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Asymptotically optimal cubature formulas on manifolds for prefixed weights

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Abstract

Given a Riemannian manifold $\mathbb{X}$ with Riemannian measure $\mu_{\mathbb{X}}$ and positive weights $\{\omega_j\}_{j=1}^N$, we study the conditions under which there exist points $\{x_j\}_{j=1}^N \subset \mathbb{X}$ so that a cubature formula of the form

$$\int_{\mathbb{X}} P d\mu_{\mathbb{X}} = \sum_{j=1}^{N} \omega_j P(x_j) \tag{1}$$

holds for all polynomials P of order less than or equal to L. The problem is studied for diffusion polynomials (linear combinations of eigenfunctions of the Laplace–Beltrami operator) in the context of abstract Riemannian manifolds and for algebraic polynomials in the context of algebraic manifolds in $\mathbb{R}^n$.

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1. Introduction

Intuitively speaking, a set of sampling points $\{x_j\}_{j=1}^N$ and positive weights $\{\omega_j\}_{j=1}^N$ is a cubature of strength L if all polynomials of degree L are exactly integrated by the weighted sums over the sampling values (a more detailed definition will be given in Section 2). Cubature formulas have been deeply studied from both theoretical and practical points of view. Their associated literature is extensive, we refer for example to the books [23,25] and references therein. Not many explicit examples of cubature formulas are known, for a compilation of cubature formulas, one can check [7]. The most studied case is the one of the so called L-designs, where all the weights are equal, see [2,8,17].

See also [15], where one can find examples of L-designs for several values of L and different manifolds, as well as examples of cubature formulas with non-equal weights.

Here we study the existence of cubature formulas on manifolds from a non-constructive point of view. We know that the existence of a cubature of strength L depends on a relation between the dimension d of the manifold, the degree L of the polynomials and the number N of points and weights. In particular, the existence of exact cubatures with precisely as many terms as the dimension of the space of polynomials of degree up to L (this dimension is essentially L^d) is given by Tchakaloff's theorem (see [22, Exercise 2.5.8, p. 85]). Unfortunately, this theorem gives no information on the location of the nodes nor the magnitude of the positive weights.

If weights are fixed a priori, then the existence of cubature points is more complicated. Bondarenko, Radchenko and Viazovska showed in [2] that there is a constant C_d such that for every $N \geq C_d L^d$ there exists an L-design in the d-dimensional sphere with exactly N nodes. Later, Etayo, Marzo and Ortega-Cerdà generalized this result in [10] to the case of a compact connected affine algebraic manifold. Gariboldi and Gigante proved the analogous result on a compact connected oriented Riemannian manifold, where polynomials are replaced by the so-called diffusion polynomials, that is finite linear combinations of eigenfunctions of the Laplace–Beltrami operator, see [13]. For the sphere, each eigenfunction of the Laplace–Beltrami operator is the restriction of a polynomial, so that both, [10] and [13] apply. In general though, the eigenfunctions of the Laplace–Beltrami operator on an algebraic manifold are not necessarily polynomials, see Section 2.3.

Actual constructions of L-designs are difficult for most manifolds. A starting point of a construction recipe could be taking a partition of the manifold and then identifying one node within each region of the partition. The weights correspond to the regions volume and uniform partitions lead to equal weights. For non-homogeneous manifolds, one may be faced with non-uniform partitions (with smaller regions where the curvature is greater), so that the weights are fixed but not uniform. From an applied point of view, cubature points, L-designs, and arranging points on some manifold in general also relate to the problem of optimal distribution (locations) of sensors in some large sensor network. For a fixed number of sensors, the latter may be solved by geometric ad-hoc constructions or by numerical optimization. It has been recognized that L-designs of minimal cardinality N look well-distributed and satisfy asymptotical equidistribution properties. The existence of asymptotically optimal L-designs, hence, suggests the existence of good sensor configurations at least for large sensor numbers. Often, information from different devices from potentially varying manufacturers need to be combined. Therefore, varying reception ranges or sensitivities of the sensors are a common issue that require weighted designs. The sensors now lead to prefixed weights and the points' locations still need to be determined. The idea of assigning weights to the sensors is not new. Recently, complex weighted network models have been applied to the study of wireless sensor networks, where the edge weights, and vertex degrees and strength are taken into consideration, see [26].

In this paper we therefore study the existence of cubature points for fixed weights that are not all equal, in the case of compact connected oriented Riemannian manifolds with diffusion polynomials, and compact connected real algebraic manifolds with algebraic polynomials. In order to do so, we prove the existence of weighted area partitions on manifolds and some Marcinkiewicz–Zygmund type inequalities for gradients of polynomials and diffusion polynomials.

1.1. Organization

We define cubature formulas on Riemannian and algebraic manifolds in Section 2. In Section 3 we state the main result and in Section 4 we provide the proof based on an existence result from Brouwer degree theory. The remaining part of the manuscript is dedicated to verify that the assumptions of the Brouwer degree theorem are satisfied. In particular, in Section 5 we prove a result that may be of independent interest: the existence of a partition of any Riemannian manifold into parts with given prefixed areas. Several differential geometry tools are used throughout the paper, we refer the reader to [9] for a detailed study on this topic.

2. Cubature formulas on Riemannian and algebraic manifolds

2.1. Cubature formulas on Riemannian manifolds

Let $\mathcal{M}$ be a d-dimensional connected compact orientable Riemannian manifold without boundary, where the Riemannian metric is normalized so that it induces a probability measure $\mu_{\mathcal{M}}$, $\mu_{\mathcal{M}}(\mathcal{M}) = 1$. Eigenfunctions of the Laplacian on $\mathcal{M}$ are the key ingredient to carry an analogue of Fourier series on manifolds. Let $\{\varphi_k\}_{k=0}^{\infty}$ be an orthonormal basis of eigenfunctions of the (positive) Laplace–Beltrami operator, with eigenvalues $0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots$, $\Delta\varphi_k = \lambda_k^2 \varphi_k$ and let $\Pi_L(\mathcal{M}) = \operatorname{span}\{\varphi_k : \lambda_k \leq L\}$ be the space of diffusion polynomials of bandwidth $L \geq 0$.

Definition 1. For N points $\{x_j\}_{j=1}^N \subset \mathcal{M}$ and weights $\{\omega_j\}_{j=1}^N \subset \mathbb{R}$, we say that $\{(x_j, \omega_j)\}_{j=1}^N$ is a cubature of strength L if

$$\int_{\mathcal{M}} P(x) d\mu_{\mathcal{M}}(x) = \sum_{j=1}^{N} \omega_j P(x_j) \quad \text{for all } P \in \Pi_L(\mathcal{M}). \tag{2}$$

Since the constant function is contained in $\Pi_L(\mathcal{M})$, (2) implies $\sum_{j=1}^N \omega_j = 1$. Moreover, by orthogonality of the eigenfunctions φ_k

$$\int_{\mathcal{M}} \varphi_k(x) d\mu_{\mathcal{M}}(x) = 0 \quad \text{for all } k \geq 1,$$

hence $\{(x_j,\omega_j)\}_{j=1}^N$ is a cubature of strength L if and only if $\sum_{j=1}^N \omega_j = 1$ and

$$\sum_{j=1}^N \omega_j P(x_j) = 0 \quad \text{for all } P \in \Pi_L^0(\mathcal{M}),$$

where $\Pi_L^0(\mathcal{M}) = \operatorname{span}\{\varphi_k : 0 < \lambda_k \le L\}$.

For each $L \ge 0$, we denote by $N(L)$ the minimal number of points in a cubature of strength L.

Proposition 2. *There exists a positive constant $c_{\mathcal{M}}$ such that if $\{(x_j,\omega_j)\}_{j=1}^{N(L)}$ is a cubature of strength L, then $N(L) \ge c_{\mathcal{M}} L^d$ for every $L \ge 0$.*

Proof. Let $\{(x_j,\omega_j)\}_{j=1}^{N(L)}$ be a cubature of strength L and $\alpha > d$. By [3, Theorem 2.12] there exists a constant $\beta > 0$ such that for every function f in the Sobolev space $W^{\alpha,1}(\mathcal{M})$ one has

$$\left| \int_{\mathcal{M}} f(x) d\mu_{\mathcal{M}}(x) - \sum_{j=1}^{N(L)} \omega_j f(x_j) \right| \le \beta L^{-\alpha} \|f\|_{W^{\alpha,1}(\mathcal{M})}.$$

By [3, Theorem 2.16], there exists also a constant $\gamma > 0$ such that for every L there exists a function $f_L \in W^{\alpha,1}(\mathcal{M})$ with

$$\left| \int_{\mathcal{M}} f_L(x) d\mu_{\mathcal{M}}(x) - \sum_{j=1}^{N(L)} \omega_j f_L(x_j) \right| \ge \gamma N(L)^{-\alpha/d} \|f_L\|_{W^{\alpha,1}(\mathcal{M})}.$$

Therefore

$$\gamma N(L)^{-\alpha/d} \|f_L\|_{W^{\alpha,1}(\mathcal{M})} \le \left| \int_{\mathcal{M}} f_L(x) d\mu_{\mathcal{M}}(x) - \sum_{j=1}^{N(L)} \omega_j f_L(x_j) \right| \le \beta L^{-\alpha} \|f_L\|_{W^{\alpha,1}(\mathcal{M})}$$

and this gives $N(L) \ge \left(\frac{\gamma}{\beta}\right)^{d/\alpha} L^d$.

The precise definition of the Sobolev spaces $W^{\alpha,1}(\mathcal{M})$ can be found for example in [3, Definitions 2.2 and 2.3]. □

In fact, the condition proved in Proposition 2 can be extended to the more general setting of quasi-metric measure spaces, and for the much larger class of approximate quadrature measures, for which it suffices that identity (2) holds up to a small error. In particular, in [20, Theorem 4], the author proves that the support of an approximate quadrature measure of order L must contain at least cL^d points. We emphasize that according to [20], the hypothesis that for every $x \in \mathcal{M}$, $\sum_{j=1}^{N(L)} |\omega_j| \chi_{B(x,1/L)}(x_j) \le cL^{-d}$ has to be required on the cubature $\{x_j, \omega_j\}$, whereas in Proposition 2 no further hypotheses on the cubature are required (here $B(x, 1/L)$ is the geodesic ball centered at x and with radius $1/L$). In the same paper a variational problem giving a construction of the points x_j so as to achieve approximate quadrature measures is presented.

