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MIXED $L^p(L^2)$ NORMS OF THE LATTICE POINT DISCREPANCY

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Abstract. We estimate some mixed $L^p\left(L^2\right)$ norms of the discrepancy between the volume and the number of integer points in $r\Omega - x$, a dilated by a factor r and translated by a vector x of a convex body Ω in $\mathbb{R}^d$ with smooth boundary with non-vanishing Gaussian curvature,

$$\left\{\int_{\mathbb{T}^d}\left(\frac{1}{H}\int_R^{R+H}\left|\sum_{k\in\mathbb{Z}^d}\chi_{r\Omega-x}(k)-r^d\left|\Omega\right|\right|^2dr\right)^{p/2}dx\right\}^{1/p}.$$

We obtain estimates for fixed values of H and $R\to\infty$, and also asymptotic estimates when $H\to\infty$.

1. Introduction

The discrepancy between the volume and the number of integer points in $r\Omega - x$, a dilated by a factor r and translated by a vector x of bounded domain Ω in $\mathbb{R}^d$, is

$$\mathcal{D}\left(r\Omega-x\right)=\sum_{k\in\mathbb{Z}^d}\chi_{r\Omega-x}(k)-r^d\left|\Omega\right|.$$

Here $\chi_{r\Omega-x}(y)$ denotes the characteristic function of $r\Omega - x$ and $|\Omega|$ the measure of Ω. A classical problem is to estimate the size of $\mathcal{D}\left(r\Omega-x\right)$, as $r\to+\infty$. For a survey see e.g. [28] and [25]. We want to estimate the mixed $L^p\left(L^2\right)$ norms of this discrepancy:

$$\left\{\int_{\mathbb{T}^d}\left[\frac{1}{H}\int_R^{R+H}\left|\mathcal{D}\left(r\Omega-x\right)\right|^2dr\right]^{p/2}dx\right\}^{1/p}.$$

In order to present our results, we need to introduce some notation. If $d\mu(r)$ is a finite Borel measure on the line $-\infty<r<+\infty$, and if $0<H<+\infty$ and $-\infty<R<+\infty$, the dilated and translated measure $d\mu_{H,R}\left(r\right)$ is defined by

$$\mu_{H,R}\left\{I\right\}=\mu\left\{H^{-1}\left(I-R\right)\right\}.$$

Alternatively, by duality with continuous bounded functions,

$$\int_{\mathbb{R}}f\left(r\right)d\mu_{H,R}\left(r\right)=\int_{\mathbb{R}}f\left(R+Hr\right)d\mu\left(r\right).$$

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With this definition, the Fourier transforms of $d\mu(r)$ and $d\mu_{H,R}(r)$ are related by the equation

$$\begin{aligned}\widehat{\mu}_{H,R}(\zeta) &= \int_{\mathbb{R}} \exp(-2\pi i\zeta r)\, d\mu_{H,R}(r) \\ &= \int_{\mathbb{R}} \exp(-2\pi i\zeta(R+Hr))\, d\mu(r) = \exp(-2\pi i R\zeta)\, \widehat{\mu}(H\zeta).\end{aligned}$$

Recall that the Fourier dimension of a compactly supported measure is the supremum of all δ such that there exists C such that $|\widehat{\mu}(\zeta)| \leq C|\zeta|^{-\delta/2}$. See [9, Section 4.4] and [30, Section 12.17]. By a classical result of D. G. Kendall, the L^2 norm of the discrepancy of an oval $\mathcal{D}(r\Omega - x)$ is of the order of $r^{(d-1)/2}$. See [27] and what follows. For this reason we shall call $r^{-(d-1)/2}\mathcal{D}(r\Omega - x)$ the normalized discrepancy. Our main result below is an estimate of the Fourier dimension of the set where this normalized discrepancy may be large.

Theorem 1.1. *Assume that $d\mu(r)$ is a Borel probability measure on $\mathbb{R}$, with support in $\varepsilon < r < \delta$, with $\delta > \varepsilon > 0$, and assume that the Fourier transform of $d\mu(r)$ has the decay*

$$|\widehat{\mu}(\zeta)| \leq B(1+|\zeta|)^{-\beta},$$

for some $\beta \geq 0$ and $B > 0$. Assume that Ω is a convex set in $\mathbb{R}^d$, with a smooth boundary with strictly positive Gaussian curvature. Finally, for given d and β, define A and α as follows:

$$\begin{cases} d=2, & 0\leq\beta<1, & A=4/(1-\beta), & \alpha=(1+\beta)/4, \\ d=2, & \beta=1, & A=+\infty, & \alpha=1, \\ d=2, & \beta>1, & A=+\infty, & \alpha=1/2, \end{cases}$$

$$\begin{cases} d=3, & 0\leq\beta\leq 1/2, & A=(3-2\beta)/(1-\beta), & \alpha=(1-\beta)/(3-2\beta), \\ d=3, & 1/2\leq\beta<1, & A=6/(2-\beta), & \alpha=(1+\beta)/6, \\ d=3, & \beta=1, & A=6, & \alpha=5/6, \\ d=3, & \beta>1 & A=6, & \alpha=1/3, \end{cases}$$

$$\begin{cases} d\geq 4, & 0\leq\beta<1, & A=(2d-4\beta)/(d-1-2\beta), & \alpha=(d-1-2\beta)/(2d-4\beta), \\ d\geq 4, & \beta=1, & A=(2d-4)/(d-3), & \alpha=(d-1)/(2d-4), \\ d\geq 4, & \beta>1, & A=(2d-4)/(d-3), & \alpha=(d-3)/(2d-4). \end{cases}$$

Then the following hold:
(1) If $p < A$, then there exists C such that for every $H, R \geq 1$,

$$\left\{\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}(r\Omega - x)\right|^2 d\mu_{H,R}(r)\right)^{p/2} dx\right\}^{1/p} \leq C\left(\frac{1}{p}-\frac{1}{A}\right)^{-\alpha}.$$

(2) If $p = A$, then there exists C such that for every $H, R \geq 1$,

$$\left\{\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}(r\Omega - x)\right|^2 d\mu_{H,R}(r)\right)^{p/2} dx\right\}^{1/p} \leq C\log^{\alpha}(1+R).$$

(3) If $\beta > 0$ and $p < A$, the family of functions indexed by R and H

$$\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}(r\Omega - x)\right|^2 d\mu_{H,R}(r)$$

has a limit $\mathcal{G}(x)$ *in the norm of* $L^{p/2}(\mathbb{T}^d)$ *as* $H \to +\infty$. *In particular, the convergence in norm implies the convergence of the norms, and this is uniform in* $R \geq 1$,

$$\lim_{H\to+\infty} \left\{ \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}(r\Omega - x) \right|^2 d\mu_{H,R}(r) \right)^{p/2} dx \right\}^{1/p} = \left\{ \int_{\mathbb{T}^d} |\mathcal{G}(x)|^{p/2} dx \right\}^{1/p}.$$

The statement of the theorem is scary, but a few words could help the reader in understanding its meaning. Consider the case $\beta > 1$, which corresponds to the case of an absolutely continuous measure μ with bounded density. The second point of Theorem 1.1 reduces to

$$\left\{ \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}(r\Omega - x) \right|^2 d\mu_{H,R}(r) \right)^{p/2} dx \right\}^{1/p}$$
$$\leq \begin{cases} C \log^{1/2}(1+R) & \text{if } d = 2, p = +\infty, \\ C \log^{1/3}(1+R) & \text{if } d = 3, p = 6, \\ C \log^{(d-3)/(2d-4)}(1+R) & \text{if } d > 3, p = (2d-4)/(d-3). \end{cases}$$

When p is smaller than the above critical indices, then there is no logarithmic loss, and one can give a precise estimate of the norms in terms of p. This is contained in point (1) of the theorem. When β is smaller than 1, then the values of the critical index A decrease with β, and when $\beta = 0$ one obtains the critical values $p = 4$ for $d = 2$, $p = 3$ for $d = 3$ and $p = 2d/(d-1)$ for $d > 3$.

The growth $(1/p - 1/A)^{-\alpha}$ of the norm of the discrepancy in the above theorem allows to extrapolate some Orlicz type estimates at the critical index $p = A$. In other words, we will be able to remove the logarithmic loss in point (2) of Theorem 1.1 by replacing the $L^A(L^2)$ norm with a slightly smaller Orlicz norm.

Corollary 1.2. *(1) Assume one of the following rows of indices:*

$$\begin{cases} d = 2, \quad \beta = 1, \quad \alpha = 2, \quad \gamma < 2/e, \\ d = 2, \quad \beta > 1, \quad \alpha = 1, \quad \gamma < 1/e. \end{cases}$$

Then there exists $C > 0$ *such that for every* $H, R \geq 1$,

$$\int_{\mathbb{T}^2} \exp\left(\gamma \left(\int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}(r\Omega - x) \right|^2 d\mu_{H,R}(r) \right)^{1/\alpha} \right) dx \leq C.$$

(2) Assume one of the following rows of indices:

$$\left\{ d = 2, \quad 0 \leq \beta < 1, \quad p = 4/(1-\beta), \quad \gamma > 2/(1-\beta), \right.$$

$$\begin{cases} d = 3, & 0 \leq \beta \leq 1/2, & p = (3-2\beta)/(1-\beta), & \gamma > 2, \\ d = 3, & 1/2 \leq \beta < 1, & p = 6/(2-\beta), & \gamma > 3/(2-\beta), \\ d = 3, & \beta = 1, & p = 6, & \gamma > 6, \\ d = 3, & \beta > 1, & p = 6, & \gamma > 3, \end{cases}$$

$$\begin{cases} d \geq 4, & 0 \leq \beta < 1, & p = (2d-4\beta)/(d-1-2\beta), & \gamma > 2, \\ d \geq 4, & \beta = 1, & p = (2d-4)/(d-3), & \gamma > (2d-4)/(d-3), \\ d \geq 4, & \beta > 1, & p = (2d-4)/(d-3), & \gamma > 2. \end{cases}$$

Then there exists C *such that for every* $H, R \geq 1$,

$$\int_{\mathbb{T}^d} \left(\int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}(r\Omega - x) \right|^2 d\mu_{H,R}(r) \right)^{p/2}$$

$$\times \log^{-\gamma}\left(2+\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}\left(r\Omega-x\right)\right|^{2}d\mu_{H,R}\left(r\right)\right)dx\leq C.$$

The existence of the limit function $\mathcal{G}(x)$ in the above Theorem 1.1 is somehow related to the quasi-periodicity of the normalized discrepancy as a function of the dilation parameter r. Indeed, for *"generic"* convex sets, the theorem can be slightly strengthened. Let us recall that the support function of a convex set $\Omega \subset \mathbb{R}^d$ is defined by $g(x) = \sup_{y\in\Omega}\{x \cdot y\}$. The point y that realizes the supremum in this definition is the point of $\partial\Omega$ where the outer normal is parallel to x. For example, if Ω is the unit ball centered at the origin, then $g(x) = |x|$.

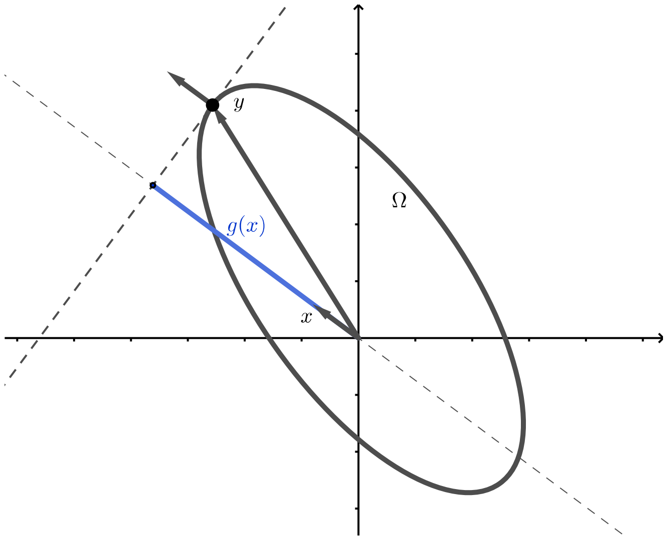

FIGURE 1. The value of $g(x)$ when $|x| = 1$.

Corollary 1.3. *Let Ω be a convex set with smooth boundary with strictly positive Gaussian curvature, let $g(x) = \sup_{y\in\Omega}\{x \cdot y\}$ be its support function, and let $\mathcal{G}(x)$ be the limit function in the statement of Theorem 1.1 (3) with $\beta > 0$.*

(1) The limit function $\mathcal{G}(x)$ is constant if and only if the support function $g(x)$ is injective when restricted to the integers, that is $g(m) \neq g(n)$ for every $m, n \in \mathbb{Z}^d$ with $m \neq n$. This constant is independent of the probability measure $d\mu(r)$ and it is explicitly given by the series

$$\frac{1}{2\pi^2}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}K(n)^{-1}\left|n\right|^{-d-1},$$

where $K(n)$ is the Gaussian curvature of $\partial\Omega$ at the points with outer unit normal $n/|n|$.

(2) If the support function $g(x)$ has the property that there exists C such that for every m in $\mathbb{Z}^d$ the equation $g(m) = g(n)$ has at most C solutions n in $\mathbb{Z}^d$, then the limit function $\mathcal{G}(x)$ is bounded and continuous in $\mathbb{T}^d$.

(3) For every convex set Ω with smooth boundary with strictly positive Gaussian curvature there exists a translation t such that the support function of the set $\Omega+t$ is not injective on the integers, and the limit function $\mathcal{G}_{\Omega+t}(x)$ associated with $\Omega+t$ is

nonconstant. On the other hand, for almost every translation t the support function of $\Omega+t$ is injective on the integers, and the limit function $\mathcal{G}_{\Omega+t}(x)$ is constant, both with respect to x and t.

A particularly interesting case of the above theorem and corollary is the sphere. For a generic translation of the unit sphere, the limit function $\mathcal{G}(x)$ is a constant that can be expressed in terms of an Epstein zeta function (see [8, page 195]). On the other hand, for the sphere centered at the origin, not only the limit function is nonconstant, but it is even unbounded.

Theorem 1.4. *Let $\Sigma = \{|x| \le 1\}$ be the unit ball in $\mathbb{R}^d$.*

(1) For almost every translation t, the limit function $\mathcal{G}_{\Sigma+t}(x)$ is the constant

$$\frac{1}{2\pi^2} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^{-d-1}.$$

(2) The limit function $\mathcal{G}_{\Sigma}(x)$ of the sphere centered at the origin is nonconstant. In particular, if the dimension of the space d is greater than or equal to 4, then this limit function is unbounded. More precisely, if $d \ge 4$ and if $d\mu(r)$ is an arbitrary Borel probability measure on $\mathbb{R}$, then for every $p > 2d/(d-3)$,

$$\limsup_{H,R\to+\infty} \left\{ \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}(r\Sigma - x) \right|^2 d\mu_{H,R}(r) \right)^{p/2} dx \right\}^{1/p} = +\infty.$$

If in addition the Fourier transform of $d\mu(r)$ vanishes at infinity, $\lim_{|\zeta|\to+\infty} \{|\widehat{\mu}(\zeta)|\} = 0$, then the supremum limit can be replaced by a limit,

$$\lim_{H,R\to+\infty} \left\{ \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}(r\Sigma - x) \right|^2 d\mu_{H,R}(r) \right)^{p/2} dx \right\}^{1/p} = +\infty.$$

Theorem 1.1 (3) and Theorem 1.4 (2) give a measure of how unbounded the limit function $\mathcal{G}(x)$ can be. Our proof of Theorem 1.4 (2) applies only to the case $d \ge 4$, but recall that G. H. Hardy proved in [13] that in dimension $d = 2$ for the disc in the plane centered at the origin there is a sequence of values of r going to infinity such that $r^{-1/2}\mathcal{D}(r\Sigma)$ is as large as $C\log^{1/4}(r)$. A similar result in dimension $d = 3$ for the sphere centered at the origin has been proved by G. Szegő in [38].

