers in order to relax the coupling between different agents. These multipliers can be seen as prices in a market mechanism, by means of which is achieved an agreement between the solution of the different subproblems.

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Model predictive control (MPC) is one of the most successful techniques of advanced control in the process industries. It is based on the solution of an optimization problem and allows the explicit consideration of hard constraints in the formulation (Camacho and Bordons, 2004). Furthermore, asymptotic stability and constraints satisfaction of the closed-loop system has been established (Mayne *et al.*, 2000; Rawlings and Mayne, 2009, Chapter 2).
Most of the results on MPC consider the regulation problem, that is steering the system to a fixed setpoint. In (Limon *et al.*, 2008; Ferramosca *et al.*, 2009) an MPC for tracking of constrained linear systems is proposed, which is able to lead the system to any admissible set point in an admissible way.
In (Ferramosca *et al.*, 2011; Ferramosca, 2011), the MPC for tracking presented in (Limon *et al.*, 2008) and (Ferramosca *et al.*, 2009) is extended to the case of large scale distributed systems. Among the different solutions presented in literature, the authors particularly focus on the cooperative formulation for distributed MPC.
In this paper, the application of this novel controller to the 4 tanks plant is presented. This plant has been used as Benchmark for the HD-MPC Project. To this aim, see (Alvarado *et al.*, 2011), where different distributed control strategy applied to the 4 tanks plant, are compared.

The paper is organized as follows. In section 2 the constrained tracking problem is stated. In section 3 the proposed cooperative distributed MPC for tracking is presented. In section 4 the application of the proposed controller to the real 4 tanks plant is illustrated. Finally, the conclusions of this work are given in section 5.

2. PROBLEM STATEMENT

Consider a system described by a linear invariant discrete time model

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the current control vector, $y \in \mathbb{R}^p$ is the controlled output and x^+ is the successor state. The solution of this system for a given sequence of control inputs $\mathbf{u}$ and initial state x is denoted as $x(j) = \phi(j; x, \mathbf{u})$ where $x = \phi(0; x, \mathbf{u})$. The state of the system and the control input applied at sampling time k are denoted as $x(k)$ and $u(k)$ respectively. The system is subject to hard constraints on state and control:

$$x(k) \in X, \quad u(k) \in U \tag{2}$$

for all $k \geq 0$. $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are compact convex polyhedra containing the origin in their interior. It is assumed that the following hypothesis hold.

Assumption 1. The pair (A,B) is stabilizable and the state is measured at each sampling time.

2.1 Characterization of the equilibrium points of the plant

The steady state, input and output of the plant (x_s, u_s, y_s) are such that (1) is fulfilled, i.e. $x_s = Ax_s + Bu_s$, and $y_s = Cx_s + Du_s$.
We define the sets of admissible equilibrium states, inputs and outputs as

$$\mathcal{Z}_s = \{(x,u) \in X \times U \mid x = Ax + Bu\} \tag{3}$$

$$\mathcal{X}_s = \{x \in X \mid \exists u \in U \text{ such that } (x,u) \in \mathcal{Z}_s\} \tag{4}$$

$$\mathcal{Y}_s = \{y = C_x + D_u \mid (x,u) \in \lambda\mathcal{Z}_s\} \tag{5}$$

Notice that $\mathcal{X}_s$ is the projection of $\mathcal{Z}_s$ onto X.

Under assumption 1 and *Lemma 1.14* in (Rawlings and Mayne, 2009, p. 83), any steady state and input of system (1) (x_s, u_s) and the relative output y_s, namely, every solution of the following equation,

$$\begin{bmatrix} A - I_n & B & \mathbf{0}_{p,1} \\ C & D & -I_p \end{bmatrix} \begin{bmatrix} x_s \\ u_s \\ y_s \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n,1} \\ \mathbf{0}_{p,1} \end{bmatrix} \tag{6}$$

are such that $(x_s, u_s) = M_y y_s$, where M_y is a suitable matrix.
If *Lemma 1.14* in (Rawlings and Mayne, 2009, p. 83) does not hold, the steady state and input (x_s, u_s) of the system can be parameterized by a linear combination of a vector $\theta \in \mathbb{R}^m$ (Limon *et al.*, 2008).