In [13], the authors proved that if $N \ge C_{\mathcal{M}} L^d$ with $C_{\mathcal{M}}$ a fixed constant depending only on $\mathcal{M}$, then there exists a set of points $\{x_j\}_{j=1}^N$ such that $\{(x_j, 1/N)\}_{j=1}^N$ is a cubature of strength L. What happens if weights are fixed, but not all equal? Existence of a cubature of strength L with the same cardinality in this case is not always guaranteed, as we show in Example 3.

Example 3. This example elaborates upon the theory exposed in [3]. Let $\mathcal{M}$ be a d-dimensional manifold as described in the beginning of this section. Let the weights be given by

$$\omega_1 = 1 - \frac{1}{N+1}, \qquad \omega_j = \frac{1}{(N+1)(N-1)}, \; 2 \leq j \leq N.$$

Assume that $\{(x_j, \omega_j)\}_{j=1}^N$ is a cubature of strength L on $\mathcal{M}$. Let f be a function such that f is supported on a ball of radius $cN^{-1/d}$ around x_1, $f(x_1) = 1$, $\int_{\mathcal{M}} f(x) d\mu_{\mathcal{M}}(x) = N^{-1}$ and

$$\|f\|_{W^{d+d\varepsilon,1}(\mathcal{M})} \leq cN^{\frac{d+d\varepsilon}{d}-1} = cN^{\varepsilon},$$

where ε is a small positive number and f is a function as in the proof of Theorem 2.16 in [3].

By [3, Theorem 2.12], we have

$$\left| \int_{\mathcal{M}} f(x) d\mu_{\mathcal{M}}(x) - \sum_{j=1}^{N(L)} \omega_j f(x_j) \right| \leq CL^{-(d+d\varepsilon)} \|f\|_{W^{d+d\varepsilon,1}(\mathcal{M})}.$$

Since

$$\left| \int_{\mathcal{M}} f(x) d\mu_{\mathcal{M}}(x) - \sum_{j=1}^{N(L)} \omega_j f(x_j) \right|$$
$$= \left| \frac{1}{N} - \left(1 - \frac{1}{N+1}\right) f(x_1) + O\left(N \frac{1}{N^2}\right) \right| = 1 + O\left(\frac{1}{N}\right)$$

and

$$L^{-(d+d\varepsilon)} \|f\|_{W^{d+d\varepsilon,1}(\mathcal{M})} \leq L^{-(d+d\varepsilon)} N^{\varepsilon},$$

one has

$$N \geq CL^{\frac{d+d\varepsilon}{\varepsilon}}.$$

Therefore $\{(x_j, \omega_j)\}_{j=1}^N$ cannot be a cubature of strength L with the above choice of weights under the only hypothesis $N \geq CL^d$.

In fact, it has been recently proved, see [4], that the following estimate holds

$$1 \geq C_{\mathcal{M}} L^d \sum_{j=1}^{N} \omega_j^2$$

for all real weights $\{\omega_j\}_{j=1}^N$ such that $\sum_{j=1}^N \omega_j = 1$.

2.2. *Cubature formulas on algebraic manifolds*

Let $\mathcal{V} \subset \mathbb{R}^n$ be a smooth, connected and compact affine algebraic manifold of dimension d

$$\mathcal{V} = \left\{ x \in \mathbb{R}^n \; : \; p_1(x) = \cdots = p_r(x) = 0 \right\},$$

where $p_1, \ldots, p_r \in \mathbb{R}[X]$ are polynomials with real coefficients and the normal space at $x \in \mathcal{V}$ is of dimension $n - d$. $\mathcal{V}$ carries a Riemannian structure with measure $\mu_{\mathcal{V}}$ inherited from the Riemannian metric induced by the embedding of $\mathcal{V}$ in $\mathbb{R}^n$. We normalize this measure so that $\mu_{\mathcal{V}}(\mathcal{V}) = 1$. Hence, the definitions and statements of the previous section do apply in principle (provided that $\mathcal{V}$ is orientable). Here, however, we replace the diffusion polynomials by the algebraic polynomials on $\mathbb{R}^n$ restricted to $\mathcal{V}$ and denoted by

$$\Pi_L(\mathcal{V}) := \{P_{|\mathcal{V}} : P \in \mathbb{R}[z_1, \dots, z_n] : \text{ total degree of } P \text{ is } \leq L\}.$$

For an algebraic manifold and in analogy to Definition 1, we now define the concept of algebraic cubatures.

Definition 4. For N points $\{x_j\}_{j=1}^N \subset \mathcal{V}$ and weights $\{\omega_j\}_{j=1}^N \subset \mathbb{R}$, we say that $\{(x_j, \omega_j)\}_{j=1}^N$ is an algebraic cubature of strength L if

$$\int_{\mathcal{V}} P(x) d\mu_{\mathcal{V}}(x) = \sum_{j=1}^N \omega_j P(x_j) \quad \text{for all } P \in \Pi_L(\mathcal{V}).$$

Let $\Pi_L^0(\mathcal{V})$ denote the orthogonal complement of the constant function within $\Pi_L(\mathcal{V})$. As before, since $\sum_{j=1}^N \omega_j = 1$, then $\{(x_j, \omega_j)\}_{j=1}^N$ is an algebraic cubature of strength L if and only if

$$\sum_{j=1}^N \omega_j P(x_j) = 0 \quad \text{for all } P \in \Pi_L^0(\mathcal{V}).$$

The analogue of Proposition 2 also holds for an algebraic manifold, i.e., there exists a positive constant $c_{\mathcal{V}}$ such that if $\{(x_j, \omega_j)\}_{j=1}^N$ is an algebraic cubature of strength L, then $N \geq c_{\mathcal{V}} L^d$ for every $L \geq 0$, the proof of [10, Proposition 2.1] for L-designs can be extended to this case step by step.

Notice that given a Riemannian algebraic manifold, we have two different definitions for a cubature of strength L, the one given in Definition 1 and the one given in Definition 4. Observe though that the first definition is intrinsic, whereas the second is extrinsic and depends on the specific embedding of the manifold in the Euclidean space $\mathbb{R}^n$.

2.3. On the relation between polynomials and diffusion polynomials

The relation between polynomials and diffusion polynomials on an algebraic manifold has not been deeply understood. Nevertheless, we know this relation for some particular manifolds. In the case of the sphere $\mathbb{S}^d$, the eigenfunctions of the Laplacian are polynomials in the ambient space $\mathbb{R}^{d+1}$ restricted to the sphere. The Grassmannian manifold $\mathbb{G}_{k,m}$, consisting of the k-dimensional subspaces of $\mathbb{R}^m$, can be isometrically embedded into $\mathbb{R}^{m^2}$ by seeing it as the set of symmetric $m \times m$ matrices which are projection operators and have trace equal to k (see [6, Section 1.3.2.]). Any diffusion polynomial on the Grassmannian manifold is then the restriction of a polynomial in the ambient space $\mathbb{R}^{m^2}$, see [5]. In general, though, eigenfunctions of the Laplacian on an algebraic manifold $\mathcal{V}$ are not necessarily restrictions of polynomials. In the following example we show that there are diffusion polynomials on the ellipse that are not restrictions of polynomials in the ambient space. Notice that the circle and the ellipse are different algebraic manifolds, but they coincide as Riemannian manifolds.

The (positive) Laplacian $\Delta_{\mathbb{R}/2\pi\mathbb{Z}}$ on $\mathbb{R}/2\pi\mathbb{Z}$ is simply $-\partial_t^2$ acting on $2\pi\mathbb{Z}$ periodic real-valued functions on $\mathbb{R}$. Its eigenvalues are k^2, for $k \in \mathbb{N}$, with associated eigenfunctions

$$t \mapsto \cos(kt), \qquad t \mapsto \sin(kt).$$

So we have two eigenfunctions associated to each eigenvalue k^2.

Definition 5. For fixed $A, B > 0$, we consider the ellipse

$$E_{A,B} = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \right\},$$

which is parametrized by

$$u_{A,B} : \mathbb{R}/2\pi\mathbb{Z} \to E_{A,B}, \quad t \mapsto (A\cos(t), B\sin(t)).$$

The circle is a particular case of the ellipse, for which the relations of Laplacian eigenfunctions and polynomials are well-studied.

Example 6 (*Circle* $\mathbb{S}^1$)**.** For $A = B = 1$, the mapping $u_{1,1}$ is an arc-length parametrization of $\mathbb{S}^1$, hence, an isometry, so that

$$\Delta_{\mathbb{S}^1} f = (\Delta_{\mathbb{R}/2\pi\mathbb{Z}}(f \circ u_{1,1})) \circ u_{1,1}^{-1} \tag{3}$$

holds for every $f \in \mathcal{C}^\infty(\mathbb{S}^1)$. In particular, the eigenvalues of $\Delta_{\mathbb{S}^1}$ are k^2 with associated eigenfunctions

$$f_k : \mathbb{S}^1 \to \mathbb{R}, \quad (x, y) \mapsto \cos(k u_{1,1}^{-1}(x, y)), \qquad g_k : \mathbb{S}^1 \to \mathbb{R}, \quad (x, y) \mapsto \sin(k u_{1,1}^{-1}(x, y)).$$

Hence, we observe, for $t \in \mathbb{R}/2\pi\mathbb{Z}$,

$$f_k(\cos(t), \sin(t)) = \cos(kt),$$
$$g_k(\cos(t), \sin(t)) = \sin(kt).$$

All eigenfunctions of $\Delta_{\mathbb{S}^1}$ are restrictions of algebraic polynomials in $\mathbb{R}^2$, which can be derived from the Chebycheff-polynomials of first and second type, T_k and U_k, via

$$\mathbb{R}^2 \ni (x, y) \mapsto T_k(x), \qquad T_k(\cos(t)) = \cos(kt),$$
$$\mathbb{R}^2 \ni (x, y) \mapsto U_{k-1}(x)y, \qquad U_{k-1}(\cos(t))\sin(t) = \sin(kt), \quad t \in \mathbb{R}.$$

We now state that the situation is very different for $A \neq B$.