The range $p > 2d/(d-3)$ in this theorem should be compared with the range $p < (2d-4)/(d-3)$ in the previous theorem. What happens in between is an open problem. In any case, observe that both these indices $(2d-4)/(d-3)$ and $2d/(d-3)$ are asymptotic to 2 as $d \to +\infty$. The proof of this theorem reduces essentially to an estimate of the norm in $L^{p/2}(\mathbb{T}^d)$ of the function $\mathcal{G}(x)$ which appears as a limit of the discrepancy in Theorem 1.1. We do not know if the statement of the above theorem for the ball also applies to a generic convex set. According to Corollary 1.3 (3), for every convex set Ω there is translation t such that the limit function $\mathcal{G}_{\Omega+t}(x)$ is nonconstant. We do not know if this limit function can be unbounded and how unbounded it can be.

The techniques used to prove the above theorems also apply to the estimates of pure L^p norms of the discrepancy:

$$\left\{ \int_{\mathbb{T}^d} \int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}(r\Omega - x) \right|^p d\mu_{H,R}(r)\, dx \right\}^{1/p}.$$

This will be addressed in another paper. Here it suffices to remark that the set of p's that give bounded pure L^p norms is more restricted than the set of p's that give bounded mixed $L^p(L^2)$ norms. See [10].

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2. Previous results

In order to put in an appropriate perspective our results, let us present a short non-exhaustive list of previous results on the discrepancy.

Studying the arithmetical function $r(n)$, the number of integer pairs (h,k) with $n=h^2+k^2$, G. H. Hardy in [12] and [13] and E. Landau in [29] proved that the discrepancy $\left|\sum_{n\leq T}r(n)-\pi T\right|$ can be larger than $CT^{1/4}\log^{1/4}(T)$. On the other hand, in [14] Hardy proved that the mean square average of this discrepancy satisfies for every $\varepsilon>0$ the estimate

$$\left\{\frac{1}{T}\int_0^T\left|\sum_{n\leq t}r(n)-\pi t\right|^2dt\right\}^{1/2}\leq CT^{1/4+\varepsilon}.$$

In our notation $\sum_{n\leq t}r(n)-\pi t$ is nothing but the discrepancy $\mathcal{D}\left(\sqrt{t}\Omega\right)$ of the disc $\Omega=\{|x|\leq 1\}$ in the plane. Hardy also stated that it is not unlikely that the supremum norm of this discrepancy is dominated by $T^{1/4+\varepsilon}$ for every $\varepsilon>0$. This is the so called Gauss circle problem, see the survey paper [25], or the books [28, 39].

H. Cramer in [7] removed the ε in the theorem of Hardy and proved the more precise asymptotic estimate

$$\lim_{T\to+\infty}\left\{T^{-3/2}\int_0^T\left|\sum_{n\leq t}r(n)-\pi t\right|^2dt\right\}^{1/2}=\left\{\frac{1}{3\pi^2}\sum_{n=1}^{+\infty}\frac{r(n)^2}{n^{3/2}}\right\}^{1/2}.$$

The distribution and higher power moment in the Gauss circle problem and the related Dirichlet divisor problem have been studied by D. R. Heath-Brown in [16] and by K. M. Tsang in [40]. See also [26] and [38] for tridimensional versions of these results.

The above results for the disc and the ball have been extended to other domains. In [31], W. Nowak proved that if Ω is a convex set in the plane with smooth boundary with strictly positive curvature, then for every $R\geq 1$,

$$\left\{\frac{1}{R}\int_0^R\left|\mathcal{D}\left(r\Omega\right)\right|^2dr\right\}^{1/2}\leq CR^{1/2}.$$

Indeed, P. Bleher proved in [2] a more precise asymptotic estimate: There exists an explicit constant C such that

$$\lim_{R\to+\infty}R^{-1/2}\left\{\frac{1}{R}\int_0^R\left|\mathcal{D}\left(r\Omega\right)\right|^2dr\right\}^{1/2}=C.$$

M. Huxley in [19] considered the mean value of the discrepancy over short intervals and proved that if Ω is a convex set in the plane with smooth boundary with strictly positive curvature, then

$$\left\{\int_R^{R+1} |\mathcal{D}(r\Omega)|^2 \, dr\right\}^{1/2} \leq CR^{1/2} \log^{1/2}(R).$$

W. Nowak in [32] proved that the above estimate remains valid also when the integration is over the interval $R \leq r \leq R + \log(R)$, while for $H \leq R$ but $H/\log(R) \to +\infty$ he proved the more precise asymptotic estimate

$$\lim_{R\to+\infty} R^{-1/2} \left\{\frac{1}{H}\int_R^{R+H} |\mathcal{D}(r\Omega)|^2 \, dr\right\}^{1/2} = C.$$

A. Iosevich, E. Sawyer, A. Seeger in [23] and [24] extended the above results to convex sets in $\mathbb{R}^d$ with smooth boundary with strictly positive Gaussian curvature,

$$\left\{\frac{1}{R}\int_R^{2R} |\mathcal{D}(r\Omega)|^2 \, dr\right\}^{1/2} \leq \begin{cases} CR^{1/2} & \text{if } d = 2, \\ CR\log(R) & \text{if } d = 3, \\ CR^{d-2} & \text{if } d > 3. \end{cases}$$

The above are results on the averages of the discrepancy under dilations. D. G. Kendall considered the mean square average of the discrepancy under translations and proved in [27] that if Ω is an oval in $\mathbb{R}^d$,

$$\left\{\int_{\mathbb{T}^d} |\mathcal{D}(R\Omega - x)|^2 \, dx\right\}^{1/2} \leq CR^{(d-1)/2}.$$

Mean square averages of the discrepancy under rotations of the domain have been considered by A. Iosevich in [21], and more general random discrepancies have been discussed in [22]. Finally, L. Brandolini, S. Hofmann and A. Iosevich proved in [6] that the average decay under rotations of the Fourier transform of the characteristic function of a convex set, without any further smoothness or curvature assumptions, satisfies the estimate

$$\left\{\int_{\mathbb{SO}(d)} |\widehat{\chi}_\Omega(\sigma\xi)|^2 \, d\sigma\right\}^{1/2} \leq C(\text{diameter}(\Omega))^{(d-1)/2}|\xi|^{-(d+1)/2}.$$

A corollary of this result is that, without smoothness or curvature assumptions,

$$\left\{\int_{\mathbb{SO}(d)}\int_{\mathbb{T}^d} |\mathcal{D}(\sigma R\Omega - x)|^2 \, dx d\sigma\right\}^{1/2} \leq C(\text{diameter}(\Omega))^{(d-1)/2}R^{(d-1)/2}.$$

See [5, Theorem 6 and Theorem 21].

The study of the L^p norm of the discrepancy with $p \neq 2$ is more recent and the results are less complete. In [3] the L^p norm of the discrepancy was studied for rotated and translated polyhedra Ω in $\mathbb{R}^d$,

$$\left\{\int_{\mathbb{SO}(d)}\int_{\mathbb{T}^d} |\mathcal{D}(\sigma R\Omega - x)|^p \, dx d\sigma\right\}^{1/p} \leq \begin{cases} C\log^d(R) & \text{if } p = 1, \\ CR^{(d-1)(1-1/p)} & \text{if } 1 < p \leq +\infty. \end{cases}$$

In the same paper it was also proved that the above inequalities can be reversed, at least for a simplex and for $p > 1$. In other words, the L^p discrepancy of a polyhedron grows with p. On the contrary, for certain domains with curvature there exists a range of indices p where the L^p discrepancy is of the same order as the L^2 discrepancy, possibly up to a logarithmic transgression. Indeed, M. Huxley in [20] proved that if Ω is a convex set in the plane with boundary with continuous positive curvature, then

$$\left\{\int_{\mathbb{T}^2} |\mathcal{D}(R\Omega - x)|^4 \, dx\right\}^{1/4} \leq CR^{1/2}\log^{1/4}(R).$$

In [4] the above result was extended to convex sets with smooth boundary with positive Gaussian curvature in higher dimensions,

$$\left\{\int_{\mathbb{T}^d} |\mathcal{D}(R\Omega - x)|^p \, dx\right\}^{1/p} \leq \begin{cases} CR^{(d-1)/2} & \text{if } p < 2d/(d-1), \\ CR^{(d-1)/2}\log^{(d-1)/2d}(R) & \text{if } p = 2d/(d-1). \end{cases}$$

The present paper continues this line of research. In particular, here we study the average of the discrepancy function with respect to translations and dilations of the convex set Ω,

$$\left\{\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}(r\Omega - x)\right|^2 d\mu_{H,R}(r)\right)^{p/2} dx\right\}^{1/p}.$$

Theorem 1.1 shows that, for example when the measure μ is the Lebesgue measure restricted to an interval, these averages present the same order of decay as in the case $p = 2$ for all values of $p < +\infty$ when $d = 2$, or $p < 6$ when $d = 3$, or $p < (2d-4)/(d-3)$ when $d > 3$. All this shows that the critical index $p = 2d/(d-1)$ obtained in [20] and [4] can be improved when the L^p norm is replaced with the smaller $L^p(L^2)$ mixed norm. The following examples describe this improvement more precisely.

Example 2.1. If $d\mu(r)$ is the unit mass concentrated at $r = 0$, then $\widehat{\mu}(\zeta) = 1$, so that $\beta = 0$, and the $L^p(L^2)$ mixed norm in Theorem 1.1 reduces to a pure L^p norm, and one obtains

$$\begin{aligned} &\left\{\int_{\mathbb{T}^d}\left|R^{-(d-1)/2}\mathcal{D}(R\Omega - x)\right|^p dx\right\}^{1/p} \\ &\leq \begin{cases} C(2d/(d-1) - p)^{-(d-1)/2d} & \text{if } d \geq 2 \text{ and } p < 2d/(d-1), \\ C\log^{(d-1)/2d}(1+R) & \text{if } d \geq 2 \text{ and } p = 2d/(d-1). \end{cases}\end{aligned}$$

In particular, one recovers some of the results in [20] and [4].

Example 2.2. If $d\mu(r)$ is the uniformly distributed measure in $\{0 < r < 1\}$, then the $L^p(L^2)$ mixed norm in Theorem 1.1 is

$$\left\{\int_{\mathbb{T}^d}\left(\frac{1}{H}\int_R^{R+H}\left|r^{-(d-1)/2}\mathcal{D}(r\Omega - x)\right|^2 dr\right)^{p/2} dx\right\}^{1/p}.$$

The Fourier transform of the uniformly distributed measure in $\{0 < r < 1\}$ has decay $\beta = 1$,

$$\widehat{\mu}(\zeta) = \int_0^1 \exp(-2\pi i\zeta r)\, dr = \exp(-\pi i\zeta)\frac{\sin(\pi\zeta)}{\pi\zeta}.$$

On the other hand, if $\psi(r)$ is a non negative smooth function with integral one and support in $0 \leq r \leq 1$, one can consider a smoothed average

$$\left\{ \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}(r\Omega - x) \right|^2 H^{-1} \psi\left(H^{-1}(r - R) \right) dr \right)^{p/2} dx \right\}^{1/p}.$$

This smoothed average is equivalent to the uniform average over $\{R < r < R + H\}$, but the decay of the Fourier transform $\widehat{\psi}(\zeta)$ is faster than any power β. Hence for the uniformly distributed measure in $\{0 < r < 1\}$ the theorem applies with the indices corresponding to $\beta > 1$:

$$\left\{ \int_{\mathbb{T}^d} \left(\frac{1}{H} \int_R^{R+H} \left| r^{-(d-1)/2} \mathcal{D}(r\Omega - x) \right|^2 dr \right)^{p/2} dx \right\}^{1/p}$$

$$\leq \begin{cases} Cp^{1/2} & \text{if } d = 2 \text{ and } p < +\infty, \\ C \log^{1/2}(1 + R) & \text{if } d = 2 \text{ and } p = +\infty, \\ C(6 - p)^{-1/3} & \text{if } d = 3 \text{ and } p < 6, \\ C \log^{1/3}(1 + R) & \text{if } d = 3 \text{ and } p = 6, \\ C((2d - 4)/(d - 3) - p)^{-(d-3)/(2d-4)} & \text{if } d \geq 4 \text{ and } p < (2d - 4)/(d - 3), \\ C \log^{(d-3)/(2d-4)}(1 + R) & \text{if } d \geq 4 \text{ and } p = (2d - 4)/(d - 3). \end{cases}$$

Observe that the range of indices in the above theorem and corollaries for which the mixed $L^p\left(L^2\right)$ norm remains uniformly bounded is larger than the range of indices in [20] and [4] quoted in the previous example.

Example 2.3. As an intermediate case between the two preceeding examples, one can consider a measure $d\mu(r) = r^{-\alpha} \chi_{\{0<r<1\}}(r)\, dr$, with $0 < \alpha < 1$. In this case $|\widehat{\mu}(\zeta)| \leq C(1 + |\zeta|)^{\alpha - 1}$, that is $\beta = 1 - \alpha$. As a more sophisticated intermediate example, recall that a compactly supported probability measure is a Salem measure if its Fourier dimension $\gamma = \sup\left\{\delta : \ |\widehat{\mu}(\zeta)| \leq C(1 + |\zeta|)^{-\delta/2} \right\}$ is equal to the Hausdorff dimension of the support. Such measures exist for every dimension $0 < \gamma < 1$. See [30, Section 12.17]. The above theorem and corollary assert that the discrepancy cannot be too large in mean on the supports of translated and dilates of these measures.

3. Proofs of theorems and corollaries

The proofs will be splitted into a number of lemmas, some of them well known. The starting point is the observation of D. G. Kendall that the discrepancy $\mathcal{D}(r\Omega - x)$ is a periodic function of the translation, and it has a Fourier expansion with coefficients that are a sampling of the Fourier transform of Ω,

$$\widehat{\chi}_\Omega(\xi) = \int_\Omega \exp(-2\pi i \xi x)\, dx.$$

Lemma 3.1. *The number of integer points in $r\Omega - x$, a translated by a vector $x \in \mathbb{R}^d$ and dilated by a factor $r > 0$ of a domain Ω in the d dimensional Euclidean space, is a periodic function of the translation with Fourier expansion*

$$\sum_{k \in \mathbb{Z}^d} \chi_{r\Omega - x}(k) = \sum_{n \in \mathbb{Z}^d} r^d \widehat{\chi}_\Omega(rn) \exp(2\pi i n x).$$

In particular,

$$\mathcal{D}\left(r\Omega - x\right) = \sum_{n\in\mathbb{Z}^d\setminus\{0\}} r^d \widehat{\chi}_\Omega\left(rn\right)\exp(2\pi inx).$$

Proof. This is a particular case of the Poisson summation formula. □

Remark 3.2. We emphasize that the Fourier expansion of the discrepancy converges at least in $L^2\left(\mathbb{T}^d\right)$, but we are not claiming that it converges pointwise. Indeed, the discrepancy is discontinuous, hence the associated Fourier expansion does not converge absolutely or uniformly. To overcome this problem, one can introduce a mollified discrepancy. If the domain Ω is convex and it contains the origin, then there exists $\varepsilon > 0$ such that if $\varphi\left(x\right)$ is a non negative smooth radial function with support in $\{|x| \le \varepsilon\}$ and with integral 1, and if $0 < \delta \le 1$ and $r \ge 1$, then

$$\begin{aligned}
|\Omega|\left(\left(r-\delta\right)^d - r^d\right) + \left(r-\delta\right)^d \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \widehat{\varphi}\left(\delta n\right)\widehat{\chi}_\Omega\left(\left(r-\delta\right)n\right)\exp\left(2\pi inx\right) \\
\le \sum_{n\in\mathbb{Z}^d} \chi_{r\Omega}(n+x) - |\Omega|\, r^d \\
\le |\Omega|\left(\left(r+\delta\right)^d - r^d\right) + \left(r+\delta\right)^d \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \widehat{\varphi}\left(\delta n\right)\widehat{\chi}_\Omega\left(\left(r+\delta\right)n\right)\exp\left(2\pi inx\right).
\end{aligned}$$

One has $\left|(r+\delta)^d - r^d\right| \le Cr^{d-1}\delta$, and one can define the mollified discrepancy

$$\left(r\pm\delta\right)^d \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \widehat{\varphi}\left(\delta n\right)\widehat{\chi}_\Omega\left(\left(r\pm\delta\right)n\right)\exp\left(2\pi inx\right).$$

Observe that the discrepancy is the limit of this mollified discrepancy as $\delta \to 0+$. Also observe that since $|\widehat{\varphi}\left(\zeta\right)| \le C\left(1+|\zeta|\right)^{-\gamma}$ for every $\gamma > 0$, the mollified Fourier expansion has no problems of convergence.