2.2 Distributed model of the plant

In this work, a distributed control framework is considered based on a suitable partition of the plant into a collection of coupled subsystems. In virtue of (Stewart *et al.*, 2010, Section 3.1.1) and (Rawlings and Mayne, 2009, Chapter 6, pp. 421-422), we consider that the plant given by (1) is partitioned in M subsystems (where $M \leq m$) modeled as follows:

$$\begin{aligned} x_i^+ &= A_i x_i + \sum_{j=1}^{M} \bar{B}_{ij} u_j \\ y_i &= C_i x_i + \sum_{j=1}^{M} \bar{D}_{ij} u_j \end{aligned} \tag{7}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_j \in \mathbb{R}^{m_j}$, $y_i \in \mathbb{R}^p_i$, $A_i \in \mathbb{R}^{n_i \times n_i}$ and $B_{ij} \in \mathbb{R}^{n_i \times m_j}$. Without loss of generality, it is considered that $u = (u_1, \cdots, u_M)$.

As proved in (Stewart *et al.*, 2010), any plant can be partitioned as proposed for a certain definition of x_i. If the couple (C_i, A_i) is observable, the inner state of the partition can be calculated or estimated from the measured output of the subsystem y_i.

For the sake of simplicity of the exposition, the results will be presented for the case of two players game. In this case, the plant can be represented in the form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{21} \end{bmatrix} u_1 + \begin{bmatrix} \bar{B}_{12} \\ \bar{B}_{22} \end{bmatrix} u_2$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 & \\ & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \bar{D}_{11} \\ \bar{D}_{21} \end{bmatrix} u_1 + \begin{bmatrix} \bar{D}_{12} \\ \bar{D}_{22} \end{bmatrix} u_2$$

The solution of this system, given the sequences of control inputs $\mathbf{u}_1$ and $\mathbf{u}_2$ and initial state x is denoted as $x(j) = \phi(j; x, \mathbf{u}_1, \mathbf{u}_2)$ where $x = \phi(0; x, \mathbf{u}_1, \mathbf{u}_2)$.

The problem we consider is the design of a cooperative distributed MPC controller to track a (possible time-varying) plant-wide target output $y_t = (y_{t,1}, y_{t,2})$. The proposed distributed controller will ensure convergence to the target if this is admissible or as close as possible if not admissible. This control law is shown in the following section.

3. COOPERATIVE MPC FOR TRACKING

The distributed control scheme proposed in (Ferramosca *et al.*, 2011; Ferramosca, 2011) extends the MPC for tracking presented in (Limon *et al.*, 2008; Ferramosca *et al.*, 2009) to a cooperative distributed framework. As in the centralized case, an artificial equilibrium point of the plant (x_s, u_s, y_s), characterized by y_s, is added as decision variable and the following modified cost function is considered:

$$V_N(x, y_t; \mathbf{u}, y_s) = \sum_{j=0}^{N-1} \|x(j) - x_s\|_Q^2 + \|u(j) - u_s\|_R^2 + \|x(N) - x_s\|_P^2 + V_O(y_s, y_t)$$

where $x = (x_1, x_2)$, $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ and (x_s, u_s, y_s) is the artificial equilibrium point of the plant given by y_s. The function $V_O(y_s, y_t)$ is the so called *offset cost function* and it is defined as follows:

Definition 1. Let $V_O(y_s, y_t)$ be a convex and positive definite function in y_s such that the minimizer of

$$\min_{y_s \in \mathcal{Y}_s} V_O(y_s, y_t)$$

is unique.