Proposition 7. *If $A \neq B$, then each nonzero eigenvalue of the Laplacian on $E_{A,B}$ has an eigenfunction that is not the restriction of any algebraic polynomial on $\mathbb{R}^2$ with complex coefficients.*

Proof. Let us consider the parametrization of the ellipse $u_{A,B}$ given in Definition 5. For $A \neq B$, $u_{A,B}$ is not an isometry. To compute the arc-length parametrization of $E_{A,B}$, we define $\ell_{A,B} := \int_{-\pi}^{\pi} \|\dot{u}_{A,B}(t)\| \mathrm{d}t$ and

$$h_{A,B} : [0, 2\pi] \to [0, \ell_{A,B}], \quad t \mapsto \int_0^t \|\dot{u}_{A,B}(s)\| \mathrm{d}s. \tag{4}$$

We now identify $h_{A,B}$ with its periodic extension $h_{A,B} : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/\ell_{A,B}\mathbb{Z}$. The arc-length parametrization of $E_{A,B}$ is

$$\psi_{A,B} : \mathbb{R}/\ell_{A,B}\mathbb{Z} \to E_{A,B}, \quad t \mapsto u_{A,B}(h_{A,B}^{-1}(t)) = (A\cos(h_{A,B}^{-1}(t)), B\sin(h_{A,B}^{-1}(t))).$$

We deduce that two linearly independent eigenfunctions on $E_{A,B}$ with respect to the eigenvalue k^2, for $0 < k \in \mathbb{N}$, are

$$f_k : E_{A,B} \to \mathbb{R}, \quad (x, y) \mapsto \cos\left(\frac{2\pi}{\ell_{A,B}} k\psi_{A,B}^{-1}(x, y)\right), \qquad g_k : E_{A,B} \to \mathbb{R}, \quad (x, y) \mapsto \sin\left(\frac{2\pi}{\ell_{A,B}} k\psi_{A,B}^{-1}(x, y)\right).$$

They span the eigenspace associated to k^2. Since $\psi_{A,B}$, $u_{A,B}$, and $h_{A,B}$ are bijections, this implies, for $t \in \mathbb{R}/2\pi\mathbb{Z}$,

$$f_k(A\cos(t), B\sin(t)) = \cos\left(\frac{2\pi}{\ell_{A,B}} kh_{A,B}(t)\right), \tag{5}$$

$$g_k(A\cos(t), B\sin(t)) = \sin\left(\frac{2\pi}{\ell_{A,B}} kh_{A,B}(t)\right). \tag{6}$$

To prove our claim, we now assume that both, f_k and g_k, are restrictions of algebraic polynomials on $\mathbb{R}^2$, i.e., for $x, y \in E_{A,B}$,

$$f_k(x, y) = \sum_{m,n\in\mathbb{N}} \alpha_{m,n} x^m y^n,$$

$$g_k(x, y) = \sum_{m,n\in\mathbb{N}} \beta_{m,n} x^m y^n,$$

with finitely many nonzero coefficients $\alpha_{m,n}, \beta_{m,n} \in \mathbb{C}$. Thus, (5) and (6) imply, for $t \in \mathbb{R}/2\pi\mathbb{Z}$,

$$\cos\left(\frac{2\pi}{\ell_{A,B}} kh_{A,B}(t)\right) = \sum_{m,n\in\mathbb{N}} \alpha_{m,n} A^m \cos^m(t) B^n \sin^n(t), \tag{7}$$

$$\sin\left(\frac{2\pi}{\ell_{A,B}} kh_{A,B}(t)\right) = \sum_{m,n\in\mathbb{N}} \beta_{m,n} A^m \cos^m(t) B^n \sin^n(t). \tag{8}$$

Trigonometric identities, in particular power reduction formulae and product to sum identities, imply that both, (7) and (8), are trigonometric polynomials, i.e., finite linear combination of $\cos(lt)$ and $\sin(mt)$, $l, m \in \mathbb{N}$. Therefore, their derivatives

$$t \mapsto -\sin\left(\frac{2\pi}{\ell_{A,B}} kh_{A,B}(t)\right) \frac{2\pi}{\ell_{A,B}} kh'_{A,B}(t), \tag{9}$$

$$t \mapsto \cos\left(\frac{2\pi}{\ell_{A,B}} kh_{A,B}(t)\right) \frac{2\pi}{\ell_{A,B}} kh'_{A,B}(t), \tag{10}$$

are also trigonometric polynomials. The obvious identity

$$h'_{A,B}(t) = \cos^2\left(\frac{2\pi}{\ell_{A,B}} kh_{A,B}(t)\right) h'_{A,B}(t) + \sin^2\left(\frac{2\pi}{\ell_{A,B}} kh_{A,B}(t)\right) h'_{A,B}(t)$$

implies that $h'_{A,B}$ is a trigonometric polynomial due to (7), (8), and (9), (10) being trigonometric polynomials and the latter being an algebra. Hence, the definition of $h_{A,B}(t)$ in (4) yields that

$$h'_{A,B}(t) = \|\dot{u}_{A,B}(t)\| = \sqrt{A^2\sin^2(t) + B^2\cos^2(t)} = \sqrt{A^2 + B^2 + (B^2 - A^2)\cos(2t)}$$

is a trigonometric polynomial. The infinite Taylor expansion of the square root implies that the above right-hand-side has an infinite Fourier series if and only if $B^2 - A^2 \neq 0$. More elementary, since $h'_{A,B}(t)$ is an even function, it must be a finite linear combination of $\cos(lt)$, $l \in \mathbb{N}$. Since the square of $h'_{A,B}(t)$ coincides with $A^2 + B^2 + (B^2 - A^2)\cos(2t)$, the largest l

that can occur with nonzero coefficient in $h'_{A,B}$ is $l = 1$ due to product to sum identities for the cosine. Since there is no $\cos(t)$ term in the square of $h'_{A,B}(t)$, we deduce $A^2 = B^2$, which contradicts the assumption of the proposition. $\square$

Remark 8. Note that in Proposition 7 we prove a result stronger than needed, since we allow the polynomials to have complex coefficients meanwhile $\Pi_L(\mathcal{V})$ is a vector space of polynomials with real coefficients.

2.4. Notation

We use the notation $\gtrsim$ meaning the right-hand side is less than or equal to the left-hand side up to a positive constant factor that is only allowed to depend on $\mathcal{M}$ or $\mathcal{V}$ and hence on d. The symbol $\lesssim$ is used analogously.

For a more compact notation, we make the convention that $\mathbb{X}$ either denotes $\mathcal{M}$ or $\mathcal{V}$ as defined in Section 2. Then $\mu_{\mathbb{X}}$ will denote respectively $\mu_{\mathcal{M}}$ or $\mu_{\mathcal{V}}$, and $\Pi_L(\mathbb{X})$ and $\Pi_L^0(\mathbb{X})$ will denote the diffusion polynomials $\Pi_L(\mathcal{M})$ and $\Pi_L^0(\mathcal{M})$ or the algebraic polynomials $\Pi_L(\mathcal{V})$ and $\Pi_L^0(\mathcal{V})$.

3. Main result

Our main result holds for the Riemannian manifold $\mathcal{M}$ with diffusion polynomials and for the algebraic manifold $\mathcal{V}$ with algebraic polynomials.

Theorem 9 (*Main Result*)**.** *Let $h = 1$ if $\mathbb{X} = \mathcal{M}$, and $h = d$ if $\mathbb{X} = \mathcal{V}$. There exists a constant $C = C_{\mathbb{X}}$ such that for all $b \geq 1$, if*

$$N \geq C b^{2h+2} L^d$$

and if the weights $\{\omega_j\}_{j=1}^N$ are such that

$$\sum_{j=1}^N \omega_j = 1 \text{ and } 0 \leq \omega_j \leq \frac{b}{N},$$

then there exists $\{x_j\}_{j=1}^N \subset \mathbb{X}$ such that $\{(x_j, \omega_j)\}_{j=1}^N$ is a cubature/algebraic cubature of strength L.

This result is a direct consequence of the following weaker version, where a lower bound on the weights is imposed.

Theorem 10. *Let $h = 1$ if $\mathbb{X} = \mathcal{M}$, and $h = d$ if $\mathbb{X} = \mathcal{V}$. There exists a constant $C = C_{\mathbb{X}}$ such that for all $0 < a \leq 1 \leq b$, if*

$$N \geq C b \left(\frac{b}{a}\right)^{2h} L^d$$

and if the weights $\{\omega_j\}_{j=1}^N$ are such that

$$\sum_{j=1}^N \omega_j = 1 \text{ and } \frac{a}{N} \leq \omega_j \leq \frac{b}{N},$$

then there exists $\{x_j\}_{j=1}^N \subset \mathbb{X}$ such that $\{(x_j, \omega_j)\}_{j=1}^N$ is a cubature/algebraic cubature of strength L.

The proof of Theorem 10 is presented in the subsequent section. Here, we show how Theorem 9 follows from Theorem 10.

Proof of Theorem 9. Assume all weights are in increasing order, $\omega_1 \le \omega_2 \le \cdots \le \omega_N$. Let us organize the set of weights in blocks with total mass at least $1/N$. Thus let j_1 be such that $\sum_{j=1}^{j_1-1} \omega_j < 1/N$ but $W_1 = \sum_{j=1}^{j_1} \omega_j \ge 1/N$. Let j_2 be such that $\sum_{j=j_1+1}^{j_2-1} \omega_j < 1/N$ but $W_2 = \sum_{j=j_1+1}^{j_2} \omega_j \ge 1/N$, and so on, up until $j_m = N$ in such a way that $\sum_{j=j_{m-1}+1}^{N-1} \omega_j < 1/N$ but $W_m = \sum_{j=j_{m-1}+1}^{N} \omega_j \ge 1/N$. Notice that the construction ends correctly since $\omega_N \ge 1/N$. By construction, for all $i = 1, \dots, m$,

$$\frac{1}{N} \le W_i \le \frac{b+1}{N}, \qquad 1 = \sum_{j=1}^{N} \omega_j = \sum_{i=1}^{m} W_i \le m\frac{b+1}{N},$$

so that $m \ge N/(b+1) \gtrsim (b+1)^{2h+1} L^d$. We can therefore apply Theorem 10 to the weights $\{W_i\}_{i=1}^m$ and we conclude that there are points $\{x_i\}_{i=1}^m$ such that $\{(x_i, W_i)\}_{i=1}^m$ is a cubature of strength L. By repeating the point x_i for all the weights ω_j with $j_{i-1}+1 \le j \le j_i$ we obtain the desired cubature $\{(x_i, \omega_i)\}_{i=1}^N$. $\square$

Remark 11. By Proposition 2, Theorem 9 is sharp, in the sense that the exponent d in the condition $N \ge Cb^{2h+2}L^d$ is best possible. On the other hand, we do not know if any of the other constants there, say C or the exponent $2h+2$, are sharp.