Lemma 3.3. *Assume that Ω is a convex body in $\mathbb{R}^d$ with smooth boundary with everywhere positive Gaussian curvature. Define the support function $g\left(x\right) = \sup_{y\in\Omega}\{x\cdot y\}$. Then, there exist functions $\{a_j\left(\xi\right)\}_{j=0}^{+\infty}$ and $\{b_j\left(\xi\right)\}_{j=0}^{+\infty}$ homogeneous of degree 0 and smooth in $\mathbb{R}^d\setminus\{0\}$ such that the Fourier transform of the characteristic function of Ω for $|\xi| \to +\infty$ has the asymptotic expansion*

$$\begin{aligned}
&\widehat{\chi}_\Omega\left(\xi\right) = \int_\Omega \exp\left(-2\pi i\xi\cdot x\right)dx \\
&= \exp\left(-2\pi ig\left(\xi\right)\right)|\xi|^{-(d+1)/2}\sum_{j=0}^{h} a_j\left(\xi\right)|\xi|^{-j} + \exp\left(2\pi ig\left(-\xi\right)\right)|\xi|^{-(d+1)/2}\sum_{j=0}^{h} b_j\left(\xi\right)|\xi|^{-j} \\
&+ \mathcal{O}\left(|\xi|^{-(d+2h+3)/2}\right).
\end{aligned}$$

The functions $a_j\left(\xi\right)$ and $b_j\left(\xi\right)$ depend on a finite number of derivatives of a parametrization of the boundary of Ω at the points with outward unit normal $\pm\xi/|\xi|$. In particular,

$$\begin{aligned}
a_0(\xi) &= (2\pi)^{-1}\exp(\pi i(d+1)/4)K\left(\xi\right)^{-1/2}, \\
b_0(\xi) &= (2\pi)^{-1}\exp(-\pi i(d+1)/4)K\left(-\xi\right)^{-1/2},
\end{aligned}$$

where $K(\pm\xi)$ *is the Gaussian curvature of* $\partial\Omega$ *at the points with outward unit normal* $\pm\xi/|\xi|$.

Proof. This is a classical result. See e.g. [11], [17], [18], [36]. Here, as an explicit example, we just recall that the Fourier transform of a ball $\left\{x\in\mathbb{R}^d : |x|\leq R\right\}$ can be expressed in terms of a Bessel function, and Bessel functions have simple asymptotic expansions in terms of trigonometric functions,

$$\begin{aligned}
&\widehat{\chi}_{\{|x|\leq R\}}(\xi) = R^d\widehat{\chi}_{\{|x|\leq 1\}}(R\xi) = R^d|R\xi|^{-d/2}J_{d/2}(2\pi|R\xi|)\\
&= \pi^{-1}R^{(d-1)/2}|\xi|^{-(d+1)/2}\cos(2\pi R|\xi| - (d+1)\pi/4)\\
&- 2^{-4}\pi^{-2}\left(d^2-1\right)R^{(d-3)/2}|\xi|^{-(d+3)/2}\sin(2\pi R|\xi| - (d+1)\pi/4) + ...\\
&+ O(R^{(d-2h-3)/2}|\xi|^{-(d+2h+3)/2})
\end{aligned}$$

More generally, also the Fourier transform of an ellipsoid, that is an affine image of a ball, can be expressed in terms of Bessel functions. See [37]. □

Lemma 3.4. *Assume that* Ω *is a convex body in* $\mathbb{R}^d$ *with smooth boundary with everywhere positive Gaussian curvature. Let* z *be a complex parameter, and for every* $j=0,1,2,...$ *and* $r\geq 1$*, with the notation of the previous lemmas, let define the tempered distributions* $\Phi_j(z,r,x)$ *via the Fourier expansions*

$$\begin{aligned}
&\Phi_j(z,r,x) = r^{-j}\sum_{n\in\mathbb{Z}^d\setminus\{0\}} a_j(n)|n|^{-z-j}\exp(-2\pi i g(n)r)\exp(2\pi i n x)\\
&+ r^{-j}\sum_{n\in\mathbb{Z}^d\setminus\{0\}} b_j(n)|n|^{-z-j}\exp(2\pi i g(-n)r)\exp(2\pi i n x).
\end{aligned}$$

(1) If $\mathrm{Re}(z)+j>d/2$ *then the Fourier expansion that defines* $\Phi_j(z,r,x)$ *converges in* $L^2\left(\mathbb{T}^d\right)$*. If* $\mathrm{Re}(z)+j>d$ *then the convergence is absolute and uniform.*

(2) Let

$$\mathcal{R}_h(r,x) = r^{-(d-1)/2}\mathcal{D}(r\Omega-x) - \sum_{j=0}^{h}\Phi_j((d+1)/2,r,x).$$

If $h>(d-3)/2$ *there exists* C *such that for every* $r\geq 1$*,*

$$|\mathcal{R}_h(r,x)| \leq Cr^{-h-1}.$$

Proof. This is a consequence of the previous lemmas. The terms $\Phi_j((d+1)/2,r,x)$ come from the terms homogeneous of degree $-(d+1)/2-j$ in the asymptotic expansion of the Fourier transform of Ω, while the remainder $\mathcal{R}_h(r,x)$ is given by an absolutely and uniformly convergent Fourier expansion. □

After these preliminary lemmas and with the above notation, we have that the normalized discrepancy has the asymptotic expansion

$$r^{-(d-1)/2}\mathcal{D}(r\Omega-x) = \sum_{j=0}^{h}\Phi_j((d+1)/2,r,x) + O(r^{-h-1}).$$

Now one can describe the strategy of the proof of Theorem 1.1 as follows: The terms that appear in the expansion of the normalized discrepancy have the form

$$r^{-j}\sum_{n\in\mathbb{Z}^d\setminus\{0\}} c(n)|n|^{-(d+1)/2-j}\exp(\pm 2\pi i g(\mp n)r)\exp(2\pi i n x).$$

The expression $\exp\left(\pm 2\pi i g\left(\mp n\right) r\right)$ above has to be intended either as $\exp\left(2\pi i g\left(-n\right) r\right)$ or $\exp\left(-2\pi i g\left(n\right) r\right)$. One can replace the real parameter $\left(d+1\right)/2$ which describes the decay of the Fourier transform by a complex parameter z, and define the function

$$\Theta_j\left(z,r,x\right) = r^{-j} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} c(n)\left|n\right|^{-z-j} \exp\left(\pm 2\pi i g\left(\mp n\right) r\right) \exp\left(2\pi i n x\right).$$

Observe that the Fourier coefficients of this function are analytic functions of the complex variable z. One can estimate the $L^2\left(L^2\right)$ norm of this function via the Parseval equality, and it can be easily seen that the norm is finite if $\operatorname{Re}\left(z\right) > d/2$. Then one can estimate the $L^p\left(L^2\right)$ norm with $p \geq 4$ via the Hausdorff Young inequality, and it turns out that if $\operatorname{Re}\left(z\right) > d\left(1-1/p\right) - 1/2$ then this norm is finite. Finally, the result for $z = \left(d+1\right)/2$, the one for the discrepancy, follows by complex interpolation.

Lemma 3.5. *Let $g\left(x\right) = \sup_{y\in\Omega}\left\{x\cdot y\right\}$ be the support function of a convex Ω which contains the origin, and with a smooth boundary with everywhere positive Gaussian curvature.*

(1) This support function is convex, homogeneous of degree one, positive and smooth away from the origin, and it is equivalent to the Euclidean norm, that is there exist $0 < A < B$ such that for every x,

$$A\left|x\right| \leq g\left(x\right) \leq B\left|x\right|.$$

(2) There exists $C > 0$ such that for all unit vectors ω and ϑ in $\mathbb{R}^d$, there exists a number $A\left(\vartheta,\omega\right)$ such that for every real τ one has

$$\left|g\left(\vartheta-\tau\omega\right) - g\left(\vartheta\right)\right| \geq C\frac{\left|\tau\right|\left|\tau - A\left(\vartheta,\omega\right)\right|}{1+\left|\tau\right|}.$$

Proof. When Ω is the sphere $\left\{\left|x\right|\leq 1\right\}$ the proof is explicit and transparent. Indeed, one has $g(x) = |x|$ and, if $|\vartheta| = |\omega| = 1$, then

$$\left|\vartheta-\tau\omega\right| - \left|\vartheta\right| = \frac{\left|\vartheta-\tau\omega\right|^2 - \left|\vartheta\right|^2}{\left|\vartheta-\tau\omega\right| + \left|\vartheta\right|} = \frac{\tau(\tau - 2\vartheta\cdot\omega)}{1+\left|\vartheta-\tau\omega\right|}.$$

The proof for a generic convex set is a bit more involved. The convexity of the support function easily follows from the convexity of Ω, and also the other properties in (1) are elementary. In order to prove (2), observe that for $|\omega| = |\vartheta| = 1$ and $-\infty < \tau < +\infty$,

$$\left|g\left(\vartheta-\tau\omega\right) - g\left(\vartheta\right)\right| = \left|\frac{g\left(\vartheta-\tau\omega\right)^2 - g\left(\vartheta\right)^2}{g\left(\vartheta-\tau\omega\right) + g\left(\vartheta\right)}\right| \geq C\frac{\left|g\left(\vartheta-\tau\omega\right)^2 - g\left(\vartheta\right)^2\right|}{1+\left|\tau\right|}.$$

It then suffices to prove that there exists a $C > 0$ such that for all unit vectors ϑ and ω, there exists a number $A\left(\vartheta,\omega\right)$ such that for every real τ one has

$$\left|g\left(\vartheta-\tau\omega\right)^2 - g\left(\vartheta\right)^2\right| \geq C\left|\tau\right|\left|\tau - A\left(\vartheta,\omega\right)\right|.$$

Let us show that the function $f\left(\tau\right) = g\left(\vartheta-\tau\omega\right)^2 - g\left(\vartheta\right)^2$ is strictly convex. If $\omega = \pm\vartheta$, then

$$f\left(\tau\right) = g\left(\left(1\pm\tau\right)\vartheta\right)^2 - g\left(\vartheta\right)^2 = \left(\left(1\pm\tau\right)^2 - 1\right) g\left(\vartheta\right)^2 = \left(\tau^2 \pm 2\tau\right) g\left(\vartheta\right)^2.$$

Therefore, if $\omega = \pm\vartheta$,

$$\frac{d^2}{d\tau^2} f(\tau) = 2g(\vartheta)^2 \geq 2C > 0.$$

If $\omega \neq \pm\vartheta$, then $\vartheta - \tau\omega \neq 0$, and

$$\frac{d}{d\tau} f(\tau) = -2g(\vartheta - \tau\omega) \nabla g(\vartheta - \tau\omega) \cdot \omega,$$

$$\frac{d^2}{d\tau^2} f(\tau) = 2(\nabla g(\vartheta - \tau\omega) \cdot \omega)^2 + 2g(\vartheta - \tau\omega)\omega^t \cdot \nabla^2 g(\vartheta - \tau\omega) \cdot \omega.$$

For notational simplicity call $\vartheta - \tau\omega = x$. The Hessian matrix $\nabla^2 g(x)$ is homogeneous of degree -1 and positive semidefinite. When $|x| = 1$ one eigenvalue is 0 and the associated eigenvector is the gradient $\nabla g(x)$, while all the other eigenvalues are the reciprocal of the principal curvatures at the point where the normal is $\nabla g(x)$. See [34, Corollary 2.5.2]. Let α be the minimum of $g(x)$ on the sphere $\{|x| = 1\}$, let $\beta > 0$ be the minimum of $|\nabla g(x)|$ on $\{|x| = 1\}$, and let $\gamma > 0$ be the minimum of the non zero eigenvalues of $\nabla^2 g(x)$ on $\{|x| = 1\}$. If one decomposes ω into $\omega_0 + \omega_1$, where ω_0 is parallel to $\nabla g(x)$ and ω_1 is orthogonal to $\nabla g(x)$, then

$$\begin{aligned}
\frac{d^2}{d\tau^2} f(\tau) &= 2(\nabla g(x) \cdot \omega_0)^2 + 2g(x)\omega_1^t \cdot \nabla^2 g(x) \cdot \omega_1 \\
&= 2|\nabla g(x)|^2 |\omega_0|^2 + 2g(x/|x|)\omega_1^t \cdot \nabla^2 g(x/|x|) \cdot \omega_1 \\
&\geq 2\beta^2 |\omega_0|^2 + 2\alpha\gamma |\omega_1|^2 \geq 2C > 0.
\end{aligned}$$

Therefore, for every ϑ and ω the function $f(\tau)$ is strictly convex, and it has exactly two zeros, one is $\tau = 0$ and the other is $\tau = A(\vartheta, \omega)$, possibly the zero is double. By the Lagrange remainder in interpolation, there exists ε such that

$$f(\tau) = \tau(\tau - A(\vartheta, \omega)) \frac{1}{2} \frac{d^2 f}{d\tau^2}(\varepsilon).$$

And since $d^2 f(\tau)/d\tau^2 \geq 2C > 0$,

$$\left| g(\vartheta - \tau\omega)^2 - g(\vartheta)^2 \right| \geq C|\tau| |\tau - A(\vartheta, \omega)|.$$

□

Lemma 3.6. *If $g(x)$ is the support function of Ω, then for every $(d+1)/2 \leq \alpha < d$ and $\beta \geq 0$, there exists C such that for every $y \in \mathbb{R}^d - \{0\}$,*

$$\int_{\mathbb{R}^d} |x|^{-\alpha} |x - y|^{-\alpha} (1 + |g(x) - g(x - y)|)^{-\beta} dx$$
$$\leq \begin{cases} C|y|^{d-2\alpha-\beta} & \text{if } 0 \leq \beta < 1, \\ C|y|^{d-2\alpha-1} \log(2 + |y|) & \text{if } \beta = 1, \\ C|y|^{d-2\alpha-1} & \text{if } \beta > 1. \end{cases}$$

Proof. Let us explain the numerology behind the lemma. If there is no cutoff $(1 + |g(x) - g(x - y)|)^{-\beta}$, then the change of variables $x = |y|z$ and $y = |y|\omega$ gives

$$\int_{\mathbb{R}^d} |x|^{-\alpha} |x - y|^{-\alpha} dx = |y|^{d-2\alpha} \int_{\mathbb{R}^d} |z|^{-\alpha} |z - \omega|^{-\alpha} dx = C|y|^{d-2\alpha}.$$