This function $V_O(y_s, y_t)$ is a measure of the (economic) cost associated to a given setpoint y_t. This function is typically chosen as a function of the distance $\|y_s - y_t\|$ (Ferramosca *et al.*, 2009).

The following assumptions are considered to prove stability of the controller:

Assumption 2.

(1) Let $R \in \mathbb{R}^{m \times m}$ be a positive semidefinite matrix and $Q \in \mathbb{R}^{n \times n}$ a positive semi-definite matrix such that the pair $(Q^{1/2}, A)$ is observable.
(2) Let $K \in \mathbb{R}^{m \times n}$ be a stabilizing control gain for the centralized system, such that $(A + BK)$ has all the eigenvalues in the unit cirle.
(3) Let $P \in \mathbb{R}^{n \times n}$ be a positive definite matrix for the centralized system such that:

$$(A+BK)'P(A+BK) - P = -(Q + K'RK) \quad (8)$$

(4) Let $\Omega_t^a \subseteq \mathbb{R}^{n+p}$ be an admissible polyhedral invariant set for tracking for system (1) subject to (2), for a given gain K (Limon *et al.*, 2008). That is, given the extended state $a = (x, y_s)$, for all $a \in \Omega_\lambda$, then $a^+ = A_a a \in \Omega_t^a$, where A_a is the closed-loop matrix given by

$$A_a = \begin{bmatrix} A + BK & BL \\ 0 & I_p \end{bmatrix}$$

and $L = [-K, I_p]M_y$. Furthermore Ω_t^a must be contained in the polyhedral set W_λ given by

$$W_\lambda = \{(x, y_s) \in X \times \mathcal{Y}_s : Kx + Ly_s \in U\}$$

The control action to be applied at each sampling time is calculated by an iterative method where an optimization problem for each agent is solved at each iteration. The optimization problem that each i-th agent solves at the $p+1$ iteration is denoted as $P_i(x, y_t, \mathbf{u}^{[p]})$ and it is given by:

$$(\mathbf{u}_i^*, y_{s,i}^*) = \arg\min_{\mathbf{u}_i, y_s} V_N(x, y_t; \mathbf{u}, y_s) \quad (9a)$$

$$\begin{aligned} s.t.\quad & \\ & x_q(j+1) = A_q x_q(j) + \sum_{\ell=1}^{2} B_{q\ell} u_\ell(j),\ q \in \mathbb{I}_{1,2} \\ & x_1(0) = x_1,\ x_2(0) = x_2 \\ & (\mathbf{u}_1^{[p]}, \mathbf{u}_2^{[p]}) = \mathbf{u}^{[p]}, \\ & u_\ell(j) = u_\ell^{[p]}(j) \quad \ell \in \mathbb{I}_{1,2} \setminus i, \\ & (x_1(j), x_2(j)) \in X, \\ & (u_1(j), u_2(j)) \in U, \quad j = 0, ..., N-1 \\ & (x(N), y_s) \in \Omega_t^a \end{aligned}$$

Based on the solution of this optimization problem for each agent, namely $\mathbf{u}_1^*$ and $\mathbf{u}_2^*$, the solution of the $p+1$-iteration is given by

$$\mathbf{u}_1^{[p+1]} = w_1 \mathbf{u}_1^* + (1 - w_1)\mathbf{u}_1^{[p]} \quad (10a)$$

$$\mathbf{u}_2^{[p+1]} = (1 - w_2)\mathbf{u}_2^{[p]} + w_2 \mathbf{u}_2^* \quad (10b)$$

$$y_s^{[p+1]} = w_1 y_{s,1}^* + w_2 y_{s,2}^* \quad (10c)$$

$$w_1 + w_2 = 1 \quad w_1, w_2 > 0$$

As in (Stewart *et al.*, 2010), once the algorithm reaches the last iteration $\bar{p}$, the inputs of the plant are $u_1(k) = u_1^{[\bar{p}]}(0; k)$ and $u_2(k) = u_2^{[\bar{p}]}(0; k)$.