Remark 12. We do not know if one can show the existence of algebraic cubatures of strength L with $N \approx L^d$ points, if the manifold is not contained in the zero set of a collection of polynomials. It is known however that one such cubature in the equal weight case in a non algebraic d-dimensional manifold should be rather pathological. Indeed, for example, by [1, Theorem 3], the nodes of the cubature should be non uniformly separated. Also, by [16, Theorem 5.1], for all L there should be a point x_0 such that replacing any node of the cubature with x_0, the error in the corresponding cubature should be greater than

$$\frac{1}{2N} \sup_{\mathcal{M}} |P|$$

for at least one polynomial P of degree not exceeding $(2d+2)N^{1/(d+1)}$.

4. Proof of Theorem 10

As in [2,10,13], the proof is based on a result from the Brouwer degree theory.

Lemma 13 (*[21, Theorem 1.2.9]*). *Let H be a finite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $f : H \to H$ be a continuous mapping and Ω an open bounded subset with boundary $\partial\Omega$ such that $0 \in \Omega \subset H$. If $\langle x, f(x)\rangle > 0$ for all $x \in \partial\Omega$, then there exists $x \in \Omega$ satisfying $f(x) = 0$.*

The following result is the main tool to define a mapping with the properties stated in Lemma 13.

Lemma 14. *Let $h = 1$ if $\mathbb{X} = \mathcal{M}$ and $h = d$ if $\mathbb{X} = \mathcal{V}$. There exists a constant $C = C_{\mathbb{X}} > 0$ such that the following holds: for all $0 < a \le 1 \le b$, for all $N \ge Cb(\frac{b}{a})^{2h} L^d$ and for all* *weights* $\{\omega_j\}_{j=1}^N$ *such that*

$$\sum_{j=1}^{N} \omega_j = 1 \text{ and } \frac{a}{N} \leq \omega_j \leq \frac{b}{N},$$

there exists a continuous mapping

$$F : \Pi_L^0(\mathbb{X}) \to \mathbb{X}^N$$
$$P \mapsto (x_1(P), \ldots, x_N(P)),$$

such that for all $P \in \Pi_L^0(\mathbb{X})$ *with* $\int_{\mathbb{X}} \|\nabla P(x)\| d\mu_{\mathbb{X}}(x) = 1$,

$$\sum_{j=1}^{N} \omega_j P(x_j(P)) > 0.$$

We postpone the proof of Lemma 14 and now verify our main result.

Proof of Theorem 10. Fix L and define

$$\Omega = \left\{ P \in \Pi_L^0(\mathbb{X}) : \int_{\mathbb{X}} \|\nabla P(x)\| d\mu_{\mathbb{X}}(x) < 1 \right\}, \tag{11}$$

which is clearly an open subset of $\Pi_L^0(\mathbb{X})$ such that $0 \in \Omega \subset \Pi_L^0(\mathbb{X})$. Since $\int_{\mathbb{X}} \|\nabla P(x)\| d\mu_{\mathbb{X}}(x)$ is a norm in the finite dimensional space $\Pi_L^0(\mathbb{X})$, it is equivalent to the L^2 norm in $\Pi_L^0(\mathbb{X})$, so Ω is also bounded in $\Pi_L^0(\mathbb{X}) \subset L^2(\mathbb{X})$. Take $C = C_{\mathbb{X}}$ as in Lemma 14, let $N \geq Cb(\frac{b}{a})^{2h} L^d$ and let $x_i(P)$ be the points defined by the map F in Lemma 14 for $P \in \partial\Omega$.

By the Riesz Representation Theorem, for each point $x \in \mathbb{X}$ there exists a unique polynomial $G_x \in \Pi_L^0(\mathbb{X})$ such that

$$\langle G_x, P \rangle = P(x)$$

for all $P \in \Pi_L^0(\mathbb{X})$. Then a set of points $\{x_j\}_{j=1}^N \subset \mathbb{X}$ together with a set of weights $\{\omega_i\}_{j=1}^N \subset \mathbb{R}_+$ is a cubature formula of strength L if and only if

$$\sum_{i=1}^{N} \omega_j G_{x_j} = 0.$$

Now let $U : \mathbb{X}^N \to \Pi_L^0(\mathbb{X})$ be the continuous map defined by $U(x_1, \ldots, x_N) = \sum_{i=1}^N \omega_j G_{x_j}$ and let us consider the composition

$$f = U \circ F : \Pi_L^0(\mathbb{X}) \to \Pi_L^0(\mathbb{X}).$$

Then, by Lemma 14, for every $P \in \partial\Omega$ we have

$$\langle P, f(P) \rangle = \sum_{j=1}^{N} \omega_j P(x_j(P)) > 0.$$

We conclude with Lemma 13, stating that there exists $Q \in \Omega$ such that $U(F(Q)) = 0$, that is, such that $\sum_{j=1}^N \omega_j G_{x_j(Q)} = 0$, which implies that $\{(x_j(Q), \omega_j)\}_{j=1}^N$ is a cubature formula of strength L. □

In order to complete the above proof, we must verify Lemma 14. We define the application F through a gradient flow with initial points that are taken from a partition of the manifold into regions with areas corresponding to the weights. To verify suitable properties of the flow, Marcinkiewicz–Zygmund inequalities for the gradient of diffusion polynomials and algebraic polynomials are required. These are the topics of the subsequent sections. We start with weighted area partitions, where the result holds for both scenarios and then we prove Marcinkiewicz–Zygmund inequalities separately for polynomials and diffusion polynomials.

5. Weighted area partitions

Here we generalize the results in [14] for the case of equal weight partitions to the case of not all equal weights.

Definition 15. Let $0 < a \leq 1 \leq b$ and $0 < c_3 < c_4$. We say that a collection of subsets of $\mathbb{X}$, $\mathcal{R} = \{R_j\}_{j=1}^N$ is a partition of $\mathbb{X}$ with constants a, b, c_3 and c_4 if the following hold:

- $\cup_{j=1}^N R_j = \mathbb{X}$ and $\mu_{\mathbb{X}}(R_i \cap R_j) = 0$ for all $1 \leq i < j \leq N$,
- $a/N \leq \mu_{\mathbb{X}}(R_j) \leq b/N$ for $j = 1, \ldots, N$,
- each R_j is contained in a geodesic ball X_j of radius $c_4 b^{1/d} N^{-1/d}$ and contains a geodesic ball Y_j of radius $c_3 \left(a^2/b\right)^{1/d} N^{-1/d}$.

We denote by $\mathcal{P}(a, b, c_3, c_4)$ the collection of all such partitions.

Proposition 16. *There exist two constants $0 < c_3 < c_4$ such that for all constants a and b with $0 < a \leq 1 \leq b$, for every $N \geq 1$ and for every choice of weights $\{\omega_j\}_{j=1}^N$ with $\sum_{j=1}^N \omega_j = 1$ and $a/N \leq \omega_j \leq b/N$, there is a partition of $\mathbb{X}$, $\mathcal{R} = \{R_j\}_{j=1}^N \in \mathcal{P}(a, b, c_3, c_4)$ such that $\mu_{\mathbb{X}}(R_j) = \omega_j$ for all $j = 1, \ldots, N$.*

The proof of Proposition 16 is based on the following lemma on non-atomic measures not having gaps in their range:

Lemma 17 (*[14, Corollary 3]*)**.** *Let S be a measurable subset of $\mathbb{X}$. Then, for any $0 \leq r \leq \mu_{\mathbb{X}}(S)$, there is $\Gamma \subset S$ such that $\mu_{\mathbb{X}}(\Gamma) = r$.*

Note that this lemma holds for more general spaces $\mathbb{X}$ than the ones we consider in the present manuscript, see [14] for a brief discussion.

Corollary 18. *Given positive weights $\{\omega_j\}_{j=1}^N \subset \mathbb{R}$, let S and $Q_1, \ldots, Q_N \subset S$ be measurable subsets of $\mathbb{X}$. If $\{Q_j\}_{j=1}^N$ are pairwise disjoint with $\mu_{\mathbb{X}}(Q_j) \leq \omega_j$ and $\mu_{\mathbb{X}}(S) \geq \sum_{j=1}^N \omega_j$, then there are pairwise disjoint $R_1, \ldots, R_N \subset S$, such that $Q_j \subset R_j$ and $\mu_{\mathbb{X}}(R_j) = \omega_j$, $j = 1, \ldots, N$.*

Proof of Corollary 18. We start with $S_1 := S \setminus \bigcup_{j=1}^N Q_j$. Since

$$\mu_{\mathbb{X}}(S_1) \geq \mu_{\mathbb{X}}(S) - \mu_{\mathbb{X}}(\cup_{j=1}^N Q_j) \geq \sum_{j=1}^N \omega_j - \sum_{j=1}^N \mu_{\mathbb{X}}(Q_j) \geq \omega_1 - \mu_{\mathbb{X}}(Q_1),$$

there is $\Gamma_1 \subset S_1$ such that $\mu_{\mathbb{X}}(\Gamma_1) = \omega_1 - \mu_{\mathbb{X}}(Q_1)$. We set $R_1 := Q_1 \cup \Gamma_1$. Next, we define $S_2 := S_1 \setminus R_1$. There is $\Gamma_2 \subset S_2$ such that $\mu_{\mathbb{X}}(\Gamma_2) = \omega_2 - \mu_{\mathbb{X}}(Q_2)$. Let $R_2 := Q_2 \cup \Gamma_2$ and so on. $\square$

Our assumptions on $\mathbb{X}$ imply that there are constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 r^d \leq \mu_{\mathbb{X}}(B(x,r)) \leq c_2 r^d, \qquad \text{for all } x \in \mathbb{X},\ 0 < r \leq \operatorname{diam}(\mathbb{X}), \tag{12}$$

where $B(x,r)$ denotes the ball of radius r centered at x. The proof of Proposition 16 proceeds as in the equal weight case in [14], with a few technical modifications. As in [14], we know that there is a family of δ-adic cubes in $\mathbb{X}$, i.e., for any $0 < \delta < 1$ there exist $0 < u_1 \leq u_2 < \infty$, a collection of open subsets $\{Q_\alpha^k : k \in \mathbb{Z},\ \alpha \in I_k\}$ in $\mathbb{X}$, where each I_k is a finite index set, and points $\{z_\alpha^k : k \in \mathbb{Z},\ \alpha \in I_k\}$ with

(i) $\mu_{\mathbb{X}}(\mathbb{X} \setminus \bigcup_{\alpha \in I_k} Q_\alpha^k) = 0$, for all $k \in \mathbb{Z}$,
(ii) for $l > k$ and $\alpha \in I_l$, there is $\beta_0 \in I_k$ such that
- $Q_\alpha^l \subset Q_{\beta_0}^k$,
- $Q_\alpha^l \cap Q_\beta^k = \emptyset$, for all $\beta \in I_k$ with $\beta \neq \beta_0$

(iii) $B(z_\alpha^k, u_1\delta^k) \subset Q_\alpha^k \subset B(z_\alpha^k, u_2\delta^k)$, for all $k \in \mathbb{Z}$, $\alpha \in I_k$.