On the other hand, the cutoff $(1 + |g(x) - g(x - y)|)^{-\beta}$ gives an extra decay. In particular, the integral with the cutoff $(1 + |g(x) - g(x - y)|)^{-\beta}$ with β large is essentially over the set $\{g(x) = g(x-y)\}$, that is the cutoff reduces the space dimension by 1. This suggests that, at least when β is large, the integral with the cutoff can be seen as the convolution in $\mathbb{R}^{d-1}$ of two homogeneous functions of degree $-\alpha$, and this gives the decay $|y|^{d-1-2\alpha}$. Hence, when $\beta = 0$ the decay is $|y|^{d-2\alpha}$, and when $\beta > 1$ the decay is $|y|^{d-1-2\alpha}$. Finally, by interpolation, when $0 < \beta < 1$ the decay is $|y|^{d-\beta-2\alpha}$. This is just the numerology, the details of the proof are more delicate. The change of variables $x = |y| z$ and $y = |y| \omega$ gives

$$\begin{aligned}
&\int_{\mathbb{R}^d} |x|^{-\alpha} |x-y|^{-\alpha} (1 + |g(x) - g(x-y)|)^{-\beta} \, dx \\
&= |y|^{d-2\alpha} \int_{\mathbb{R}^d} |z|^{-\alpha} |z-\omega|^{-\alpha} (1 + |y| \, |g(z) - g(z-\omega)|)^{-\beta} \, dz.
\end{aligned}$$

If ε is positive and suitably small, there exists $\delta > 0$ such that for every ω and z with $|\omega| = 1$ and $|z| < \varepsilon$ one has $g(z-\omega) - g(z) > \delta$. This suggests to split the domain of integration into

$$\{|z| \le \varepsilon\} \cup \{|z-\omega| \le \varepsilon\} \cup \{|z| \ge \varepsilon, \ |z-\omega| \ge \varepsilon\}.$$

The integral over $\{|z| \le \varepsilon\}$ is bounded by

$$\begin{aligned}
&\int_{\{|z|\le\varepsilon\}} |z|^{-\alpha} |z-\omega|^{-\alpha} (1 + |y| \, |g(z) - g(z-\omega)|)^{-\beta} \, dz \\
&\le (1-\varepsilon)^{-\alpha} (1 + \delta |y|)^{-\beta} \int_{\{|z|\le\varepsilon\}} |z|^{-\alpha} \, dz \\
&\le C (1 + |y|)^{-\beta}.
\end{aligned}$$

The integral over $\{|z-\omega| \le \varepsilon\}$ is bounded similarly,

$$\int_{\{|z-\omega|\le\varepsilon\}} |z|^{-\alpha} |z-\omega|^{-\alpha} (1 + |y| \, |g(z) - g(z-\omega)|)^{-\beta} \, dz \le C (1 + |y|)^{-\beta}.$$

It remains to estimate the integral over $\{|z| \ge \varepsilon, \ |z-\omega| \ge \varepsilon\}$. First observe that

$$\begin{aligned}
&\int_{\{|z|\ge\varepsilon, \ |z-\omega|\ge\varepsilon\}} |z|^{-\alpha} |z-\omega|^{-\alpha} (1 + |y| \, |g(z) - g(z-\omega)|)^{-\beta} \, dz \\
&\le C \int_{\{|z|\ge\varepsilon\}} |z|^{-2\alpha} (1 + |y| \, |g(z) - g(z-\omega)|)^{-\beta} \, dz.
\end{aligned}$$

In spherical coordinates write $z = \rho\vartheta$, with $\varepsilon \le \rho < +\infty$ and $|\vartheta| = 1$, and $dz = \rho^{d-1} d\rho d\vartheta$, with $d\vartheta$ the surface measure on the $d-1$ dimensional sphere $\mathbb{S}^{d-1} = \{|\vartheta| = 1\}$. Then by the above lemma and the change of variables $\rho = 1/\tau$, recalling that $|\omega| = 1$ and $2\alpha - d - 1 \ge 0$,

$$\begin{aligned}
&\int_{\{|z|\ge\varepsilon\}} |z|^{-2\alpha} (1 + |y| \, |g(z) - g(z-\omega)|)^{-\beta} \, dz \\
&= \int_{\mathbb{S}^{d-1}} \int_{\varepsilon}^{+\infty} \rho^{d-1-2\alpha} (1 + |y| \, |g(\rho\vartheta) - g(\rho\vartheta - \omega)|)^{-\beta} \, d\rho d\vartheta \\
&= \int_{\mathbb{S}^{d-1}} \int_0^{1/\varepsilon} \tau^{2\alpha-d-1} \left(1 + |y| \left| \frac{g(\vartheta) - g(\vartheta - \tau\omega)}{\tau} \right| \right)^{-\beta} d\tau d\vartheta \\
&\le \varepsilon^{d+1-2\alpha} \int_{\mathbb{S}^{d-1}} \int_0^{1/\varepsilon} \left(1 + |y| \left| \frac{g(\vartheta) - g(\vartheta - \tau\omega)}{\tau} \right| \right)^{-\beta} d\tau d\vartheta
\end{aligned}$$

$$\leq C\int_{\mathbb{S}^{d-1}}\int_{0}^{1/\varepsilon}\left(1+|y|\left|\tau-A\left(\vartheta,\omega\right)\right|\right)^{-\beta}d\tau d\vartheta.$$

Finally, if $\sup_{|\vartheta|=|\omega|=1}\{|A(\vartheta,\omega)|\}=\gamma$, then

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}}\int_{0}^{1/\varepsilon}\left(1+|y|\left|\tau-A\left(\vartheta,\omega\right)\right|\right)^{-\beta}d\tau d\vartheta\\&\leq\left|\mathbb{S}^{d-1}\right|\int_{-\gamma-1/\varepsilon}^{\gamma+1/\varepsilon}\left(1+|y|\,|\tau|\right)^{-\beta}d\tau\\&=\left|\mathbb{S}^{d-1}\right||y|^{-1}\int_{-(\gamma+1/\varepsilon)|y|}^{(\gamma+1/\varepsilon)|y|}\left(1+|\tau|\right)^{-\beta}d\tau\\&\leq\begin{cases}C\left(1+|y|\right)^{-\beta}&\text{if }0\leq\beta<1,\\C\left(1+|y|\right)^{-1}\log\left(2+|y|\right)&\text{if }\beta=1,\\C\left(1+|y|\right)^{-1}&\text{if }\beta>1.\end{cases}\end{aligned}$$

□

By Lemma 3.4, the normalized discrepancy $r^{-(d-1)/2}\mathcal{D}\left(r\Omega-x\right)$ is a sum of terms $\Phi_j\left(\left(d+1\right)/2,r,x\right)$. The following lemma studies the two terms $\Theta_j\left(z,r,x\right)$ that appear in the definition of $\Phi_j\left(z,r,x\right)$.

Lemma 3.7. *Let $d\mu\left(r\right)$ be a Borel probability measure on $\mathbb{R}$ with support in $0<\varepsilon<r<\delta<+\infty$, and with $\left|\widehat{\mu}\left(\zeta\right)\right|\leq B\left(1+|\zeta|\right)^{-\beta}$. Let $c\left(\xi\right)$ be a bounded homogeneous function of degree 0, let $\left(d+1\right)/2\leq\operatorname{Re}\left(z\right)<d$, and for $j=0,1,2,...$ and any choice of $\pm$, define*

$$\Theta_j\left(z,r,x\right)=r^{-j}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}c\left(n\right)|n|^{-z-j}\exp\left(\pm2\pi ig\left(\mp n\right)r\right)\exp\left(2\pi inx\right).$$

Moreover, for $H,R\geq1$, define

$$\mathcal{F}_j\left(z,H,R,x\right)=\int_{\mathbb{R}}\left|\Theta_j\left(z,r,x\right)\right|^2d\mu_{H,R}\left(r\right).$$

Expand this last function into a Fourier series in the variable x,

$$\mathcal{F}_j\left(z,H,R,x\right)=\sum_{k\in\mathbb{Z}^d}\widehat{\mathcal{F}}_j\left(z,H,R,k\right)\exp\left(2\pi ikx\right).$$

(1) If $j=0$ there exists a constant C, which may depend on d, B, β and $\sup_{n\in\mathbb{Z}^d}\{|c(n)|\}$, but it is independent of the complex parameter z and of the real parameters H and R, such that for every $H,R\geq1$ and $k\in\mathbb{Z}^d$ the Fourier coefficients of $\mathcal{F}_0\left(z,H,R,x\right)$ satisfy the estimates

$$\left|\widehat{\mathcal{F}}_0\left(z,H,R,k\right)\right|\leq\begin{cases}C\left(1+|k|\right)^{d-\beta-2\operatorname{Re}(z)}&\text{if }0\leq\beta<1,\\C\left(1+|k|\right)^{d-1-2\operatorname{Re}(z)}\log\left(2+|k|\right)&\text{if }\beta=1,\\C\left(1+|k|\right)^{d-1-2\operatorname{Re}(z)}&\text{if }\beta>1.\end{cases}$$

(2) If $j\geq1$ then there exists a constant C such that for every $H,R\geq1$ and $k\in\mathbb{Z}^d$ the Fourier coefficients of $\mathcal{F}_j\left(z,H,R,x\right)$ satisfy the estimates

$$\left|\widehat{\mathcal{F}}_j\left(z,H,R,k\right)\right|\leq C\left(R+H\right)^{-2j}\left(1+|k|\right)^{d-1-2\operatorname{Re}(z)}.$$

Proof. Let us fix a choice of $\pm$,

$$\Theta_j(z,r,x) = r^{-j} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} c(n)\,|n|^{-z-j} \exp(-2\pi i g(n)r)\exp(2\pi i n x).$$

Expanding the product $\Theta_j(z,r,x)\cdot\overline{\Theta_j(z,r,x)}$ and integrating against $d\mu_{H,R}(r)$, one obtains

$$\mathcal{F}_j(z,H,R,x) = \int_{\mathbb{R}} |\Theta_j(z,r,x)|^2\, d\mu_{H,R}(r)$$

$$= \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d\setminus\{0,k\}} c(n)\,\overline{c(n-k)}\,|n|^{-z-j}\,|n-k|^{-\overline{z}-j}\exp(2\pi i k x)$$

$$\times \exp(2\pi i(g(n-k)-g(n))R)\int_{\mathbb{R}}(R+Hr)^{-2j}\exp(2\pi i H(g(n-k)-g(n))r)\,d\mu(r).$$

The product term by term of the series and the integration term by term can be justified with a suitable summation method, which amounts to introduce a cutoff in the the series that defines $\Theta_j(z,r,x)$. See Remark 3.2. In particular, the Fourier coefficients of $\mathcal{F}_j(z,H,R,x)$ are

$$\widehat{\mathcal{F}}_j(z,H,R,k) = \sum_{n\in\mathbb{Z}^d\setminus\{0,k\}} c(n)\,\overline{c(n-k)}\,|n|^{-z-j}\,|n-k|^{-\overline{z}-j}$$

$$\times \exp(2\pi i(g(n-k)-g(n))R)\int_{\mathbb{R}}(R+Hr)^{-2j}\exp(2\pi i H(g(n-k)-g(n))r)\,d\mu(r).$$

When $j=0$, by the assumption on the Fourier transform of the measure $d\mu(r)$,

$$\left|\widehat{\mathcal{F}}_0(z,H,R,k)\right|$$

$$\leq B\sup_{n\in\mathbb{Z}^d}\left\{|c(n)|^2\right\}\sum_{n\in\mathbb{Z}^d\setminus\{0,k\}}|n|^{-\operatorname{Re}(z)}\,|n-k|^{-\operatorname{Re}(z)}\,(1+|g(n-k)-g(n)|)^{-\beta}$$

$$\leq C\int_{\mathbb{R}^d}|x|^{-\operatorname{Re}(z)}\,|x-k|^{-\operatorname{Re}(z)}\,(1+|g(x-k)-g(x)|)^{-\beta}\,dx.$$

It then suffices to apply Lemma 3.6. The substitution of the series with an integral can be justified observing that all functions involved are slowly varying when one replaces n with an x such that $|n-x|\leq C$.

When $j\geq 1$,

$$\left|\widehat{\mathcal{F}}_j(z,H,R,k)\right|$$

$$\leq \sup_{n\in\mathbb{Z}^d}\left\{|c(n)|^2\right\}\sum_{n\in\mathbb{Z}^d\setminus\{0,k\}}|n|^{-\operatorname{Re}(z)-j}\,|n-k|^{-\operatorname{Re}(z)-j}\int_{\mathbb{R}}(R+Hr)^{-2j}\,d\mu(r)$$

$$\leq \begin{cases} C(R+H)^{-2j}(1+|k|)^{d-2j-2\operatorname{Re}(z)} & \text{if } \operatorname{Re}(z)+j<d,\\ C(R+H)^{-2j}(1+|k|)^{-d}\log(2+|k|) & \text{if } \operatorname{Re}(z)+j=d,\\ C(R+H)^{-2j}(1+|k|)^{-\operatorname{Re}(z)-j} & \text{if } \operatorname{Re}(z)+j>d.\end{cases}$$

Finally observe that when $\alpha<d$ and $j\geq 1$, then all these estimates are dominated by $C(R+H)^{-2j}(1+|k|)^{d-1-2\operatorname{Re}(z)}$.

□

Lemma 3.8. *Let $0\leq\beta<1$ and let $\mathcal{F}_0(z,H,R,x)$ be as in Lemma 3.7. Then there exists C such that for every $H,R\geq 1$ the following hold.*

(1) *If* $d/2 < \operatorname{Re}(z) \leq (3d-2\beta)/4$ *and* $2 \leq p < (d-2\beta)/(d-\beta-\operatorname{Re}(z))$, *then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C\left(\frac{d-2\beta}{d-\beta-\operatorname{Re}(z)}-p\right)^{-(d-\beta-\operatorname{Re}(z))/(d-2\beta)}.$$

(2) *If* $(3d-2\beta)/4 < \operatorname{Re}(z) < d-\beta/2$ *and* $4 \leq p < d/(d-\beta/2-\operatorname{Re}(z))$, *then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C\left(\frac{d}{d-\beta/2-\operatorname{Re}(z)}-p\right)^{(d-\beta/2-\operatorname{Re}(z))/d-1/2}.$$

(3) *If* $\operatorname{Re}(z) = d-\beta/2$ *and* $p<+\infty$, *then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq Cp^{1/2}.$$

Proof. In this and in the following proofs, we shall repeatedly use an elementary inequality. If $\delta > d$, then

$$\sum_{k\in\mathbb{Z}^d}(1+|k|)^{-\delta} \leq C(\delta-d)^{-1}.$$

If $p=2$ and $\operatorname{Re}(z) > d/2$ then, by Parseval equality,

$$\begin{aligned}
&\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \\
&= \left\{\int_{\mathbb{T}^d}\int_{\mathbb{R}} |\Theta_0(z,r,x)|^2\,d\mu_{H,R}(r)\,dx\right\}^{1/p} \\
&= \left\{\int_{\mathbb{R}} d\mu(r) \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |c(n)|^2\,|n|^{-2\operatorname{Re}(z)}\right\}^{1/2} \\
&\leq C\left(\operatorname{Re}(z)-\frac{d}{2}\right)^{-1/2}.
\end{aligned}$$