To proceed with the analysis of the proposed controller, we will denote

$$\mathbf{v} = (\mathbf{u}_1, \mathbf{u}_2, y_s)$$

$\mathbf{v}$ is said to be feasible at x if each optimization problem $P_i(x, y_t, (\mathbf{u}_1, \mathbf{u}_2))$ is feasible for all $i \in \mathbb{I}_{1:2}$. The set of states for which there exists a feasible $\mathbf{v}$ is denoted as $\mathcal{X}_N$. Notice that this set is equal to the feasible set of the centralized MPC for tracking (Limon *et al.*, 2008), i.e. the set of states that can be admissibly steered to $Proj_x(\Omega_t^a)$ in N steps.

Since the proposed distributed MPC can be considered as a suboptimal formulation of the centralized MPC, a *warm start* of the suboptimal MPC has to be defined in order to determines recursive feasibility and convergence of the control algorithm. In (Ferramosca *et al.*, 2011; Ferramosca, 2011), the initial solution $\mathbf{v}^{[0]}$ of the iterative procedure (10) is obtained by means of the following algorithm:

Algorithm 1.

Given the solution $\mathbf{v}(k)$, the objective is to calculate the warm start at sampling time $k+1$, denoted as

$\mathbf{v}(k+1)^{[0]} = (\mathbf{u}_1(k+1)^{[0]}, \mathbf{u}_1(k+1)^{[0]}, y_s(k+1)^{[0]})$.

(1) Define the first candidate initial solution:

$$\tilde{\mathbf{u}}_1(k+1) = \{u_1(1;k), ..., u_1(N-1;k), u_{c,1}(N)\}$$
$$\tilde{\mathbf{u}}_2(k+1) = \{u_2(1;k), ..., u_2(N-1;k), u_{c,2}(N)\}$$

where

$$u_c(N) = (u_{c,1}(N), u_{c,2}(N)) = Kx(N) + Ly_s^0(k)$$

is the centralized solution given by the centralized terminal control law, and $x(N) = \phi(N; x(k), \mathbf{u}_1(k), \mathbf{u}_2(k))$.

(2) Define the second candidate initial solution:

$$\hat{\mathbf{u}}_1(k+1) = \{\hat{u}_{c,1}(0), ..., \hat{u}_{c,1}(N-1)\}$$
$$\hat{\mathbf{u}}_2(k+1) = \{\hat{u}_{c,2}(0), ..., \hat{u}_{c,2}(N-1)\}$$

where $(\hat{u}_{c,1}(j), \hat{u}_{c,2}(j)) = \hat{u}_c(j)$ and

$$\hat{x}(0) = x(k+1)$$
$$\hat{x}(j+1) = (A+BK)\hat{x}(j) + BLy_s^0(k), \ j \in \mathbb{I}_{1:N-2}$$
$$\hat{u}_c(j) = K\hat{x}(j) + Ly_s^0(k)$$

(3) **IF** $(x(k+1), y_s^0(k)) \in \Omega_t^a$ **AND** $V_N(x(k+1), y_t, \hat{\mathbf{u}}) \leq V_N(x(k+1), y_t, \tilde{\mathbf{u}})$
SET

$$\mathbf{v}(k+1)^{[0]} = (\hat{\mathbf{u}}_1(k+1), \hat{\mathbf{u}}_2(k+1), y_s^0(k))$$

ELSE

$$\mathbf{v}(k+1)^{[0]} = (\tilde{\mathbf{u}}_1(k+1), \tilde{\mathbf{u}}_2(k+1), y_s^0(k))$$

Remark 2. As usual in the suboptimal MPC, the proposed *warm start* for the first optimization iteration $p = 0$ is given by the previous optimal sequence, shifted one step ahead, and with the last control move given by the centralized terminal control law, that is $(\tilde{\mathbf{u}}_1(k+1), \tilde{\mathbf{u}}_2(k+1), y_s(k))$. Once the system reaches the invariant set for tracking, that is $(x(k+1), y_s^0(k)) \in \Omega_t^a$, which is the region where constraints are not active, it is desirable that the distributed controller achieves better performance than the centralized one. If this is not possible, then the *warm-start* will be given by the centralized terminal control law.
With this choice, convergence to the optimal centralized target and controllability of the solution are ensured.