Assume first

$$N \geq \frac{2b}{c_1 \delta^d \operatorname{diam}(\mathbb{X})^d}. \tag{13}$$

Choose $k \in \mathbb{Z}$ such that

$$u_1 \delta^{k+1} < \left(\frac{2}{c_1}\frac{b}{N}\right)^{1/d} \leq u_1 \delta^k, \tag{14}$$

so that we obtain the estimates

$$\mu_{\mathbb{X}}(Q_\alpha^k) \geq \mu_{\mathbb{X}}(B(z_\alpha^k, u_1\delta^k)) \geq c_1 u_1^d \delta^{kd} \geq 2\frac{b}{N}. \tag{15}$$

Here, we have used (12), so that we still need to ensure $u_1\delta^k \leq \operatorname{diam}(\mathbb{X})$. Indeed, using (13) we derive

$$u_1\delta^k = \frac{u_1\delta^{k+1}}{\delta} \leq \frac{(\frac{2}{c_1}\frac{b}{N})^{1/d}}{\delta} \leq \operatorname{diam}(\mathbb{X}).$$

Thus, (15) is a valid estimate.

Similarly, we derive an upper bound

$$\mu_{\mathbb{X}}(Q_\alpha^k) \leq \mu_{\mathbb{X}}(B(z_\alpha^k, u_2\delta^k)) \leq c_2 u_2^d \delta^{kd} < \frac{c_2}{c_1}\left(\frac{u_2}{u_1}\right)^d \frac{2}{\delta^d}\frac{b}{N} = \frac{c_2}{c_1}\left(\frac{u_2}{u_1}\right)^d \frac{2}{\delta^d}\frac{b}{a}\frac{a}{N}.$$

With $C := \frac{c_2}{c_1}\left(\frac{u_2}{u_1}\right)^d \frac{2}{\delta^d} 3^d \frac{b}{a}$, we have checked

$$2\frac{b}{N} \leq \mu_{\mathbb{X}}(Q_\alpha^k) \leq \frac{C}{3^d}\frac{a}{N}. \tag{16}$$

For the cube generation k, we now build a graph with vertices I_k. For $\alpha, \beta \in I_k$, we put an edge (α, β) if and only if $B(z_\alpha^k, u_1\delta^k) \cap B(z_\beta^k, u_1\delta^k) \neq \emptyset$. This graph is connected, see [14, Proof of Theorem 2], so that we can extract a spanning tree with leaf nodes, intermediate nodes, and one root node. We create the directed tree $\mathcal{T}$ by directing the edges from the root towards the leaves, so that $(\alpha, \beta) \in \mathcal{T}$ is the directed edge between α and its child β.

The triangular inequality yields

$$Q_\alpha^k \cup \bigcup_{(\alpha,\beta)\in\mathcal{T}} Q_\beta^k \quad \subset \quad B(z_\alpha^k, 3u_2\delta^k),$$

cf. [14, Corollary 2]. Hence, we obtain the volume estimate

$$\mu_{\mathbb{X}}(Q_\alpha^k \cup \bigcup_{(\alpha,\beta)\in\mathcal{T}} Q_\beta^k) \le \mu_{\mathbb{X}}(B(z_\alpha^k, 3u_2\delta^k)) \le c_2(3u_2\delta^k)^d \le C\frac{a}{N}. \tag{17}$$

We now aim to take a younger generation of δ-adic cubes, say $l = k + m$, such that all cubes of generation l have measure smaller than $\frac{1}{C}\frac{a}{N}$. Indeed, let m be the positive integer such that

$$\delta^m \le 3C^{-2/d} < \delta^{m-1}. \tag{18}$$

Notice that $3C^{-2/d} < 1$, so that this choice is possible. Thus, for all $\alpha \in I_l$, we get from (18) and (14)

$$\begin{aligned}\mu_{\mathbb{X}}(Q_\alpha^l) \le \mu_{\mathbb{X}}(B(z_\alpha^l, u_2\delta^l)) &\le (c_2u_2^d\delta^{kd})\delta^{md} \\ &\le (c_2u_2^d\delta^{kd})3^dC^{-2} \\ &\le \frac{Ca}{3^dN}3^dC^{-2} \le \frac{1}{C}\frac{a}{N}.\end{aligned}$$

We now construct the partition by running through the directed tree $\mathcal{T}$ and using the above estimates, which are overkill for the leaves but are more appropriate for the remaining nodes. Let us denote the weights by $\Omega := \{\omega_j\}_{j=1}^N$.

Leaves

Start with a leaf node $\alpha \in I_k$. Take a maximal set of weights from Ω such that their sum is not bigger than $\mu_{\mathbb{X}}(Q_\alpha^k)$. Denote this maximal set with Ω_α and its cardinality with N_α. Each cube of generation l has measure at most $\frac{1}{C}\frac{a}{N}$, so that the volume of N_α cubes of generation l is bounded by

$$N_\alpha\frac{1}{C}\frac{a}{N} \le \frac{1}{C}\mu_{\mathbb{X}}(Q_\alpha^k) \le \frac{1}{C}\mu_{\mathbb{X}}\left(Q_\alpha^k \cup \bigcup_{(\alpha,\beta)\in T} Q_\beta^k\right) \le \frac{a}{N},$$

where we have used (17). According to (16), Q_α^k has sufficient volume that we can choose N_α cubes of generation l inside of Q_α^k. Let us denote them by $Q_{\beta_1}^l, \dots, Q_{\beta_{N_\alpha}}^l$. By Corollary 18, we enlarge each of such cubes within Q_α^k, so that their measure matches the weights in Ω_α, so that we obtain $\{R_{\beta_i}\}_{i=1}^{N_\alpha}$. The remainder in Q_α^k, i.e., $W_\alpha := Q_\alpha^k \setminus \bigcup_{i=1}^{N_\alpha} R_{\beta_i}$ has volume less than b/N, because we took the maximal number of weights.

We repeat the above steps for each leaf node but only allow weights in Ω that have not been chosen previously. After having finished all leaves, we have remainders $W_\alpha \subset Q_\alpha^k$, for each $\alpha \in I_k$ that corresponds to a leaf.

Intermediate nodes

For each $\alpha \in I_k$ that is neither a leaf nor the root, start with $X_\alpha = Q_\alpha^k \cup \bigcup_{(\alpha,\beta)\in\mathcal{T}} W_\beta$, that is we add all the remainders coming from the children of α. Note that we can proceed with the intermediate nodes in an ordering such that the remainders W_β with (α, β) have indeed all been already computed. Note also that we can assume $W_\beta \subset Q_\beta^k$, for all $(\alpha, \beta) \in \mathcal{T}$ (take this as an induction hypothesis. It is clearly true if β is a leaf node, and will follow at the end of this paragraph for the intermediate nodes). Now repeat the same argument as before with X_α in place of Q_α^k. Take a maximal set of the remaining weights from Ω such that their sum is not bigger than $\mu_{\mathbb{X}}(X_\alpha^k)$. Again, denote this maximal set with Ω_α and its cardinality with N_α. As we saw before, the entire volume of N_α cubes of generation l is at most a/N, so that they can be chosen within Q_α^k. Let us denote these cubes by $Q_{\beta_1}^l, \dots, Q_{\beta_{N_\alpha}}^l$. The volume of

$Q_\alpha^k \setminus \left(\bigcup_{i=1}^{N_\alpha} Q_{\beta_i}^l\right)$ is still at least b/N. According to Lemma 17, there is $W_\alpha \subset Q_\alpha^k \setminus \left(\bigcup_{i=1}^{N_\alpha} Q_{\beta_i}^l\right)$ with volume

$$\mu_{\mathbb{X}}(W_\alpha) = \mu_{\mathbb{X}}(X_\alpha) - \sum_{\omega \in \Omega_\alpha} \omega < b/N.$$

By Corollary 18, we extend the cubes $Q_{\beta_1}^l, \ldots, Q_{\beta_{N_\alpha}}^l$ within $X_\alpha \setminus W_\alpha$, so that the volumes match the weights in Ω_α, yielding subsets $\{R_{\beta_i}\}_{i=1}^{N_\alpha}$. By comparing volumes, the union of the extensions now covers the neighboring remainders W_β (at least up to a set of measure zero), and the new remainder W_α is indeed contained in Q_α^k.

We proceed with the remaining weights for each of the intermediate nodes in a suitable order.

Root

We do the same as for intermediate nodes but comparing volumes yields that the remainder of the root node must have measure zero.

After having treated each node in $\mathcal{T}$, we have collected a partition $\{R_j\}_{j=1}^N$, so that we obtain, with a suitable reordering, $\mu_{\mathbb{X}}(R_j) = \omega_j$, for $j = 1, \ldots, N$.