If $4 \leq p \leq +\infty$ and $\operatorname{Re}(z) > d(1-1/p)-\beta/2$ then $p/2>2$ and, by the Hausdorff Young inequality,

$$\begin{aligned}
&\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \\
&\leq \left\{\sum_{k\in\mathbb{Z}^d} \left|\widehat{\mathcal{F}}_0(z,H,R,k)\right|^{p/(p-2)}\right\}^{(p-2)/2p} \\
&\leq C\left\{\sum_{k\in\mathbb{Z}^d} \left|(1+|k|)^{d-\beta-2\operatorname{Re}(z)}\right|^{p/(p-2)}\right\}^{(p-2)/2p} \\
&\leq C\left((2\operatorname{Re}(z)+\beta-d)\frac{p}{p-2}-d\right)^{-(p-2)/2p} \\
&= C\left(\frac{p-2}{2p}\right)^{(p-2)/2p}\left(\operatorname{Re}(z)+\frac{\beta}{2}-d\left(1-\frac{1}{p}\right)\right)^{-(p-2)/2p}
\end{aligned}$$

$$\leq C\left(\mathrm{Re}\,(z)+\frac{\beta}{2}-d\left(1-\frac{1}{p}\right)\right)^{1/p-1/2}.$$

This proves the cases $p=2$ and $p\geq 4$. The case $2<p<4$ follows from these cases via complex interpolation of vector valued $L\,(p)$ spaces. The complex interpolation method is a generalization of the classical Riez-Thorin and Stein theorems of interpolation of linear operators. For the definition of the complex interpolation method, see for example [1, Chapter 4 and Chapter 5]. Here we recall the relevant result [1, Theorem 5.1.2]: Let $\mathbb{H}$ be a Hilbert space and $\mathbb{X}$ a measure space, let $1\leq a<b\leq+\infty$, $-\infty<A<B<+\infty$, and let $\Theta\,(z)$ be a function with values in the vector valued space $L^{a}\,(\mathbb{X},\mathbb{H})+L^{b}\,(\mathbb{X},\mathbb{H})$, continuous and bounded on the closed strip $\{A\leq\mathrm{Re}\,(z)\leq B\}$ and analytic on the open strip $\{A<\mathrm{Re}\,(z)<B\}$. Assume that there exist constants M and N such that for every $-\infty<t<+\infty$,

$$\begin{cases}\|\Theta\,(A+it)\|_{L^{a}(\mathbb{X},\mathbb{H})}\leq M,\\ \|\Theta\,(B+it)\|_{L^{b}(\mathbb{X},\mathbb{H})}\leq N.\end{cases}$$

If $1/p=(1-\vartheta)\,/a+\vartheta/b$, with $0<\vartheta<1$, then

$$\|\Theta\,((1-\vartheta)\,A+\vartheta B)\|_{L^{p}(\mathbb{X},\mathbb{H})}\leq M^{1-\vartheta}N^{\vartheta}.$$

Here the analytic function is $\Theta_0\,(z,r,x)$, the Hilbert space $\mathbb{H}$ is $L^{2}\,(\mathbb{R},d\mu_{H,R}\,(r))$, the measure space $\mathbb{X}$ is the torus $\mathbb{T}^{d}$, $a=2$, $A=d/2+\varepsilon$, $b=4$, $B=3d/4-\beta/2+\varepsilon$, with $\varepsilon>0$, $M=C\varepsilon^{-1/2}$ and $N=C\varepsilon^{-1/4}$. By the above computations,

$$\left\{\int_{\mathbb{T}^{d}}|\mathcal{F}_0\,(z,H,R,x)|^{p/2}\,dx\right\}^{1/p}$$
$$\leq\begin{cases}C\varepsilon^{-1/2} & \text{if } p=2 \text{ and } \mathrm{Re}\,(z)=d/2+\varepsilon,\\ C\varepsilon^{-1/4} & \text{if } p=4 \text{ and } \mathrm{Re}\,(z)=3d/4-\beta/2+\varepsilon.\end{cases}$$

By complex interpolation with

$$\begin{cases}1/p=(1-\vartheta)\,/2+\vartheta/4,\\ \mathrm{Re}\,(z)=(1-\vartheta)\,(d/2+\varepsilon)+\vartheta\,(3d/4-\beta/2+\varepsilon)\,,\end{cases}$$

that is, $2<p<4$ and $\mathrm{Re}\,(z)=d\,(1-1/p)+2\beta/p-\beta+\varepsilon$, one obtains

$$\left\{\int_{\mathbb{T}^{d}}|\mathcal{F}_0\,(z,H,R,x)|^{p/2}\,dx\right\}^{1/p}\leq C\left(\mathrm{Re}\,(z)+\beta-\frac{2\beta}{p}-d\left(1-\frac{1}{p}\right)\right)^{-1/p}.$$

The estimates that we have obtained blow up when $\mathrm{Re}\,(z)\to\mathrm{critical}\,(z)+$,

$$\mathrm{Re}\,(z)\to\begin{cases}d\left(1-\dfrac{1}{p}\right)+\dfrac{2\beta}{p}-\beta+ & \text{if } p\leq 4,\\ d\left(1-\dfrac{1}{p}\right)-\dfrac{\beta}{2}+ & \text{if } p\geq 4.\end{cases}$$

This is the same as $p\to\mathrm{critical}\,(p)-$,

$$p\to\begin{cases}\dfrac{d-2\beta}{d-\beta-\mathrm{Re}\,(z)}- & \text{if } d/2<\mathrm{Re}(z)\leq(3d-2\beta)/4,\\ \dfrac{d}{d-\beta/2-\mathrm{Re}\,(z)}- & \text{if } (3d-2\beta)/4\leq\mathrm{Re}(z)\leq d-\beta/2.\end{cases}$$

In order to complete the proof of the lemma, it suffices to translate the estimates for $\mathrm{Re}(z)\to\mathrm{critical}\,(z)+$ into estimates for $p\to\mathrm{critical}\,(p)-$.

If $d/2 < \operatorname{Re}(z) \leq (3d-2\beta)/4$ and $2 \leq p < (d-2\beta)/(d-\beta-\operatorname{Re}(z))$, then $2 \leq p \leq 4$ and $\operatorname{Re}(z) > d(1-1/p) + 2\beta/p - \beta$, and one has

$$\begin{aligned}
&\left(\operatorname{Re}(z) + \beta - \frac{2\beta}{p} - d\left(1-\frac{1}{p}\right)\right)^{-1/p} \\
&= p^{1/p}(d-\beta-\operatorname{Re}(z))^{-1/p}\left(\frac{d-2\beta}{d-\beta-\operatorname{Re}(z)} - p\right)^{(d-\beta-\operatorname{Re}(z))/(d-2\beta)-1/p} \\
&\times \left(\frac{d-2\beta}{d-\beta-\operatorname{Re}(z)} - p\right)^{-(d-\beta-\operatorname{Re}(z))/(d-2\beta)} \\
&\leq C\left(\frac{d-2\beta}{d-\beta-\operatorname{Re}(z)} - p\right)^{-(d-\beta-\operatorname{Re}(z))/(d-2\beta)}.
\end{aligned}$$

We have used the inequalities $x^{1/x} \leq e^{1/e}$ for every $x > 0$, and $(x-y)^{1/x-1/y} = \left((x-y)^{-(x-y)}\right)^{1/xy} \leq \left(e^{1/e}\right)^{1/xy} \leq e^{1/e}$ for every $x > y \geq 1$. Observe that the above constant C may depend on d, β, $\operatorname{Re}(z)$, but it is independent of p.

If $(3d-2\beta)/4 < \operatorname{Re}(z) < d-\beta/2$ and $4 \leq p < d/(d-\beta/2-\operatorname{Re}(z))$, then $\operatorname{Re}(z) > d(1-1/p) - \beta/2$, and one has

$$\begin{aligned}
&\left(\operatorname{Re}(z) + \frac{\beta}{2} - d\left(1-\frac{1}{p}\right)\right)^{1/p-1/2} \\
&= p^{1/2-1/p}(d-(d-\beta/2-\operatorname{Re}(z))p)^{1/p-1/2} \\
&= p^{1/2-1/p}(d-\beta/2-\operatorname{Re}(z))^{1/p-1/2}\left(\frac{d}{d-\beta/2-\operatorname{Re}(z)} - p\right)^{1/p-(d-\beta/2-\operatorname{Re}(z))/d} \\
&\times \left(\frac{d}{d-\beta/2-\operatorname{Re}(z)} - p\right)^{(d-\beta/2-\operatorname{Re}(z))/d-1/2} \\
&\leq C\left(\frac{d}{d-\beta/2-\operatorname{Re}(z)} - p\right)^{(d-\beta/2-\operatorname{Re}(z))/d-1/2}.
\end{aligned}$$

If $\operatorname{Re}(z) = d-\beta/2$ and $p < +\infty$, then one has

$$\left(\operatorname{Re}(z) + \frac{\beta}{2} - d\left(1-\frac{1}{p}\right)\right)^{1/p-1/2} = d^{1/p-1/2}p^{-1/p}p^{1/2} \leq Cp^{1/2}.$$

□

Lemma 3.9. *Let $\beta = 1$ and let $\mathcal{F}_0(z,H,R,x)$ be as in Lemma 3.7. Then there exists C such that for every $H,R \geq 1$ the following hold.*

(1) *If $d/2 < \operatorname{Re}(z) \leq (3d-2)/4$ and $2 \leq p < (d-2)/(d-1-\operatorname{Re}(z))$, then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C\left(\frac{d-2}{d-1-\operatorname{Re}(z)} - p\right)^{-(\operatorname{Re}(z)-1)/(d-2)}.$$

(2) *If $(3d-2)/4 < \operatorname{Re}(z) < d-1/2$ and $4 \leq p < d/(d-\operatorname{Re}(z)-1/2)$, then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C\left(\frac{d}{d-1/2-\operatorname{Re}(z)} - p\right)^{-(2\operatorname{Re}(z)+1)/2d}.$$

(3) *If* $\operatorname{Re}(z) = d - 1/2$ *and* $p < +\infty$, *then*

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p} \leq Cp.$$

Proof. The proof is, *mutatis mutandis*, as in the previous lemma, only observe the different exponents in the right hand side of the estimates. If $p = 2$ and $\operatorname{Re}(z) > d/2$, then

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p} \leq C \left(\operatorname{Re}(z) - \frac{d}{2} \right)^{-1/2}.$$

If $4 \leq p \leq +\infty$ and $\operatorname{Re}(z) > d(1 - 1/p) - 1/2$ then, by the Hausdorff Young inequality,

$$\begin{aligned} &\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p} \\ &\leq \left\{ \sum_{k \in \mathbb{Z}^d} \left| \widehat{\mathcal{F}}_0(z, H, R, k) \right|^{p/(p-2)} \right\}^{(p-2)/2p} \\ &\leq C \left\{ \sum_{k \in \mathbb{Z}^d} \left| (1 + |k|)^{d-1-2\operatorname{Re}(z)} \log(2 + |k|) \right|^{p/(p-2)} \right\}^{(p-2)/2p}. \end{aligned}$$

The series can be compared with the integral

$$\begin{aligned} &\left\{ \int_{\mathbb{R}^d} \left| (1 + |x|)^{d-1-2\operatorname{Re}(z)} \log(2 + |x|) \right|^{p/(p-2)} dx \right\}^{(p-2)/2p} \\ &= \left\{ |\{|\vartheta| = 1\}| \int_0^{+\infty} (1 + \rho)^{(d-1-2\operatorname{Re}(z))p/(p-2)} \log^{p/(p-2)}(2 + \rho) \, \rho^{d-1} d\rho \right\}^{(p-2)/2p}. \end{aligned}$$

The last integral can be compared to another integral,

$$\int_1^{+\infty} t^{-\alpha} \log^{\beta}(t) \, dt = (\alpha - 1)^{-(\beta+1)} \int_0^{+\infty} s^{\beta} e^{-s} ds = (\alpha - 1)^{-(\beta+1)} \Gamma(\beta + 1).$$

Hence,

$$\begin{aligned} &\left\{ \int_0^{+\infty} (1 + \rho)^{(d-1-2\operatorname{Re}(z))p/(p-2)+d-1} \log^{p/(p-2)}(2 + \rho) \, d\rho \right\}^{(p-2)/2p} \\ &\leq C \left((2\operatorname{Re}(z) + 1 - d) \frac{p}{p-2} - d \right)^{-(1+p/(p-2))(p-2)/2p} \\ &\leq C \left(\operatorname{Re}(z) - d\left(1 - \frac{1}{p}\right) + \frac{1}{2} \right)^{1/p-1}. \end{aligned}$$

This proves the cases $p = 2$ and $p \geq 4$. The case $2 < p < 4$ follows from these cases via complex interpolation. By the above computations,

$$\begin{aligned} &\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p} \\ &\leq \begin{cases} C\varepsilon^{-1/2} & \text{if } p = 2 \text{ and } \operatorname{Re}(z) = d/2 + \varepsilon, \\ C\varepsilon^{-3/4} & \text{if } p = 4 \text{ and } \operatorname{Re}(z) = (3d - 2)/4 + \varepsilon. \end{cases} \end{aligned}$$

By complex interpolation, if $2 < p < 4$ and $\mathrm{Re}\,(z) = d\,(1-1/p) + 2/p - 1 + \varepsilon$,

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0\,(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C\left(\mathrm{Re}\,(z) - \left(d\left(1-\frac{1}{p}\right) + \frac{2}{p} - 1\right)\right)^{1/p-1}.$$

The estimates that we have obtained blow up when $\mathrm{Re}\,(z) \to \mathrm{critical}\,(z)+$,

$$\mathrm{Re}\,(z) \to \begin{cases} d\left(1-\dfrac{1}{p}\right) + \dfrac{2}{p} - 1+ & \text{if } p \leq 4, \\ d\left(1-\dfrac{1}{p}\right) - \dfrac{1}{2}+ & \text{if } p \geq 4. \end{cases}$$

This is the same as $p \to \mathrm{critical}\,(p)-$,

$$p \to \begin{cases} \dfrac{d-2}{d-1-\mathrm{Re}\,(z)}- & \text{if } d/2 < \mathrm{Re}(z) \leq (3d-2)/4, \\ \dfrac{d}{d-1/2-\mathrm{Re}\,(z)}- & \text{if } (3d-2)/4 \leq \mathrm{Re}(z) \leq d-1/2. \end{cases}$$

In order to complete the proof it suffices to translate, as in the previous lemma, the estimates for $\mathrm{Re}(z) \to \mathrm{critical}\,(z)+$ into estimates for $p \to \mathrm{critical}\,(p)-$. □

Lemma 3.10. *Let $\beta > 1$ and let $\mathcal{F}_0\,(z,H,R,x)$ be as in the Lemma 3.7. Then there exists C such that for every $H, R \geq 1$ the following hold.*

(1) *If $d/2 < \mathrm{Re}\,(z) \leq (3d-2)\,/4$ and $2 \leq p < (d-2)(d-1-\mathrm{Re}\,(z))$, then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0\,(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C\left(\frac{d-2}{d-1-\mathrm{Re}\,(z)} - p\right)^{-(d-1-\mathrm{Re}(z))/(d-2)}.$$

(2) *If $(3d-2)\,/4 < \mathrm{Re}\,(z) < d - 1/2$ and $4 \leq p < d/\,(d-\mathrm{Re}\,(z) - 1/2)$, then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0\,(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C\left(\frac{d}{d-1/2-\mathrm{Re}\,(z)} - p\right)^{(d-1/2-\mathrm{Re}(z))/d-1/2}.$$

(3) *If $\mathrm{Re}\,(z) = d - 1/2$ and $p < +\infty$, then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0\,(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq Cp^{1/2}.$$

Proof. The proof is as in the previous two lemmas. If $p = 2$ and $\mathrm{Re}\,(z) > d/2$ then, by Parseval equality,

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0\,(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C\left(\mathrm{Re}\,(z) - \frac{d}{2}\right)^{-1/2}.$$

If $4 \leq p \leq +\infty$ and $\mathrm{Re}\,(z) > d\,(1-1/p) - 1/2$ then, by the Hausdorff Young inequality,