Hence, at each sampling time k, the initial *warm start* $\mathbf{v}^{[0]}(k)$ is calculated using algorithm 1 and then, $\mathbf{v}^{[p]}(k)$ is obtained from the iterative procedure given by (9) and (10). At a certain number of iteration $\bar{p}$, the final solution, denoted as

$$\mathbf{v}(k) = (\mathbf{u}_1^{[\bar{p}]}(k), \mathbf{u}_2^{[\bar{p}]}(k), \hat{y}_s^{[\bar{p}]}(k)),$$

is achieved. This solution is a function of: (i) the current state $x(k)$ and (ii) the initial feasible solution $\mathbf{v}^{[0]}(k)$ that depends on $\mathbf{v}(k-1)$. Then, the overall control law can be posed as

$$\mathbf{v}(k+1) = g(x(k), \mathbf{v}(k)) \quad (11a)$$
$$x(k+1) = Ax(k) + BH\mathbf{v}(k) \quad (11b)$$

where g is a suitable function and H is an appropriate constant matrix.

In (Ferramosca, 2011)[Theorem 5.6], the stabilizing properties of this controller are stated. It is, in fact, proved that for all $x(0) \in \mathcal{X}_N$ and for all y_t, the closed-loop system is asymptotic stable and converges to an equilibrium point $(x_s^*, u_s^*) = M_y y_s^*$ such that

$$y_s^* = \arg\min_{y_s \in \mathcal{Y}_s} V_O(y_s, y_t)$$

Moreover, if $y_t \in \mathcal{Y}_s$, then $y_s^* = y_t$.

3.1 Properties of the proposed controller

The proposed controller provides the following properties to the closed-loop system:

- The domain of attraction of the proposed controller is (potentially) larger than the domain of the standard distributed MPC, since this set is defined for any equilibrium point.
- The proposed controller is able to track any changing setpoint, maintaining the recursive feasibility and constraint satisfaction, since the optimization problem is feasible for any y_t.
- In cooperative MPC, the target problem solved in a distributed way, converges to the centralized optimum only if the constraints are uncoupled. In case case of coupled constraints, it is recommended to solve a centralized target problem

(Rawlings and Mayne, 2009, Section 6.3.4).
The proposed controller ensures convergence to the centralized optimal equilibrium point, since every agent solves an optimization problem with a centralized offset cost function. This property holds for any suboptimal solution provided by the controller due, for instance, to the effect of coupled constraints between agents, or to a small number of iterations $\bar{p}$. Furthermore, this equilibrium point is the admissible equilibrium which minimizes the offset cost function.