Since each R_j contains a cube of generation l, it contains a ball of radius $u_1\delta^l$. A short calculation yields $\delta^l \gtrsim \left(\frac{a^2}{b}\right)^{1/d} N^{-1/d}$. On the other hand, each R_j is contained in a ball of radius $3u_2\delta^k \lesssim b^{1/d} N^{-1/d}$.

Assume now that

$$1 \le N \le \frac{2b}{c_1 \delta^d \operatorname{diam}(\mathbb{X})^d}.$$

Let k be now the unique integer such that

$$c_2(u_2\delta^k)^d \le \frac{a}{N} < c_2(u_2\delta^{k-1})^d.$$

This implies that all the cubes of generation k have measure smaller than all the values ω_j. Then take any N distinct cubes of generation k and extend them by means of Corollary 18 to disjoint sets $\{R_j\}_{j=1}^N$ with measures ω_j, respectively. Each R_j contains its corresponding cube of generation k and therefore a ball with radius

$$u_1\delta^k > \frac{u_1\delta}{u_2 c_2^{1/d}} \frac{a^{1/d}}{N^{1/d}}.$$

On the other hand, every R_j is trivially contained in a (any) ball with radius

$$\operatorname{diam}(\mathbb{X}) \le \frac{2^{1/d}}{c_1^{1/d}\delta} \frac{b^{1/d}}{N^{1/d}}.$$

This concludes the proof of Proposition 16.

Remark 19. Proposition 16 and its proof hold for any complete, connected metric measure spaces that satisfy (12), cf. [14] for further details.

6. Marcinkiewicz–Zygmund inequalities

6.1. MZ-inequalities on Riemannian manifolds

The Marcinkiewicz–Zygmund inequality for diffusion polynomials on manifolds has been proved by Maggioni, Mhaskar and Filbir throughout the papers [11,12,19]. For our proof we need the Marcinkiewicz–Zygmund inequality for the gradient of diffusion polynomials. Note that when $\mathcal{M}$ is the d-dimensional sphere, then the gradient of a polynomial is again a polynomial. In the case of a general Riemannian manifold, this fails (see [13]). Here we prove a Marcinkiewicz–Zygmund inequality for gradients of diffusion polynomials in the case of a Riemannian manifold and with prefixed weights. Throughout this section, let $\mathcal{M}$, $\mu_{\mathcal{M}}$, $\Pi_L(\mathcal{M})$ and $\Pi_L^0(\mathcal{M})$ be as defined in Section 2.1.

Proposition 20. *Let $\mathcal{M}$ be as in Section 2.1 and let $0 < c_3 < c_4$. Then, there exists a constant $C = C_{\mathcal{M}}(c_3, c_4) \geq 1$ such that for all $0 < a \leq 1 \leq b$, for all integers $N \geq Cb(\frac{b}{a})^2 L^d$, for all partitions $\mathcal{R} = \{R_j\}_{j=1}^N \in \mathcal{P}(a, b, c_3, c_4)$, for all $x_j \in R_j$, for all $P \in \Pi_L^0(\mathcal{M})$ it holds*

$$\left| \int_{\mathcal{M}} \|\nabla P(x)\| d\mu_{\mathcal{M}}(x) - \sum_{j=1}^N \omega_j \|\nabla P(x_j)\| \right| \leq \frac{1}{2} \int_{\mathcal{M}} \|\nabla P(x)\| d\mu_{\mathcal{M}}(x), \tag{19}$$

where $\omega_j = \mu_{\mathcal{M}}(R_j)$, for all $j = 1, \ldots, N$.

Proof. This proof follows the sketch of the proof of [13, Theorem 5]. Fix $\varepsilon > 0$ and let $v_\varepsilon : [0, +\infty] \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function such that

$$v_\varepsilon(u) = \begin{cases} u & \text{if } u \geq \varepsilon \\ \varepsilon/2 & \text{if } u \leq \varepsilon/4 \end{cases} \tag{20}$$

and $v_\varepsilon(u) \geq u$ for all $u \geq 0$. Let $P \in \Pi_L^0(\mathcal{M})$ and let T and S be the vector fields defined as

$$T(x) = \frac{\nabla P(x)}{v_\varepsilon\left(\|\nabla P(x)\|\right)}, \qquad S(x) = \frac{\nabla T P(x)}{v_\varepsilon\left(\|\nabla T P(x)\|\right)}.$$

Therefore,

$$T P(x) = \left\langle \nabla P(x), \frac{\nabla P(x)}{v_\varepsilon\left(\|\nabla P(x)\|\right)} \right\rangle \tag{21}$$

$$S T P(x) = \left\langle \nabla T P(x), \frac{\nabla T P(x)}{v_\varepsilon\left(\|\nabla T P(x)\|\right)} \right\rangle. \tag{22}$$

We define also for every $L \geq 0$ the kernel W_L as

$$W_L(x, y) = \sum_{\lambda_k > 0} \frac{1}{\lambda_k^2} H\left(\frac{\lambda_k}{L}\right) \varphi_k(x) \varphi_k(y), \tag{23}$$

where H is a $\mathcal{C}^\infty$ even function such that

$$H(u) = \begin{cases} 1 & \text{if } u \in [-1, 1] \\ 0 & \text{if } |u| \geq 2. \end{cases}$$

Let $\Psi_L(x, y)$ be a reproducing kernel for $\Pi_L^0(\mathcal{M})$ defined as

$$\Psi_L(x, y) = \Delta_y W_L(x, y) = \sum_{0 < \lambda_k} H\left(\frac{\lambda_k}{L}\right) \varphi_k(x) \varphi_k(y). \tag{24}$$

Here and in the following formulas, the index y or x of the operators Δ, ∇, etc. specifies which variable the operator is acting on.

It has been proved in [13, proof of Theorem 5] that there exists a constant $\kappa > 0$ such that

$$\|\nabla_y S_x T_x W_L(x, y)\| \leq \kappa L^{d+1} \left(1 + L|x - y|\right)^{-d-1}. \tag{25}$$

Notice that

$$TP(x) = \left\langle \nabla P(x), \frac{\nabla P(x)}{v_\varepsilon(\|\nabla P(x)\|)} \right\rangle = \frac{\|\nabla P(x)\|^2}{v_\varepsilon(\|\nabla P(x)\|)} \le \|\nabla P(x)\|$$

and therefore

$$\begin{aligned}
&\left| \int_{\mathcal{M}} \|\nabla P(x)\| \, d\mu_{\mathcal{M}}(x) - \sum_{j=1}^{N} \omega_j \left\|\nabla P\left(x_j\right)\right\| \right| \\
&\le \left| \int_{\mathcal{M}} (\|\nabla P(x)\| - TP(x)) \, d\mu_{\mathcal{M}}(x) \right| + \left| \int_{\mathcal{M}} TP(x) \, d\mu_{\mathcal{M}}(x) - \sum_{j=1}^{N} \omega_j TP\left(x_j\right) \right| \\
&+ \left| \sum_{j=1}^{N} \omega_j \left(TP\left(x_j\right) - \left\|\nabla P\left(x_j\right)\right\|\right) \right| \\
&\le 2\varepsilon + \left| \int_{\mathcal{M}} TP(x) \, d\mu_{\mathcal{M}}(x) - \sum_{j=1}^{N} \omega_j TP\left(x_j\right) \right|.
\end{aligned}$$

Let δ be the maximum diameter of the balls X_j as in Definition 15, so $\delta \le 2c_4 b^{1/d} N^{-1/d}$. Hence,

$$\begin{aligned}
\left| \int_{\mathcal{M}} TP(x) \, d\mu_{\mathcal{M}}(x) - \sum_{j=1}^{N} \omega_j TP\left(x_j\right) \right| &\le \sum_{j=1}^{N} \int_{R_j} \left|TP(x) - TP\left(x_j\right)\right| d\mu_{\mathcal{M}}(x) \\
&\le \sum_{j=1}^{N} \omega_j \sup_{x,z \in R_j} |TP(x) - TP(z)| \\
&\le \sum_{j=1}^{N} \omega_j \sup_{x,z \in R_j} \sup_{t \in [0,|x-z|]} \|\nabla TP(\alpha(t))\| \, |x - z|.
\end{aligned}$$

where α is a normalized geodesic joining x and z. Since R_j is contained in the ball X_j, the geodesic α is contained in the ball $2X_j$ with the same center as X_j and radius twice the radius of X_j. It follows that

$$\left| \int_{\mathcal{M}} TP(x) \, d\mu_{\mathcal{M}}(x) - \sum_{j=1}^{N} \omega_j TP\left(x_j\right) \right| \le \delta \sum_{j=1}^{N} \omega_j \sup_{x \in 2X_j} \|\nabla TP(x)\|.$$

From Eq. (22) one has that

$$STP(x) = \frac{\|\nabla TP(x)\|^2}{v_\varepsilon(\|\nabla TP(x)\|)} \le \|\nabla TP(x)\|$$

and therefore

$$\delta \sum_{j=1}^{N} \omega_j \sup_{x \in 2X_j} \|\nabla TP(x)\|$$

$$\leq \delta \sum_{j=1}^{N} \omega_j \sup_{x \in 2X_j} \left| \|\nabla T P(x)\| - S T P(x) \right| + \delta \sum_{j=1}^{N} \omega_j \sup_{x \in 2X_j} |S T P(x)|$$

$$\leq \delta\varepsilon + \delta \sum_{j=1}^{N} \omega_j \sup_{x \in 2X_j} |S T P(x)|.$$

Hence we have obtained

$$\left| \int_{\mathcal{M}} \|\nabla P(x)\| \, d\mu_{\mathcal{M}}(x) - \sum_{j=1}^{N} \omega_j \left\|\nabla P\left(x_j\right)\right\| \right| \leq (2+\delta)\varepsilon + \delta \sum_{j=1}^{N} \omega_j \sup_{x \in 2X_j} |S T P(x)|.$$