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0\,(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C\left(\mathrm{Re}\,(z) - d\left(1-\frac{1}{p}\right) + \frac{1}{2}\right)^{1/p-1/2}.$$

The cases $2 < p < 4$ follow from these cases via complex interpolation. By the above computations,

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0\,(z,H,R,x)|^{p/2}\,dx\right\}^{1/p}$$
$$\leq \begin{cases} C\varepsilon^{-1/2} & \text{if } p = 2 \text{ and } \mathrm{Re}\,(z) = d/2 + \varepsilon, \\ C\varepsilon^{-1/4} & \text{if } p = 4 \text{ and } \mathrm{Re}\,(z) = (3d-2)\,/4 + \varepsilon. \end{cases}$$

By complex interpolation, with $2<p<4$ and $\operatorname{Re}(z)=d(1-1/p)+2/p-1+\varepsilon$,

$$\left\{\int_{\mathbb{T}^d}\left|\mathcal{F}_0(z,H,R,x)\right|^{p/2}dx\right\}^{1/p}\leq C\left(\operatorname{Re}(z)-\left(d\left(1-\frac{1}{p}\right)+\frac{2}{p}-1\right)\right)^{-1/p}.$$

As before, one can translate the estimates for $\operatorname{Re}(z)\to\operatorname{critical}(z)+$ into estimates for $p\to\operatorname{critical}(p)-$. □

Lemma 3.11. *Let $\beta\geq 0$ and $j\geq 1$, and let $\mathcal{F}_j(z,H,R,x)$ be as in the Lemma 3.7. Then there exists C such that for every $H,R\geq 1$ the following hold.*

(1) *If $d/2<\operatorname{Re}(z)\leq(3d-2)/4$ and $2\leq p<(d-2)(d-1-\operatorname{Re}(z))$, then*

$$\left\{\int_{\mathbb{T}^d}\left|\mathcal{F}_j(z,H,R,x)\right|^{p/2}dx\right\}^{1/p}\leq C(H+R)^{-j}\left(\frac{d-2}{d-1-\operatorname{Re}(z)}-p\right)^{-(d-1-\operatorname{Re}(z))/(d-2)}.$$

(2) *If $(3d-2)/4<\operatorname{Re}(z)<d-1/2$ and $4\leq p<d/(d-\operatorname{Re}(z)-1/2)$, then*

$$\left\{\int_{\mathbb{T}^d}\left|\mathcal{F}_j(z,H,R,x)\right|^{p/2}dx\right\}^{1/p}\leq C(H+R)^{-j}\left(\frac{d}{d-1/2-\operatorname{Re}(z)}-p\right)^{(d-1/2-\operatorname{Re}(z))/d-1/2}.$$

(3) *If $\operatorname{Re}(z)=d-1/2$ and $p<+\infty$, then*

$$\left\{\int_{\mathbb{T}^d}\left|\mathcal{F}_j(z,H,R,x)\right|^{p/2}dx\right\}^{1/p}\leq C(H+R)^{-j}p^{1/2}.$$

Proof. The proof is as in the previous lemmas. If $p=2$ and $\operatorname{Re}(z)>d/2$ then, by Parseval equality,

$$\begin{aligned}&\left\{\int_{\mathbb{T}^d}\left|\mathcal{F}_j(z,H,R,x)\right|^{p/2}dx\right\}^{1/p}\\&=\left\{\int_{\mathbb{R}}(R+Hr)^{-2j}\,d\mu(r)\sum_{n\in\mathbb{Z}^d\setminus\{0\}}|c(n)|^2|n|^{-2\operatorname{Re}(z)-2j}\right\}^{1/2}\\&\leq C(H+R)^{-j}\left(\operatorname{Re}(z)+j-\frac{d}{2}\right)^{-1/2}\leq C(H+R)^{-j}\left(\operatorname{Re}(z)-\frac{d}{2}\right)^{-1/2}.\end{aligned}$$

If $4\leq p\leq+\infty$ and $\operatorname{Re}(z)>d(1-1/p)-1/2$ then, by the Hausdorff Young inequality,

$$\begin{aligned}&\left\{\int_{\mathbb{T}^d}\left|\mathcal{F}_j(z,H,R,x)\right|^{p/2}dx\right\}^{1/p}\\&\leq\left\{\sum_{k\in\mathbb{Z}^d}\left|\widehat{\mathcal{F}}_j(z,H,R,k)\right|^{p/(p-2)}\right\}^{(p-2)/2p}\\&\leq C(H+R)^{-j}\left\{\sum_{k\in\mathbb{Z}^d}\left|(1+|k|)^{d-1-2\operatorname{Re}(z)}\right|^{p/(p-2)}\right\}^{(p-2)/2p}\\&\leq C(H+R)^{-j}\left(\operatorname{Re}(z)-d\left(1-\frac{1}{p}\right)+\frac{1}{2}\right)^{1/p-1/2}.\end{aligned}$$

By complex interpolation, if $2 < p < 4$ and $\mathrm{Re}\,(z) = d\,(1 - 1/p) + 2/p - 1 + \varepsilon$, then

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_j\,(z, H, R, x)|^{p/2}\,dx\right\}^{1/p} \le C\,(H+R)^{-j}\left(\mathrm{Re}\,(z) - \left(d\left(1 - \frac{1}{p}\right) + \frac{2}{p} - 1\right)\right)^{1/p-1}.$$

As before, one can translate the estimates for $\mathrm{Re}(z) \to \mathrm{critical}\,(z)+$ into estimates for $p \to \mathrm{critical}\,(p)\,-$. □

The above lemmas are enough for an upper bound for the norms of the discrepancy in Theorem 1.1(1), for p below the critical index A.

Proof of Theorem 1.1(1). By Lemma 3.4,

$$\begin{aligned}
&\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}\,(r\Omega - x)\right|^2 d\mu_{H,R}\,(r)\right]^{p/2} dx\right\}^{1/p} \\
&\le \sum_{j=0}^{h}\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}|\Phi_j\,((d+1)\,/2, r, x)|^2\,d\mu_{H,R}\,(r)\right]^{p/2} dx\right\}^{1/p} \\
&+\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}|\mathcal{R}_h\,(r,x)|^2\,d\mu_{H,R}\,(r)\right]^{p/2} dx\right\}^{1/p}.
\end{aligned}$$

Since the $\Phi_j\,((d+1)\,/2, r, x)$'s are sums of two $\Theta_j\,((d+1)\,/2, r, x)$'s to which the above lemmas apply, under appropriate relations between p and β the mixed norm of the discrepancy is uniformly bounded, and (1) follows from the estimates in Lemma 3.8, 3.9, 3.10, 3.11. □

In order to reach the critical index $p = A$ in Theorem 1.1(2), one needs an easy lemma suggested by the Yano extrapolation theorem. See [41] or [42, Chapter XII-4.41].

Lemma 3.12. *For every function $\mathcal{F}(x)$ defined on the torus $\mathbb{T}^d$ the following hold:*

(1) *Let $\alpha \ge 0$, $A \ge 1$, $K \ge 2$, and assume that*

$$\sup_{x\in\mathbb{T}^d}\{|\mathcal{F}\,(x)|\} \le K,$$

$$\left\{\int_{\mathbb{T}^d}|\mathcal{F}\,(x)|^p\,dx\right\}^{1/p} \le (A-p)^{-\alpha} \quad \text{for every } 0 < p < A.$$

Then there exists C independent of K and of $\mathcal{F}\,(x)$ such that

$$\left\{\int_{\mathbb{T}^d}|\mathcal{F}\,(x)|^A\,dx\right\}^{1/A} \le C\log^{\alpha}(K)\,.$$

(2) *Assume that $\alpha > 0$ and that for every $p < A < +\infty$,*

$$\left\{\int_{\mathbb{T}^d}|\mathcal{F}\,(x)|^p\,dx\right\}^{1/p} \le (A-p)^{-\alpha}\,.$$

Then for every $\gamma > 1 + \alpha A$ there exists C independent of $\mathcal{F}\,(x)$ such that

$$\int_{\mathbb{T}^d}|\mathcal{F}\,(x)|^A \log^{-\gamma}(2 + |\mathcal{F}\,(x)|)\,dx \le C.$$

(3) *Assume that* $\alpha > 0$ *and that for every* $p < +\infty$,

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}(x)|^p \, dx \right\}^{1/p} \leq p^{\alpha}.$$

Then for every $\gamma < \alpha/e$ *there exists* $C > 0$ *independent of* $\mathcal{F}(x)$ *such that*

$$\int_{\mathbb{T}^d} \exp\left(\gamma |\mathcal{F}(x)|^{1/\alpha} \right) dx \leq C.$$

Proof. (1) If $\alpha \geq 0$ and $A \geq 1$ and $0 < p < A$,

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}(x)|^A \, dx \right\}^{1/A} \leq \sup_{x \in \mathbb{T}^d} \left\{ |\mathcal{F}(x)|^{(A-p)/A} \right\} \left\{ \int_{\mathbb{T}^d} |\mathcal{F}(x)|^p \, dx \right\}^{1/A}$$
$$\leq K^{(A-p)/A} (A-p)^{-\alpha p/A} = A^{-\alpha p/A} (1 - p/A)^{\alpha(1-p/A)} K^{1-p/A} (1 - p/A)^{-\alpha}$$
$$\leq K^{1-p/A} (1 - p/A)^{-\alpha}.$$

Then, with $1 - p/A = t$,

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}(x)|^A \, dx \right\}^{1/A} \leq \inf_{0<t<1} \left\{ K^t t^{-\alpha} \right\} = e^{\alpha} \alpha^{-\alpha} \log^{\alpha}(K).$$

(2) Let $\mathcal{F}_0(x) = \mathcal{F}(x) \chi_{\{|\mathcal{F}(x)|<2\}}(x)$ and $\mathcal{F}_j(x) = \mathcal{F}(x) \chi_{\{2^j \leq |\mathcal{F}(x)| < 2^{j+1}\}}(x)$ if $j \geq 1$, and let ε_j be the measure of the set where $\mathcal{F}_j(x) \neq 0$. Then, if $j \geq 1$ and $p < A$,

$$2^{jp} \varepsilon_j \leq \int_{\mathbb{T}^d} |\mathcal{F}_j(x)|^p \, dx \leq \int_{\mathbb{T}^d} |\mathcal{F}(x)|^p \, dx \leq (A-p)^{-\alpha p}.$$

Hence, $\varepsilon_j \leq 2^{-jp} (A-p)^{-\alpha p} = 2^{-Aj} 2^{j(A-p)} (A-p)^{-\alpha A}$, and the minimum of this expression is attained at $p = A - \alpha A / j \log(2)$. This gives

$$\varepsilon_j \leq C 2^{-Aj} j^{\alpha A}.$$

Hence, if $\gamma > 1 + \alpha A$,

$$\int_{\mathbb{T}^d} |\mathcal{F}(x)|^A \log^{-\gamma}(2 + |\mathcal{F}(x)|) \, dx$$
$$= \sum_{j=0}^{+\infty} \int_{\mathbb{T}^d} |\mathcal{F}_j(x)|^A \log^{-\gamma}(2 + |\mathcal{F}_j(x)|) \, dx$$
$$\leq 2^A \log^{-\gamma}(2) + \sum_{j=1}^{+\infty} 2^{(j+1)A} \log^{-\gamma}\left(2 + 2^j\right) \varepsilon_j$$
$$\leq C + C \sum_{j=1}^{+\infty} j^{\alpha A - \gamma} \leq C.$$

(3) Let $\mathcal{F}_j(x) = \mathcal{F}(x) \chi_{\{j \leq |\mathcal{F}(x)| < j+1\}}(x)$, and let ε_j be the measure of the set where $\mathcal{F}_j(x) \neq 0$. Then, if $j \geq 1$ and $p < A$,

$$j^p \varepsilon_j \leq \int_{\mathbb{T}^d} |\mathcal{F}_j(x)|^p \, dx \leq \int_{\mathbb{T}^d} |\mathcal{F}(x)|^p \, dx \leq p^{\alpha p}.$$

Hence $\varepsilon_j \leq j^{-p} p^{\alpha p}$. The minimum of this expression is attained at $p = e^{-1} j^{1/\alpha}$, and this gives

$$\varepsilon_j \leq \exp\left(-(\alpha/e) j^{1/\alpha} \right).$$

Hence, if $\gamma < \alpha/e$,

$$\int_{\mathbb{T}^d} \exp\left(\gamma\left|\mathcal{F}(x)\right|^{1/\alpha}\right) dx = \sum_{j=0}^{+\infty} \int_{\mathbb{T}^d} \exp\left(\gamma\left|\mathcal{F}_j(x)\right|^{1/\alpha}\right) dx$$

$$\leq \sum_{j=0}^{+\infty} \varepsilon_j \exp\left(\gamma (j+1)^{1/\alpha}\right) \leq e^{\gamma} + \sum_{j=1}^{+\infty} \exp\left(-\left(\alpha/e - \gamma(1+1/j)^{1/\alpha}\right) j^{1/\alpha}\right) \leq C.$$

□

Proof of Theorem 1.1(2). This follows from part (1) of the theorem via the extrapolation Lemma 3.12. In the cases $p < +\infty$ one has just to recall that the discrepancy satisfy the trivial bound $|\mathcal{D}(r\Omega - x)| \leq Cr^d$ for every $r \geq 1$. The case $d = 2$ and $p = +\infty$ and $d\mu(x) = \chi_{\{0<r<1\}}(r)$ is proved in [19]. An alternative proof of all cases can also be obtained via the mollified discrepancy defined in Remark 3.2. For example, when $d = 2$, with the techniques in the above lemmas, one can prove that if $1 \leq H \leq R$, and $\delta \leq 1/R$,

$$\sup_{x\in\mathbb{T}^2} \left\{ \int_{\mathbb{R}} \left| (r \pm \delta)^{-3/2} \sum_{n\in\mathbb{Z}^2-\{0\}} \widehat{\varphi}(\delta n)\, \widehat{\chi}_{\Omega}((r\pm\delta) n) \exp(2\pi i n x) \right|^2 d\mu_{H,R}(r) \right\}^{1/2}$$

$$\leq \begin{cases} C\left\{ \sum_{n\in\mathbb{Z}^2} (1+|\delta n|)^{-\gamma} (1+|k|)^{-\beta-1} \right\}^{1/2} & \text{if } 0 \leq \beta < 1, \\ C\left\{ \sum_{n\in\mathbb{Z}^2} (1+|\delta n|)^{-\gamma} (1+|k|)^{-2} \log(2+|k|) \right\}^{1/2} & \text{if } \beta = 1, \\ C\left\{ \sum_{n\in\mathbb{Z}^2} (1+|\delta n|)^{-\gamma} (1+|k|)^{-2} \right\}^{1/2} & \text{if } \beta > 1, \end{cases}$$

$$\leq \begin{cases} C\delta^{(1-\beta)/2} & \text{if } 0 \leq \beta < 1, \\ C\log(1/\delta) & \text{if } \beta = 1, \\ C\log^{1/2}(1/\delta) & \text{if } \beta > 1, \end{cases}$$

$$\leq \begin{cases} CR^{(1-\beta)/2} & \text{if } 0 \leq \beta < 1, \\ C\log(R) & \text{if } \beta = 1, \\ C\log^{1/2}(R) & \text{if } \beta > 1. \end{cases}$$

□

In order to prove the asymptotics of the norms as $H \to +\infty$ in Theorem 1.1 (3), one has to work a bit more. It follows from the previous proofs that the main term in the asymptotic expansion of the discrepancy is given by

$$\Phi_0(z,r,x) = \sum_{n\in\mathbb{Z}^d\setminus\{0\}} a_0(n)\,|n|^{-z} \exp(-2\pi i g(n) r) \exp(2\pi i n x)$$
$$+ \sum_{n\in\mathbb{Z}^d\setminus\{0\}} b_0(n)\,|n|^{-z} \exp(2\pi i g(-n) r) \exp(2\pi i n x).$$

The following lemma is similar to the previous ones, just observe that one integrates the square of this function, and not the square of the modulus.