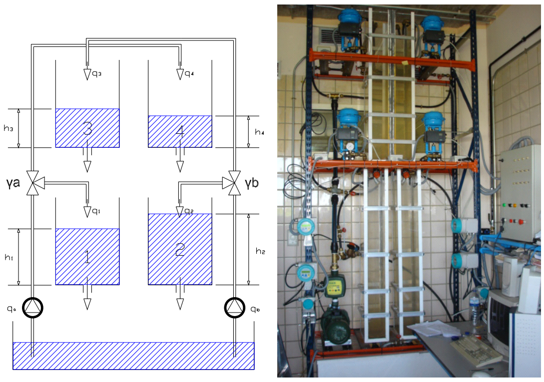
4. APPLICATION TO THE 4 TANKS SYSTEM

4.1 The 4 tanks process

The four tanks plant (Johansson, 2000) is a multivariable laboratory plant of interconnected tanks with nonlinear dynamics and subject to state and input constraints. A scheme of this plant is presented in Figure 1(a). The inputs are the voltages of the two pumps and the outputs are the water levels in the lower two tanks.
A real experimental plant developed at the University of Seville is presented in Figure 1(b). The real plant can be modified to offer a wide variety of configurations such as one single tank, two or three cascaded tanks, a mixture process and hybrid dynamics. Moreover the parameters that define the dynamics of each tank can be modified by tuning the cross-section of the outlet hole of the tank.
The real plant has been implemented using industrial instrumentation (Alvarado, 2007). The inlet flow of each tank is measured by an electromagnetic flowmeter (Siemens Sitrans FM Flow sensor 711/S and transmitters Intermag/transmag) and regulated by a pneumatic valve (Siemens VC 101 with a positioner Sipart PS2 PA). This allows the plant to emulate the three-way valve of Johansson's quadruple-tank process by providing suitable set-points to the flow controllers. The level of each tank is measured by means of a pressure sensor (Siemens Sitrans P 7MF4020 and 7MF4032). All the measurements and commands are 4-20mA current signals transmitted from/to a PLC Siemens S7-200. In order to achieve a safe operation of the plant and to prevent the overflow of tanks, each tank has a high-level switching sensor used as an alarm to switch off the pumps.
Supervision and control of the plant is carried out in a computer by means of OPC (Ole for Process Control) which allows one to connect the plant with a wide range of control programs such as LabView, Matlab or an industrial SCADA.

4.2 Model of the plant

A state space continuous time model of the quadruple tank process system (Johansson, 2000) can be derived from first principles as follows



(a)Scheme of the 4 tank process. (b)The real plant.

Figure 1. The 4 tanks process.

$$\begin{aligned}
\frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_a}{A_1}q_a \quad (12)\\
\frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_b}{A_2}q_b\\
\frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_b)}{A_3}q_b\\
\frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_a)}{A_4}q_a
\end{aligned}$$

The plant parameters, estimated on the real plant are shown in table 1.

The minimum level of the tanks has been taken greater than zero to prevent eddy effects in the discharge of the tank. An important property of this plant is that the dynamics present multivariable transmission zeros which can be located in the right hand side of the s plane for some operating conditions. Hence, the values of γ_a and γ_b have been chosen in order to obtain a system with non-minimum phase multivariable zeros.

Linearizing the model at an operating point given by h_i^0 and defining the deviation variables $x_i = h_i - h_i^o$ and $u_j = q_j - q_j^o$ where $j = a, b$ and $i = 1, \cdots, 4$ we have that:

$$\frac{dx}{dt} = \begin{bmatrix} \frac{-1}{\tau_1} & 0 & \frac{A_3}{A_1\tau_3} & 0 \\ 0 & \frac{-1}{\tau_2} & 0 & \frac{A_4}{A_2\tau_4} \\ 0 & 0 & \frac{-1}{\tau_3} & 0 \\ 0 & 0 & 0 & \frac{-1}{\tau_4} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma_a}{A_1} & 0 \\ 0 & \frac{\gamma_b}{A_2} \\ 0 & \frac{(1-\gamma_b)}{A_3} \\ \frac{(1-\gamma_a)}{A_4} & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x$$

where $\tau_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}} \geq 0$, $i = 1, \cdots, 4$, are the time constants of each tank. This model has been discretized using the zero-order hold method with a sampling time of 15 seconds.

4.3 Application of the proposed controller to the real plant

The proposed controller has been applied to the 4 tanks plant, following the guide lines of the HD-MPC project Benchmark (Alvarado *et al.*, 2011).

To this aim, the original plant has been divide in two subsystem (Fig. 2), coupled through the control action, that is the flows from the pumps.