We need to estimate the sum above. Notice that by Green's formula we have

$$\begin{aligned} P(x) &= \int_{\mathcal{M}} P(y)\, \Psi_L(x,y)\, d\mu_{\mathcal{M}}(y) = \int_{\mathcal{M}} P(y)\, \Delta_y W_L(x,y)\, d\mu_{\mathcal{M}}(y) \\ &= \int_{\mathcal{M}} \left\langle \nabla_y P(y), \nabla_y W_L(x,y) \right\rangle d\mu_{\mathcal{M}}(y), \end{aligned}$$

where W_L is defined in (23) and Ψ_L in (24). Therefore by (25) we have

$$\begin{aligned} &\delta \sum_{j=1}^{N} \omega_j \sup_{x \in 2X_j} |S T P(x)| \\ &= \delta \sum_{j=1}^{N} \omega_j \sup_{x \in 2X_j} \left| \int_{\mathcal{M}} \left\langle \nabla_y P(y), \nabla_y S_x T_x W_L(x,y) \right\rangle d\mu_{\mathcal{M}}(y) \right| \\ &\leq \delta \sum_{j=1}^{N} \omega_j \sup_{x \in 2X_j} \int_{\mathcal{M}} \left\| \nabla_y P(y) \right\| \left\| \nabla_y S_x T_x W_L(x,y) \right\| d\mu_{\mathcal{M}}(y) \\ &\leq \kappa\delta \sum_{j=1}^{N} \omega_j \sup_{x \in 2X_j} \int_{\mathcal{M}} \left\| \nabla_y P(y) \right\| L^{d+1} (1 + L|x-y|)^{-d-1} d\mu_{\mathcal{M}}(y) \\ &\leq \kappa\delta \int_{\mathcal{M}} \left\| \nabla_y P(y) \right\| \left(\sum_{j=1}^{N} L^{d+1} \omega_j \sup_{x \in 2X_j} (1 + L|x-y|)^{-d-1} \right) d\mu_{\mathcal{M}}(y). \end{aligned}$$

We reduce to estimate the sum in the integral. To do this, for any fixed y, let $J = \left\{ j : \operatorname{dist}\left(2X_j, y\right) \geq 2\delta \right\}$ and J' its complement. We start considering $j \in J$. If we call q_j the point in $2X_j$ closest to y, and p_j the point in $2X_j$ farthest from y, then

$$1 + \frac{L}{2}\left|p_j - y\right| \leq 1 + \frac{L}{2}\left(\left|q_j - y\right| + 2\delta\right) \leq 1 + L\left|q_j - y\right|$$

and therefore for the sum over J, we have

$$\begin{aligned} &\sum_{j \in J} L^{d+1} \omega_j \sup_{x \in 2X_j} (1 + L|x-y|)^{-d-1} \\ &= \sum_{j \in J} L^{d+1} \omega_j \left(1 + L\left|q_j - y\right|\right)^{-d-1} \leq \sum_{j \in J} L^{d+1} \omega_j \left(1 + \frac{L}{2}\left|p_j - y\right|\right)^{-d-1} \\ &= \sum_{j \in J} \int_{R_j} L^{d+1} \left(1 + \frac{L}{2}\left|p_j - y\right|\right)^{-d-1} d\mu_{\mathcal{M}}(x) \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j\in J}\int_{R_j} L^{d+1}\left(1+\frac{L}{2}\left|x-y\right|\right)^{-d-1} d\mu_{\mathcal{M}}\left(x\right)\\
&\leq \int_{\mathcal{M}} L^{d+1}\left(1+\frac{L}{2}\left|x-y\right|\right)^{-d-1} d\mu_{\mathcal{M}}\left(x\right)\\
&\leq c_5 L^{d+1}\int_0^{+\infty}\left(1+\frac{L}{2}s\right)^{-d-1} s^{d-1}ds\\
&\leq c_5 L^{d+1}\left(\int_0^{1/L} s^{d-1}ds+\left(\frac{2}{L}\right)^{d+1}\int_{1/L}^{+\infty} s^{-2}ds\right)\\
&\leq (d^{-1}+2^{d+1})c_5 L.
\end{aligned}
$$

Now we consider J'. We have that its cardinality is bounded above by the number of inner balls Y_j that are contained in the ball $B(y,4\delta)$, and this number is bounded above by the ratio

$$
\frac{\mu_{\mathcal{M}}(B(y,4\delta))}{\min_{j=1,\dots,N}\mu_{\mathcal{M}}(Y_j)} \leq \frac{8^d c_2 c_4^d b^2}{c_1 c_3^d a^2}.
$$

Thus, since $\frac{a}{N}\leq\omega_j\leq\frac{b}{N}$, and assuming $N\geq b^3a^{-2}L^d$,

$$
\sum_{j\in J'} L^{d+1}\omega_j \sup_{x\in 2X_j}\left(1+L\left|x-y\right|\right)^{-d-1} \leq \sum_{j\in J'} L^{d+1}\omega_j \leq \frac{8^d c_2 c_4^d b^3}{c_1 c_3^d a^2}\frac{L^{d+1}}{N} \leq \frac{8^d c_2 c_4^d}{c_1 c_3^d}L.
$$

We have obtained

$$
\begin{aligned}
&\left|\int_{\mathcal{M}}\left\|\nabla P\left(x\right)\right\| d\mu_{\mathcal{M}}\left(x\right)-\sum_{j=1}^N\omega_j\left\|\nabla P\left(x_j\right)\right\|\right|\\
&\leq\left(2+\delta\right)\varepsilon+\kappa\left((d^{-1}+2^{d+1})c_5+\frac{8^d c_2 c_4^d}{c_1 c_3^d}\right)\delta L\int_{\mathcal{M}}\left\|\nabla P\left(y\right)\right\| d\mu_{\mathcal{M}}\left(y\right).
\end{aligned}
$$

If we take

$$
\varepsilon=\frac{\kappa\left((d^{-1}+2^{d+1})c_5+\frac{8^d c_2 c_4^d}{c_1 c_3^d}\right)\delta L}{2+\delta}\int_{\mathcal{M}}\left\|\nabla P\left(y\right)\right\| d\mu_{\mathcal{M}}\left(y\right),
$$

we obtain

$$
\left|\int_{\mathcal{M}}\left\|\nabla P\left(x\right)\right\| d\mu_{\mathcal{M}}\left(x\right)-\sum_{j=1}^N\omega_j\left\|\nabla P\left(x_j\right)\right\|\right|\leq\tilde{C}b^{1/d}LN^{-1/d}\int_{\mathcal{M}}\left\|\nabla P\left(y\right)\right\| d\mu_{\mathcal{M}}\left(y\right),
$$

where

$$
\tilde{C}=4c_4\kappa\left((d^{-1}+2^{d+1})c_5+\frac{8^d c_2 c_4^d}{c_1 c_3^d}\right). \tag{26}
$$

Assuming now $N\geq 2^d\tilde{C}^d bL^d$ one obtains (19). The Proposition now follows with $C=\max\{1,2^d\tilde{C}^d\}$. $\square$

6.2. MZ-inequalities on algebraic manifolds

For an algebraic manifold with fixed weights, the Marcinkiewicz–Zygmund inequality does not follow from [11,12,19] as in Section 6.1 since we are not dealing with diffusion polynomials but with algebraic polynomials. Instead, one can apply arguments that use complexification of the variety $\mathcal{V}$ as in [10]. Throughout this section, let $\mathcal{V}$, $\mu_{\mathcal{V}}$, $\Pi_L(\mathcal{V})$ and $\Pi_L^0(\mathcal{V})$ be as defined in Section 2.2. The following proposition states the Marcinkiewicz–Zygmund inequality for algebraic polynomials and their gradients.

Proposition 21. *Let $\mathcal{V}$ be as in Section 2.2 and let $0 < c_3 < c_4$. Then, there exists a constant $C = C_{\mathcal{V}}(c_3, c_4) \geq 1$ such that for all $0 < a \leq 1 \leq b$, for all integers $N \geq Cb(\frac{b}{a})^{2d} L^d$, for all partitions $\{R_j\}_{j=1}^N \in \mathcal{P}(a, b, c_3, c_4)$, for all $x_j \in R_j$, for all $P \in \Pi_L^0(\mathcal{V})$ it holds*

$$\Big|\int_{\mathcal{V}} |P(x)| d\mu_{\mathcal{V}}(x) - \sum_{j=1}^N \omega_j |P(x_j)|\Big| \leq \frac{1}{2}\int_{\mathcal{V}} |P(x)| d\mu_{\mathcal{V}}(x), \tag{27}$$

$$\Big|\int_{\mathcal{V}} |\nabla P(x)| d\mu_{\mathcal{V}}(x) - \sum_{j=1}^N \omega_j |\nabla P(x_j)|\Big| \leq \frac{1}{2}\int_{\mathcal{V}} |\nabla P(x)| d\mu_{\mathcal{V}}(x), \tag{28}$$

where $\omega_j = \mu_{\mathcal{V}}(R_j)$ for all $j = 1, \dots, N$.

Proof of Proposition 21. We follow the equal weight case in [10] with minor technical modifications. We start with (27). Triangle and reverse triangle inequalities lead to

$$\begin{aligned}\Big|\int_{\mathcal{V}} |P(x)| d\mu_{\mathcal{V}}(x) - \sum_{j=1}^N \omega_j |P(x_j)|\Big| &= \Big|\sum_{j=1}^N \int_{R_j} |P(x)| d\mu_{\mathcal{V}}(x) - \sum_{j=1}^N \int_{R_j} |P(x_j)| d\mu_{\mathcal{V}}(x)\Big| \\ &\leq \sum_{j=1}^N \int_{R_j} \big||P(x)| - |P(x_j)|\big| d\mu_{\mathcal{V}}(x) \\ &\leq \sum_{j=1}^N \int_{R_j} \big|P(x) - P(x_j)\big| d\mu_{\mathcal{V}}(x).\end{aligned}$$

There are $x'_j \in 2X_j$, $j = 1, \dots, N$, such that $\|\nabla P(x'_j)\| \geq \|\nabla P(x)\|$, for all $x \in 2X_j$. Since $\operatorname{diam}(X_j) \leq 2c_4 b^{1/d} N^{-1/d}$,

$$\begin{aligned}\Big|\int_{\mathcal{V}} |P(x)| d\mu_{\mathcal{V}}(x) - \sum_{j=1}^N \omega_j |P(x_j)|\Big| &\leq \sum_{j=1}^N \omega_j \operatorname{diam}(2X_j) \|\nabla P(x'_j)\| \\ &\leq 2c_4 b^{1/d} N^{-1/d} \sum_{j=1}^N \omega_j \|\nabla P(x'_j)\|.\end{aligned}$$

Let $\mathbb{Y}$ denote the complexification of $\mathcal{V}$, which consists of the complex zeros of the ideal defining $\mathcal{V}$. Assume $N \geq 8^d c_4^d b L^d$, so that $2X_j \subset B(x'_j, L^{-1})$. From [1, Section 2.4], see also [10], we know that since $P \in \Pi_L^0(\mathcal{V})$,

$$\|\nabla P(x'_j)\| \lesssim L^{2d+1} \int_{B_{\mathbb{Y}}(x'_j, L^{-1})} |P(z)| d\mu_{\mathbb{Y}}(z),$$

where $B_{\mathbb{Y}}(x'_j, L^{-1})$ denotes the ball in $\mathbb{Y}$ of radius L^{-1} centered at x'_j and $\mu_{\mathbb{Y}}$ denotes the measure on the complexification.