Lemma 3.13. *Define* $\mathcal{G}(z,x)$ *and* $\mathcal{P}(z,H,R,x)$ *by*

$$\mathcal{G}(z,x)=\sum_{k\in\mathbb{Z}^d}\left(2\sum_{\substack{n\in\mathbb{Z}^d\setminus\{0,k\}\\ g(n-k)=g(n)}}a_0(n)\,b_0(k-n)\,|n|^{-z}\,|k-n|^{-z}\right)\exp(2\pi ikx),$$

$$\int_{\mathbb{R}}\Phi_0(z,r,x)^2\,d\mu_{H,R}(r)=\mathcal{G}(z,x)+\mathcal{P}(z,H,R,x).$$

(1) *Under the relations between* p *and* z *in Lemma 3.10, the function* $\mathcal{G}(z,x)$ *is in* $L^{p/2}(\mathbb{T}^d)$,

$$\left\{\int_{\mathbb{T}^d}|\mathcal{G}(z,x)|^{p/2}\,dx\right\}^{1/p}\leq C.$$

(2) *Under the relations between* p *and* z *in Lemma 3.8 if* $0\leq\beta<1$, *or in Lemma 3.9 if* $\beta=1$, *or in Lemma 3.10 if* $\beta>1$, *also the function* $\mathcal{P}(z,H,R,x)$ *is in* $L^{p/2}(\mathbb{T}^d)$, *and there exists* C *such that for every* $H,R\geq1$,

$$\left\{\int_{\mathbb{T}^d}|\mathcal{P}(z,H,R,x)|^{p/2}\,dx\right\}^{1/p}\leq C.$$

Moreover, if $\beta>0$ *then this function vanishes as* $H\to+\infty$, *uniformly in* $R\geq1$,

$$\lim_{H\to+\infty}\left\{\int_{\mathbb{T}^d}|\mathcal{P}(z,H,R,x)|^{p/2}\,dx\right\}^{1/p}=0.$$

Proof. Expanding the product $\Phi_0(z,r,x)\cdot\Phi_0(z,r,x)$ and integrating, one obtains

$$\int_{\mathbb{R}}\Phi_0(z,r,x)^2\,d\mu_{H,R}(r)$$
$$=\mathcal{G}(z,x)+\mathcal{P}_1(z,H,R,x)+\mathcal{P}_2(z,H,R,x)+\mathcal{P}_3(z,H,R,x),$$

where

$$\mathcal{G}(z,x)=2\sum_{k\in\mathbb{Z}^d}\sum_{\substack{n\in\mathbb{Z}^d\setminus\{0,k\}\\ g(n-k)=g(n)}}a_0(n)\,b_0(k-n)\,|n|^{-z}\,|k-n|^{-z}\exp(2\pi ikx),$$

$$\mathcal{P}_1(z,H,R,x)$$
$$=2\sum_{k\in\mathbb{Z}^d}\sum_{\substack{n\in\mathbb{Z}^d\setminus\{0,k\}\\ g(n-k)\neq g(n)}}a_0(n)\,b_0(k-n)\,|n|^{-z}\,|k-n|^{-z}\exp(2\pi ikx)$$
$$\times\exp(2\pi i(g(n-k)-g(n))R)\int_{\mathbb{R}}\exp(2\pi iH(g(n-k)-g(n))r)\,d\mu(r),$$

$$\mathcal{P}_2(z,H,R,x)$$
$$=\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d\setminus\{0,k\}}a_0(n)\,a_0(k-n)\,|n|^{-z}\,|k-n|^{-z}\exp(2\pi ikx)$$
$$\times\exp(-2\pi i(g(n)+g(k-n))R)\int_{\mathbb{R}}\exp(-2\pi iH(g(n)+g(k-n))r)\,d\mu(r),$$

$$\mathcal{P}_3(z,H,R,x)$$

$$= \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d\setminus\{0,k\}} b_0(n)\, b_0(k-n)\, |n|^{-z}\, |k-n|^{-z} \exp(2\pi i k x)$$
$$\times \exp\left(2\pi i \left(g(-n)+g(n-k)\right) R\right) \int_{\mathbb{R}} \exp\left(2\pi i H \left(g(-n)+g(n-k)\right) r\right) d\mu(r).$$

Let us consider the Fourier coefficients of function $\mathcal{G}(z,x)$. First observe that these coefficients do not depend on H and R. Since $a_0(n)$ and $b_0(-n)$ are bounded, the Fourier coefficient with $k=0$ is bounded by

$$\left|\widehat{\mathcal{G}}(z,0)\right| = \left|2 \sum_{n\in\mathbb{Z}^d\setminus\{0\}} a_0(n) b_0(-n)\, |n|^{-2z}\right| \le C \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^{-2\operatorname{Re}(z)} \le C.$$

For an arbitrary $\gamma>1$, the Fourier coefficients with $k\neq 0$ can be bounded by

$$\left|\widehat{\mathcal{G}}(z,k)\right| = \left|2 \sum_{\substack{n\in\mathbb{Z}^d\setminus\{0,k\}\\ g(n-k)=g(n)}} a_0(n)\, b_0(k-n)\, |n|^{-z}\, |k-n|^{-z}\right|$$
$$\le C \sum_{n\in\mathbb{Z}^d-\{0,k\}} |n|^{-\operatorname{Re}(z)}\, |k-n|^{-\operatorname{Re}(z)} \left(1+|g(n-k)-g(n)|\right)^{-\gamma}$$
$$\le C \int_{\mathbb{R}^d} |x|^{-\operatorname{Re}(z)}\, |k-x|^{-\operatorname{Re}(z)} \left(1+|g(x-k)-g(x)|\right)^{-\gamma} dx.$$

Hence, by Lemma 3.6, the last integral is dominated by $C\,|k|^{d-1-2\operatorname{Re}(z)}$. Therefore, for every k,

$$\left|\widehat{\mathcal{G}}(z,k)\right| \le C(1+|k|)^{d-1-2\operatorname{Re}(z)}.$$

The estimates of the Fourier coefficients of $\mathcal{P}_1(z,H,R,x)$ are similar to the ones of $\mathcal{G}(z,x)$. First observe that $\widehat{\mathcal{P}}_1(z,H,R,0)=0$. Then, by the assumption on the measure $d\mu(r)$, if $k\neq 0$ there exists C such that for every $H\ge 1$,

$$\left|\widehat{\mathcal{P}}_1(z,H,R,k)\right|$$
$$= \Bigg|2 \sum_{\substack{n\in\mathbb{Z}^d\setminus\{0,k\}\\ g(n-k)\neq g(n)}} a_0(n)\, b_0(k-n)\, |n|^{-z}\, |k-n|^{-z}$$
$$\times \exp\left(2\pi i \left(g(n-k)-g(n)\right) R\right) \int_{\mathbb{R}} \exp\left(2\pi i H \left(g(n-k)-g(n)\right) r\right) d\mu(r) \Bigg|$$
$$\le C \sum_{n\in\mathbb{Z}^d\setminus\{0,k\}} |n|^{-\operatorname{Re}(z)}\, |k-n|^{-\operatorname{Re}(z)} \left(1+H\,|g(n-k)-g(n)|\right)^{-\beta}$$
$$\le C \int_{\mathbb{R}^d} |x|^{-\operatorname{Re}(z)}\, |k-x|^{-\operatorname{Re}(z)} \left(1+H\,|g(x-k)-g(x)|\right)^{-\beta} dx.$$

Hence, by Lemma 3.6, for every $k\neq 0$,

$$\left|\widehat{\mathcal{P}}_1(z,H,R,k)\right| \le \begin{cases} C\,|k|^{d-2\alpha-\beta} & \text{if } 0\le\beta<1,\\ C\,|k|^{d-2\alpha-1}\log(2+|k|) & \text{if } \beta=1,\\ C\,|k|^{d-2\alpha-1} & \text{if } \beta>1. \end{cases}$$

These estimates are independent of $H, R \geq 1$. Hence, by dominated convergence applied to the sum that defines $\widehat{\mathcal{P}}_1(z, H, R, k)$, if $\beta > 0$ then

$$\lim_{H \to +\infty} \left\{ \widehat{\mathcal{P}}_1(z, H, R, k) \right\} = 0.$$

The estimates of the Fourier coefficients of $\mathcal{P}_2(z, H, R, x)$ and $\mathcal{P}_3(z, H, R, x)$ are easier. Since $g(x) \geq A|x|$ with $A > 0$,

$$\begin{aligned}
&\left| \widehat{\mathcal{P}}_2(z, H, R, k) \right| \\
&= \Big| \sum_{n \in \mathbb{Z}^d \setminus \{0,k\}} a_0(n)\, a_0(k-n)\, |n|^{-z} |k-n|^{-z} \\
&\times \exp(-2\pi i (g(n) + g(k-n)) R) \int_{\mathbb{R}} \exp(-2\pi i H (g(n) + g(k-n)) r)\, d\mu(r) \Big| \\
&\leq C H^{-\beta} \sum_{n \in \mathbb{Z}^d \setminus \{0,k\}} |n|^{-\operatorname{Re}(z)} |k-n|^{-\operatorname{Re}(z)} (|n| + |k-n|)^{-\beta} \\
&\leq C H^{-\beta} (1+|k|)^{-\beta} \sum_{n \in \mathbb{Z}^d \setminus \{0,k\}} |n|^{-\operatorname{Re}(z)} |k-n|^{-\operatorname{Re}(z)} \\
&\leq C H^{-\beta} (1+|k|)^{d-\beta-2\operatorname{Re}(z)}.
\end{aligned}$$

Moreover, by this estimate, if $\beta > 0$ then

$$\lim_{H \to +\infty} \left\{ \widehat{\mathcal{P}}_2(z, H, R, k) \right\} = 0.$$

The estimates of the Fourier coefficients of $\mathcal{P}_3(z, H, R, x)$ are analogous to the ones of $\mathcal{P}_2(z, H, R, x)$. The estimates of the norms in $L^{p/2}(\mathbb{T}^d)$ of these functions in the cases $p = 2$ and $p \geq 4$ follow from the estimates of the Fourier coefficients of the functions involved, the Parseval or Hausdorff Young inequality, and dominated convergence. Finally, the cases $2 < p < 4$ follow by complex interpolation. The details are as in the proof of Lemmas 3.8, 3.9, 3.10. □

Proof of Theorem 1.1(3). With the notation of the previous lemmas, since the discrepancy is real one can replace the square of a modulus with a plain square, and write

$$\begin{aligned}
&\left\{ \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}(r\Omega - x) \right|^2 d\mu_{H,R}(r) \right)^{p/2} dx \right\}^{1/p} \\
&= \left\{ \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}} \left(\sum_{j=0}^{h} \Phi_j((d+1)/2, r, x) + \mathcal{R}_h(r, x) \right)^2 d\mu_{H,R}(r) \right)^{p/2} dx \right\}^{1/p}.
\end{aligned}$$

The inner integral is equal to

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\sum_{j=0}^{h} \Phi_j((d+1)/2, r, x) + \mathcal{R}_h(r, x) \right)^2 d\mu_{H,R}(r) \\
&= \mathcal{G}((d+1)/2, x) + \mathcal{P}((d+1)/2, H, R, x)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{0\le i,j\le h,\ i+j>0}\int_{\mathbb{R}}\Phi_i\left((d+1)/2,r,x\right)\Phi_j\left((d+1)/2,r,x\right)d\mu_{H,R}(r)\\
&+2\sum_{0\le j\le h}\int_{\mathbb{R}}\Phi_j\left((d+1)/2,r,x\right)\mathcal{R}_h(r,x)\,d\mu_{H,R}(r)\\
&+\int_{\mathbb{R}}\mathcal{R}_h(r,x)^2\,d\mu_{H,R}(r).
\end{aligned}
$$

By the above lemmas, all these terms give a bounded contribution. The main term is $\mathcal{G}\left((d+1)/2,x\right)$, and it is independent of H, R and $d\mu(r)$. This is the function $\mathcal{G}(x)$ in the statement. The contributions of the other terms is negligible when $\beta>0$ and $H\to+\infty$. For example, let us estimate the integral with the mixed product $\Phi_i\left((d+1)/2,r,x\right)\Phi_j\left((d+1)/2,r,x\right)$. A repeated application of the Cauchy Schwarz inequality gives

$$
\begin{aligned}
&\int_{\mathbb{T}^d}\left|\int_{\mathbb{R}}\Phi_i\left((d+1)/2,r,x\right)\Phi_j\left((d+1)/2,r,x\right)d\mu_{H,R}(r)\right|^{p/2}dx\\
&\le\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}}\left|\Phi_i\left((d+1)/2,r,x\right)\right|^2d\mu_{H,R}(r)\right)^{p/4}\\
&\times\left(\int_{\mathbb{R}}\left|\Phi_j\left((d+1)/2,r,x\right)\right|^2d\mu_{H,R}(r)\right)^{p/4}dx\\
&\le\left\{\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}}\left|\Phi_i\left((d+1)/2,r,x\right)\right|^2d\mu_{H,R}(r)\right)^{p/2}dx\right\}^{1/2}\\
&\times\left\{\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}}\left|\Phi_j\left((d+1)/2,r,x\right)\right|^2d\mu_{H,R}(r)\right)^{p/2}dx\right\}^{1/2}.
\end{aligned}
$$

By Lemma 3.8, or Lemma 3.9 or Lemma 3.10, the terms with $i=0$ or $j=0$ give a bounded contribution. By Lemma 3.11, the terms with $i>0$ or with $j>0$ converge to 0 when $H+R\to+\infty$. □

Proof of Corollary 1.2. The corollary is an immediate consequence of part (1) of the theorem, and of the extrapolation Lemma 3.12. See [41] or [42, Chapter XII-4.41]. □

Proof of Corollary 1.3. By the expression of the function $\mathcal{G}(x)$ in the proof of Theorem 1.1 (3), and by the expression of $a_0(\xi)$ and $b_0(\xi)$ in terms of the curvature in Lemma 3.3,

$$
\begin{aligned}
\mathcal{G}(x)=\sum_{k\in\mathbb{Z}^d}&\left(\frac{1}{2\pi^2}\sum_{\substack{n\in\mathbb{Z}^d\setminus\{0,k\}\\ g(n-k)=g(n)}}K(n)^{-1/2}K(n-k)^{-1/2}\left|n\right|^{-(d+1)/2}\left|n-k\right|^{-(d+1)/2}\right)\\
&\times\exp\left(2\pi ikx\right).
\end{aligned}
$$

It follows that the k-th Fourier coefficient in the above expression of $\mathcal{G}(x)$ is equal to zero if and only if there are no integer solutions n to the equation $g(n-k)=g(n)$. In particular, if $g(m)\neq g(n)$ for every $m,n\in\mathbb{Z}^d$ with $m\neq n$, all Fourier coefficients with $k \neq 0$ vanish, and this function reduces to the constant

$$\frac{1}{2\pi^2} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} K(n)^{-1} |n|^{-d-1}.$$

This proves (1).

Since $A|x| \leq g(x) \leq B|x|$, if $g(n-k) = g(n)$ then $|k| \leq C|n|$ and

$$a_0(n)\, b_0(k-n)\, |n|^{-(d+1)/2} |k-n|^{-(d+1)/2} \leq C|n|^{-d-1} \leq C|k|^{-d-1}.$$

Hence, under the assumption that that for every $m \in \mathbb{Z}^d$ the equation $g(m) = g(n)$ has at most C solutions in $\mathbb{Z}^d$, the Fourier coefficients of $\mathcal{G}(x)$ are bounded by

$$2 \sum_{\substack{n \in \mathbb{Z}^d \setminus \{0,k\} \\ g(n-k)=g(n)}} a_0(n)\, b_0(k-n)\, |n|^{-(d+1)/2} |k-n|^{-(d+1)/2} \leq C|k|^{-d-1}.$$

It follows that the Fourier expansion that defines $\mathcal{G}(x)$ is absolutely and uniformly convergent, and this implies that $\mathcal{G}(x)$ is bounded and continuous. This proves (2).