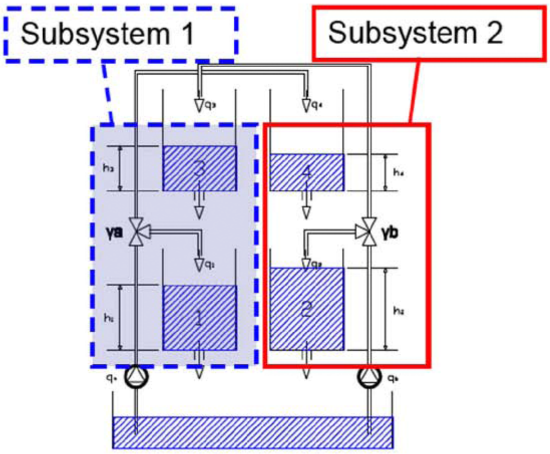


Figure 2. The 4 tanks process: 2 distributed subsystems.

The test on the plant has been run considering 4 changes of reference: $y_{t,1} = (0.65, 0.65)$, $y_{t,2} = (0.35, 0.35)$, $y_{t,3} = (0.50, 0.70)$ and $y_{t,4} = (0.90, 0.70)$. The initial state is $x_0 = (0.47, 0.49, 0.44, 0.46)$. Notice also that the constraints on the model are coupled due to the dynamic. The setups for the two distributed controllers are the followings:

- Agent 1: $Q_1 = I_2$, $R_1 = 0.01I_1$, N=5, $w_1 = 0.5$.
- Agent 2: $Q_2 = I_2$, $R_2 = 0.01I_1$, N=5, $w_2 = 0.5$.

Table 1. 4 tanks plant parameters

	Value	Unit	Description
H_{1max}	1.36	m	Maximum level of the tank 1
H_{2max}	1.36	m	Maximum level of the tank 2
H_{3max}	1.30	m	Maximum level of the tank 3
H_{4max}	1.30	m	Maximum level of the tank 4
H_{min}	0.2	m	Minimum level in all cases
V_{max}	0.2226	m^3	Maximum water volume
Q_{amax}	3.6	m^3/h	Maximum flow of pump A
Q_{bmax}	4	m^3/h	Maximum flow of pump B
Q_{min}	0	m^3/h	Minimal flow
Q_a^0	1.63	m^3/h	Equilibrium flow (Q_a)
Q_b^0	2.0000	m^3/h	Equilibrium flow (Q_b)
a_1	1.2977e-4	m^2	Discharge constant of tank 1
a_2	1.5033e-4	m^2	Discharge constant of tank 2
a_3	1.0155e-4	m^2	Discharge constant of tank 3
a_4	9.4191e-5	m^2	Discharge constant of tank 4
A	0.06	m^2	Cross-section of all tanks
γ_a	0.3		Parameter of the 3-ways valve
γ_b	0.4		Parameter of the 3-ways valve
h_1^0	0.6662	m	Equilibrium level of tank 1
h_2^0	0.6556	m	Equilibrium level of tank 2
h_3^0	0.5492	m	Equilibrium level of tank 3
h_4^0	0.5771	m	Equilibrium level of tank 4

The number of iterations of the suboptimal optimization algorithm has been chosen as $\bar{p} = 1$. The gain K is chosen as the one of the LQR and the matrix P is the solution of the Riccati equation. The invariant set for tracking has been calculating for the gain matrix K. The chosen offset cost function is $V_O(y_s, y_t) = \|y_s - y_t\|_T^2$ where $T = 100I$. The optimization has been run in Matlab. The calculated control inputs have been injected into the plant by means of OPC.

The result of the experiment are presented in figures 3, 4 and 5.

In particular, in figure 3, the set of admissible equilibrium output, $\mathcal{Y}_s$, and the state-space evolution of the output are depicted. The dots represent the desired setpoints. Notice how the controller always steers the system to the desired target.