The support function of $\Omega + t$ is

$$g_{\Omega+t}(x) = \sup_{y \in \Omega+t} \{x \cdot y\} = \sup_{y \in \Omega} \{x \cdot y\} + x \cdot t = g_\Omega(x) + x \cdot t.$$

It follows from point (1) that the 0-th Fourier coefficient of the limit function $\mathcal{G}_{\Omega+t}(x)$ does not depend on t. The equation $g_{\Omega+t}(m) = g_{\Omega+t}(n)$ is equivalent to

$$g_\Omega(m) - g_\Omega(n) = (n-m) \cdot t.$$

For every given m and n the above equation defines a hyperplane. In particular, this proves that there exists a t such that $g_{\Omega+t}(x)$ is not injective on the integers. On the other hand, if t avoids the countable union over all $m \neq n$ of these hyperplanes, which have Lebesgue measure zero, then the support function $g_{\Omega+t}(x)$ is injective. This proves (3). □

In order to prove Theorem 1.4 we need an easy algebraic lemma.

Lemma 3.14. *If $(A, B, C, D, ...)$ is a vector with integers coordinates, then the integer vectors $(x, y, z, w, ...)$ which are solutions to the equation $Ax + By + Cz + Dw + ... = 0$ are a lattice. If A and B are coprimes, so that there exist integers u and v such that $Au + Bv = 1$, then a basis of the lattice $\{Ax + By + Cz + Dw + ... = 0\}$ is*

$$\{(B, -A, 0, 0, ...), (uC, vC, -1, 0, ...), (uD, vD, 0, -1, ...), ...\}.$$

The area of a fundamental domain of this lattice is the length of the vector $(A, B, C, D, ...)$,

$$\sqrt{A^2 + B^2 + C^2 + D^2 + ...}$$

Proof. The solutions to the equation $Ax + By + Cz + Dw + ... = 0$ are a sum of a particular solution to the non homogeneous equation $Ax + By = -Cz - Dw - ...$, plus all solutions to the homogeneous equation $Ax + By = 0$. The solutions to the homogeneous equation $Ax + By = 0$ are $x = Br$ and $y = -Ar$, and a particular solution to the equation $Ax + By = -Cz - Dw - ...$ is $x = -u(Cz + Dw + ...)$ and $y = -v(Cz + Dw + ...)$. Hence, all integral solutions to $Ax + By + Cz + Dw + ... = 0$ are

$$(x, y, z, w, ...) = r(B, -A, 0, 0, ...) + s(uC, vC, -1, 0, ...) + t(uD, vD, 0, -1, ...) + ...$$

If $\{\mathbf{e}_j\}$ is the standard basis of orthogonal unit vectors, the area of a fundamental domain of the lattice $\{Ax + By + Cz + Dw + ... = 0\}$ is the length of the vector

$$\begin{aligned} &\det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & ... \\ B & -A & 0 & 0 & ... \\ uC & vC & -1 & 0 & ... \\ uD & vD & 0 & -1 & ... \\ ... & ... & ... & ... & ... \end{bmatrix} \\ &= \pm A\mathbf{e}_1 \pm B\mathbf{e}_2 \pm C\left(Au + Bv\right)\mathbf{e}_3 \pm D\left(Au + Bv\right)\mathbf{e}_4 \pm ... \\ &= \left(\pm A, \pm B, \pm C, \pm D, ...\right). \end{aligned}$$

□

Proof of Theorem 1.4. Part (1) is an immediate consequence of the Corollary 1.3.

Let us now prove part (2), and assume first that $\widehat{\mu}\left(\zeta\right)$ vanishes at infinity. If Ω is a convex set as in Theorem 1.1, set

$$\mathcal{K}\left(H, R, x\right) = \int_{\mathbb{R}} \left| r^{-(d-1)/2}\mathcal{D}\left(r\Omega - x\right)\right|^2 d\mu_{H,R}\left(r\right).$$

If the statement of part (2) of the theorem fails, then there exist $2d/\left(d-3\right) < p < +\infty$ and sequences $\{R_n\} \to +\infty$ and $\{H_n\} \to +\infty$ such that

$$\limsup_{n\to+\infty} \left\{ \int_{\mathbb{T}^d} \left|\mathcal{K}\left(H_n, R_n, x\right)\right|^{p/2} dx \right\}^{2/p} < +\infty.$$

Then a suitable subsequence converges weakly in $L^{p/2}\left(\mathbb{T}^d\right)$, and this weak convergence implies the convergence of Fourier coefficients. By the assumption that $\lim_{|\zeta|\to+\infty} \{|\widehat{\mu}\left(\zeta\right)|\} = 0$ and by Lemma 3.7, the subsequence converges weakly to the function $\mathcal{G}\left(x\right) = \mathcal{G}\left(\left(d+1\right)/2, x\right)$ defined in Lemma 3.13,

$$\mathcal{G}\left(x\right) = \sum_{k\in\mathbb{Z}^d} \left(2 \sum_{\substack{n\in\mathbb{Z}^d\setminus\{0,k\} \\ g(n-k)=g(n)}} a_0\left(n\right) b_0\left(k-n\right) |n|^{-(d+1)/2} |k-n|^{-(d+1)/2} \right) \exp\left(2\pi ikx\right).$$

Recall that, by Theorem 1.1, this function $\mathcal{G}\left(x\right)$ is in $L^{p/2}\left(\mathbb{T}^d\right)$ for every $p < \left(2d-4\right)/\left(d-3\right)$. In order to prove the theorem, it suffices to show that when Ω is the ball $\Sigma = \{|x| \leq 1\}$ this function is not in $L^{p/2}\left(\mathbb{T}^d\right)$ if $p > 2d/\left(d-3\right)$. In order to give an estimate of the norm from below, one can test this function against a Bessel potential of order $\alpha > 0$,

$$\mathcal{B}\left(x\right) = \sum_{k\in\mathbb{Z}^d} \left(1 + 4\pi^2 |k|^2\right)^{-\alpha/2} \exp\left(2\pi ikx\right).$$

This Bessel potential is a positive integrable function, which blows up as $x \to 0$ with an asymptotic expansion

$$\mathcal{B}\left(x\right) \approx \begin{cases} C\,|x|^{\alpha-d} & \text{if } 0 < \alpha < d, \\ C\log\left(1/\left|x\right|\right) & \text{if } \alpha = d, \\ C & \text{if } \alpha > d. \end{cases}$$

This follows from the Poisson summation formula, see [37, Chapter VII.2], and the asymptotic estimate of the Bessel potentials in $\mathbb{R}^d$, see [35, Chapter V 3.1]. It follows that if $1 \leq r \leq +\infty$ and $\alpha > d(1-1/r)$, then

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{B}(x)|^r \, dx \right\}^{1/r} < +\infty.$$

By the way, when $2 \leq r \leq +\infty$ and $1/r + 1/s = 1$ and $\alpha > d(1-1/r) = d/s$, this also follows via the Hausdorff Young inequality:

$$\left\{ \int_{\mathbb{T}^d} \left| \sum_{k \in \mathbb{Z}^d} \left(1 + 4\pi^2 |k|^2\right)^{-\alpha/2} \exp(2\pi i k x) \right|^r dx \right\}^{1/r}$$

$$\leq \left\{ \sum_{k \in \mathbb{Z}^d} \left| \left(1 + 4\pi^2 |k|^2\right)^{-\alpha/2} \right|^s \right\}^{1/s} < +\infty.$$

If $1/r + 1/s = 1$, then

$$2 \sum_{k \in \mathbb{Z}^d} \left(1 + 4\pi^2 |k|^2\right)^{-\alpha/2} \sum_{\substack{n \in \mathbb{Z}^d \setminus \{0,k\} \\ g(n-k) = g(n)}} a_0(n) \, b_0(k-n) \, |n|^{-(d+1)/2} \, |k-n|^{-(d+1)/2}$$

$$= \int_{\mathbb{T}^d} \mathcal{B}(x) \, \mathcal{G}(x) \, dx \leq \left\{ \int_{\mathbb{T}^d} |\mathcal{B}(x)|^r \, dx \right\}^{1/r} \left\{ \int_{\mathbb{T}^d} |\mathcal{G}(x)|^s \, dx \right\}^{1/s}.$$

Recall that $g(n) \approx |n|$ and that the products $a_0(n)b_0(k-n)$ are positive and bounded from below. If $g(n-k) = g(n)$, then $|n|^{-(d+1)/2}|n-k|^{-(d+1)/2} \approx |n|^{-d-1}$. Hence for every $\alpha > d(1-1/r) = d/s$ one obtains

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{G}(x)|^s \, dx \right\}^{1/s} \geq C \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^{-\alpha} \sum_{\substack{n \in \mathbb{Z}^d \setminus \{0\} \\ g(n-k) = g(n)}} |n|^{-d-1}.$$

Up to this point we have not assumed that the domain is a ball. Now assume that Ω is the ball $\Sigma = \{|x| \leq 1\}$. Then $a_0(n)$ and $b_0(n)$ are constants and $g(n) = |n|$, and the above inequality takes the more explicit form

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{G}(x)|^s \, dx \right\}^{1/s} \geq C \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^{-\alpha} \sum_{|n-k| = |n|} |n|^{-d-1}.$$

In order to bound this expression from below, one can restrict the sum to the k even,

$$\sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^{-\alpha} \sum_{|n-k| = |n|} |n|^{-d-1} \geq \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |2k|^{-\alpha} \sum_{|n-2k| = |n|} |n|^{-d-1}.$$

The equation $|m-2k| = |m|$ is the same as $k \cdot m = k \cdot k$, and with the change of variables $m = k + n$ one obtains $k \cdot n = 0$, so that for every $\alpha > d/s$,

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{G}(x)|^s \, dx \right\}^{1/s} \geq C \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^{-\alpha} \sum_{k \cdot n = 0} \left(|k|^2 + |n|^2\right)^{-(d+1)/2}.$$

By the above lemma, when two entries of the vector k are coprimes, the area of a fundamental domain of the $(d-1)$-dimensional lattice $\{k \cdot n = 0\}$ is $|k|$, the density of the lattice is $|k|^{-1}$ and, as a consequence of the classical theorem of Blichfeldt in the geometry of numbers (see e.g. [33, Theorem 9.5] for a proof in two dimensions which immediately extends to any dimension), for some constant C independent of k, one has

$$|\{k\cdot n=0,|n|\leq|k|\}|\geq C|k|^{d-2}.$$

Hence, when two entries of the vector k are coprimes,

$$\sum_{k\cdot n=0}\left(|k|^2+|n|^2\right)^{-(d+1)/2}\geq\left(2\,|k|^2\right)^{-(d+1)/2}|\{k\cdot n=0,\ |n|\leq|k|\}|\geq C\,|k|^{-3}.$$

By a theorem of E. Cesàro, see [15, Theorem 332], the probability that two random non negative integers are coprime is $6/\pi^2$, then the probability that two entries of the vector k are coprime is positive. This implies that if we call A the set of $k\in\mathbb{Z}^d$ with two coprime entries and if ε is sufficiently small and η is sufficiently large, then every shell $\{\eta^j\leq|k|<\eta^{j+1}\}$ contains at least $\varepsilon\eta^{jd}$ integer points in A. Hence, if $\alpha\leq d-3$,

$$\sum_{k\in\mathbb{Z}^d\setminus\{0\}}|k|^{-\alpha}\sum_{k\cdot n=0}\left(|k|^2+|n|^2\right)^{-(d+1)/2}\geq C\sum_{k\in A}|k|^{-\alpha-3}=+\infty.$$

In particular, recalling that $s=p/2$ and $\alpha>d/s=2d/p$, if $p>2d/\left(d-3\right)$ then

$$\left\{\int_{\mathbb{T}^d}|\mathcal{G}\left(x\right)|^{p/2}\,dx\right\}^{1/p}=+\infty.$$

This proves (2). Finally, (1) follows from (2) by replacing the measure $d\mu\left(r\right)$ with a convolution $\varphi*\mu\left(r\right)dr$, with $\varphi\left(r\right)$ a non negative smooth function on $\mathbb{R}$ with integral one. This convolution is a probability measure with Fourier transform that vanishes at infinity. Observe that

$$\begin{aligned}&\left\{\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}\left(r\Omega-x\right)\right|^2d\left(\varphi*\mu\right)_{H,R}\left(r\right)\right)^{p/2}dx\right\}^{2/p}\\
&=\left\{\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\left(R+H\left(r+t\right)\right)^{-(d-1)/2}\mathcal{D}\left(\left(R+H\left(r+t\right)\right)\Omega-x\right)\right|^2d\mu\left(r\right)\varphi\left(t\right)dt\right)^{p/2}dx\right\}^{2/p}\\
&\leq\int_{\mathbb{R}}\left\{\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}}\left|\left(R+H\left(r+t\right)\right)^{-(d-1)/2}\mathcal{D}\left(\left(R+H\left(r+t\right)\right)\Omega-x\right)\right|^2d\mu\left(r\right)\right)^{p/2}dx\right\}^{2/p}\varphi\left(t\right)dt.\end{aligned}$$

Hence, if

$$\left\{\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}\left(r\Omega-x\right)\right|^2d\mu_{H,R}\left(r\right)\right)^{p/2}dx\right\}^{2/p}\leq C<+\infty,$$

then also

$$\left\{\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}\left(r\Omega-x\right)\right|^2d\left(\varphi*\mu\right)_{H,R}\left(r\right)\right)^{p/2}dx\right\}^{2/p}\leq C<+\infty,$$

and the argument used to prove (2) applies. ◻

4. Concluding remarks

Remark 4.1. In Corollary 1.3 (3) it is proved that for a generic translation of a convex set the support function is injective when restricted to the integers. Indeed, it can be proved that the family of convex sets with injective support function is of second category in space of compact convex sets endowed with the Hausdorff metric. If $A+\Omega$ is the Minkowski sum of A and Ω, then $g_{rA+\Omega}(x) = rg_A(x) + g_\Omega(x)$. For a fixed x in $\mathbb{R}^d$, the function $\Omega \to g_\Omega(x)$ is continuous in the Hausdorff metric. For fixed $m, n \in \mathbb{Z}^d$ with $m \neq n$, the collection of convex sets Ω with $g_\Omega(m) \neq g_\Omega(n)$ is open in the Hausdorff metric. On the other hand, if $g_\Omega(m) = g_\Omega(n)$, and if A is a convex set with $g_A(m) \neq g_A(n)$, as in the previous remark, then $rA + \Omega \to \Omega$ as $r \to 0+$, and $g_{rA+\Omega}(m) \neq g_{rA+\Omega}(n)$. This implies that the set of Ω with $g_\Omega(m) \neq g_\Omega(n)$ is open and dense. Hence the set of Ω with $g_\Omega(m) \neq g_\Omega(n)$ for every $m, n \in \mathbb{Z}^d$ with $m \neq n$ is the intersection of a countable family of open dense sets.

Remark 4.2. Corollary 1.3 (3) may seem contradictory, but observe that the function $\mathcal{G}_\Omega(x)$ is a limit of an r average of the discrepancy of $r\Omega - x$, while the function $\mathcal{G}_{\Omega+t}(x)$ is a limit of an r average of the discrepancy of $r\Omega + (rt - x)$. These averages are different.

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