In figure 4 the time evolution of the output is presented. The desired setpoint, the artificial references and the real output are depicted respectively in blue dashed-dotted, red dashed and black solid lines. Notice how the controller steers the system to the desired setpoint, always fulfilling the constraints. Notice also the role played by the artificial reference in maintaining feasibility when the setpoint changes: even in case of a large change of setpoint, the artificial reference provide to the controller a feasible steady state (reachable in N steps), ensuring that feasibility is never lost. The centralized offset cost function ensures convergence to the desired setpoint. See in particular in figure 4, the forth change of reference of the output h_1.
Notice that the offset between references and output is due to the mismatches between the nonlinear plant and the linearized model used for the predictions.

In figure 5, the time evolution of h_3 and h_4 and the control input q_a and q_b is presented in solid black line, while the steady state value of the input (the artificial reference) is plotted in red dashed line. Notice how, when the setpoint changes, the control action tends to saturate, but then it is quickly driven to the steady state value by the artificial reference.

5. CONCLUSIONS

In (Ferramosca *et al.*, 2011; Ferramosca, 2011) a cooperative MPC for tracking is presented, able to steer the system to any admissible setpoint in an admissible way. The controller ensures convergence to the centralized optimal equilibrium point, since every agent solves an optimization problem with a centralized offset cost function.
In this paper, this controller has been applied to a real plant: the four tanks plant situated in the lab of the University of Seville. The test shows how the controller is capable to steer the system to the desired setpoint, even in presence of coupled constraints. The test shows some offset between desired setpoint and

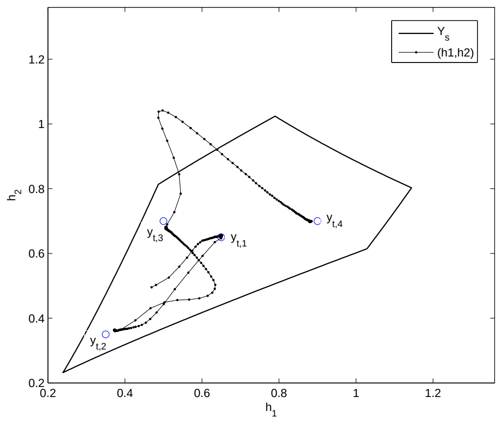


Figure 3. State-space evolution of the output h_1 and h_2. The blue dots are the setpoints.

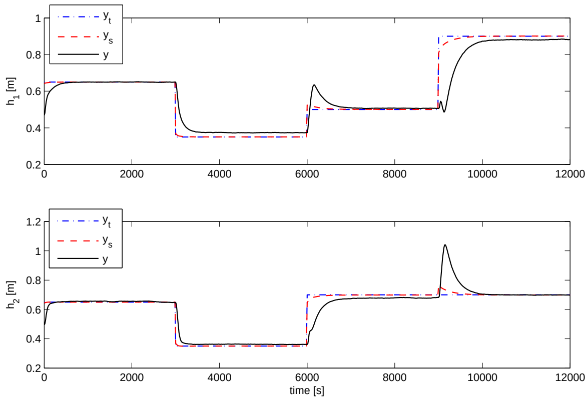


Figure 4. Time evolution of the output h_1 and h_2.

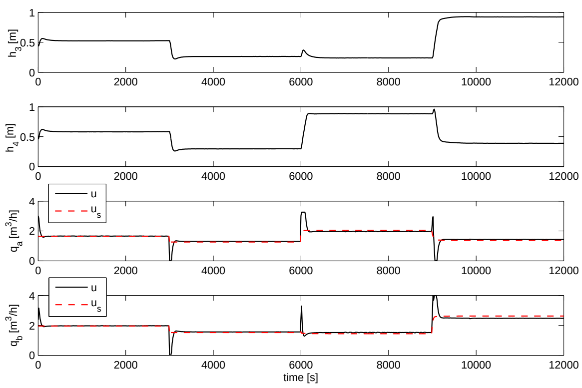


Figure 5. Time evolution of h_3 and h_4 and of the control input q_a and q_b.

real output. This offset is due to the mismatches between the nonlinear plant and the linearized model used for the predictions.

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