If π is any permutation of $\{1, \ldots, N\}$, for which

$$\bigcap_{i=1}^{m} B_{\mathbb{Y}}(x'_{\pi(i)}, L^{-1}) \neq \emptyset,$$

then, as in [10], for a constant $\beta \geq 1$ depending only on $\mathcal{V}$,

$$\bigcap_{i=1}^{m} B(x'_{\pi(i)}, \beta L^{-1}) \neq \emptyset.$$

Let z_π be a point in this intersection. A volume comparison implies

$$m \leq \frac{\mu_{\mathcal{V}}(B(z_\pi, 2\beta L^{-1}))}{\min_{i=1,\ldots,m}\{\mu_{\mathcal{V}}(Y_{\pi(i)})\}} \leq \frac{2^d c_2 \beta^d b N}{c_1 c_3^d a^2 L^d}.$$

Therefore, we derive

$$\sum_{j=1}^{N} \int_{B_{\mathbb{Y}}(x'_j, L^{-1})} |P(z)| d\mu_{\mathbb{Y}}(z) \leq \frac{2^d c_2 \beta^d b N}{c_1 c_3^d a^2 L^d} \int_{\bigcup_{j=1}^{N} B_{\mathbb{Y}}(x'_j, L^{-1})} |P(z)| d\mu_{\mathbb{Y}}(z)$$

According to [10, Lemma 3.1], this leads to

$$\sum_{j=1}^{N} \int_{B_{\mathbb{Y}}(x'_j, L^{-1})} |P(z)| d\mu_{\mathbb{Y}}(z) \lesssim \frac{2^d c_2 \beta^d b}{c_1 c_3^d a^2} \frac{N}{L^{2d}} \int_{\mathcal{V}} |P(x)| d\mu_{\mathcal{V}}(x),$$

so that we obtain

$$\sum_{j=1}^{N} \omega_j \|\nabla P(x'_j)\| \lesssim \frac{2^d c_2 \beta^d b^2}{c_1 c_3^d a^2} L \int_{\mathcal{V}} |P(x)| d\mu_{\mathcal{V}}(x).$$

Putting all this together, we derive

$$\begin{aligned} \left| \int_{\mathcal{V}} |P(x)| d\mu_{\mathcal{V}}(x) - \sum_{j=1}^{N} \omega_j |P(x_j)| \right| &\lesssim 2c_4 b^{1/d} N^{-1/d} \frac{2^d c_2 \beta^d b^2}{c_1 c_3^d a^2} L \int_{\mathcal{V}} |P(x)| d\mu_{\mathcal{V}}(x) \\ &\lesssim \frac{2^{d+1} c_2 c_4 \beta^d b^{2+1/d}}{c_1 c_3^d a^2} L N^{-1/d} \int_{\mathcal{V}} |P(x)| d\mu_{\mathcal{V}}(x) \\ &\leq \frac{1}{2} \int_{\mathcal{V}} |P(x)| d\mu_{\mathcal{V}}(x) \end{aligned}$$

as long as one adjusts the constant in the assumption $N \gtrsim b(\frac{b}{a})^{2d} L^d$.

The inequality (28) follows from (27) as in [10]. We omit the details. □

7. Gradient flow

We now apply Propositions 20 and 21 to verify Lemma 14.

Proof of Lemma 14. Here we follow closely the proof from [2, Section 4]. According to Proposition 16 there exists a partition of $\mathbb{X}$ in $\mathcal{P}(a, b, c_3, c_4)$, $\mathcal{R} = \{R_j\}_{j=1}^N$, such that $\mu_{\mathcal{M}}(R_j) = \omega_j$ for all $j = 1, \ldots, N$. Recall that $h = 1$ if $\mathbb{X} = \mathcal{M}$ and $h = d$ if $\mathbb{X} = \mathcal{V}$.

Let now $N \geq C_{\mathbb{X}}(c_3, 13c_4)b(b/a)^{2h}L^d$, where $C_{\mathbb{X}}(\cdot,\cdot)$ is as in Propositions 20 and 21. We start choosing an arbitrary $x_j \in R_j$ for all $1 \leq j \leq N$ and consider the map $U : \Pi_L^0(\mathbb{X}) \to \mathcal{X}(\mathbb{X})$

$$U(P)(y) = \frac{\nabla P(y)}{v_\varepsilon\left(\|\nabla P(y)\|\right)},$$

where $\mathcal{X}(\mathbb{X})$ is the space of differentiable vector fields on $\mathbb{X}$ and v_ε is as in Eq. (20). For each $1 \leq j \leq N$ let $y_j : \Pi_L^0(\mathbb{X}) \times [0, +\infty) \to \mathbb{X}$ be the map satisfying the differential equation

$$\begin{cases} \dfrac{d}{dt} y_j(P,t) = U(P)(y_j(P,t)) \\ y_j(P,0) = x_j \end{cases}$$

for each $P \in \Pi_L^0(\mathbb{X})$. For every $P \in \Pi_L^0(\mathbb{X})$ and for every j, the map $t \to y_j(P,t)$ is defined and smooth on the whole real line (see [24, Theorem 6, p. 147]). Furthermore, $U(P)(y)$ is Lipschitz continuous with respect to P. It follows that for each j the map $P \to y_j(P,\cdot)$ is continuous in P (see [18, Corollary 1.6, p. 68]). Now set

$$F(P) = (x_1(P), \ldots, x_N(P)) = \left(y_1\left(P, 12c_4 b^{\frac{1}{d}} N^{-\frac{1}{d}}\right), \ldots, y_N\left(P, 12c_4 b^{\frac{1}{d}} N^{-\frac{1}{d}}\right)\right).$$

By the above considerations, we have that F is continuous on $\Pi_L^0(\mathbb{X})$. Let us take $P \in \Pi_L^0(\mathbb{X})$ such that

$$\int_{\mathbb{X}} \|\nabla P(x)\| d\mu_{\mathbb{X}}(x) = 1,$$

which means $P \in \partial\Omega$, where Ω is defined as (11). Then we can split

$$\begin{aligned}\sum_{j=1}^N \omega_j P(x_j(P)) &= \sum_{j=1}^N \omega_j P\left(y_j\left(P, 12c_4 b^{\frac{1}{d}} N^{-\frac{1}{d}}\right)\right) \\ &= \sum_{j=1}^N \omega_j P(x_j) + \int_0^{12c_4 b^{\frac{1}{d}} N^{-\frac{1}{d}}} \frac{d}{dt}\left(\sum_{j=1}^N \omega_j P(y_j(P,t))\right) dt.\end{aligned}$$

Observe first that

$$\begin{aligned}\left|\sum_{j=1}^N \omega_j P(x_j)\right| &= \left|\sum_{j=1}^N \int_{R_j} (P(x_j) - P(x)) d\mu_{\mathbb{X}}(x)\right| \leq \sum_{j=1}^N \int_{R_j} \left|P(x_j) - P(x)\right| d\mu_{\mathbb{X}}(x) \\ &\leq \sum_{j=1}^N \omega_j \operatorname{diam}(R_j) \max_{z \in 2X_j} \|\nabla P(z)\| \leq \frac{2c_4 b^{\frac{1}{d}}}{N^{\frac{1}{d}}} \sum_{j=1}^N \omega_j \|\nabla P(z_j)\|,\end{aligned}$$

where z_j is the point that realizes the maximum and X_j is the geodesic ball described in Definition 15. We consider now the partition $\mathcal{R}' = \{R_1', \ldots, R_N'\} \in \mathcal{P}(a, b, c_3, 2c_4)$ where $R_j' = R_j \cup \{z_j\}$. Notice that this is a partition of $\mathbb{X}$ and $\mu_{\mathcal{M}}(R_j') = \omega_j$ for all $j = 1, \ldots, N$. Therefore, by Propositions 20 and 21 applied to $P \in \partial\Omega$ and the partition $\mathcal{R}'$, we have

$$\left|\sum_{j=1}^N \omega_j P(x_j)\right| \leq \frac{2c_4 b^{\frac{1}{d}}}{N^{\frac{1}{d}}} \sum_{j=1}^N \omega_i \|\nabla P(z_j)\|$$

$$\leq \frac{2c_4 b^{\frac{1}{d}}}{N^{\frac{1}{d}}} \left| \sum_{j=1}^{N} \omega_i \|\nabla P(z_j)\| - \int_{\mathbb{X}} \|\nabla P(z)\| d\mu_{\mathbb{X}}(z) \right| + \frac{2c_4 b^{\frac{1}{d}}}{N^{\frac{1}{d}}} \int_{\mathbb{X}} \|\nabla P(z)\| d\mu_{\mathbb{X}}(z)$$

$$\leq \frac{3c_4 b^{\frac{1}{d}}}{N^{\frac{1}{d}}} \int_{\mathbb{X}} \|\nabla P(z)\| d\mu_{\mathbb{X}}(z) = \frac{3c_4 b^{\frac{1}{d}}}{N^{\frac{1}{d}}}.$$

Furthermore, for $t \in [0, 12c_4 b^{\frac{1}{d}} N^{-\frac{1}{d}}]$, we have

$$\frac{d}{dt}\left(\sum_{j=1}^{N} \omega_j P(y_j(P,t))\right) = \sum_{j=1}^{N} \omega_j \frac{\|\nabla P(y_j(P,t))\|^2}{v_\epsilon(\|\nabla P(y_j(P,t))\|)}$$

$$\geq \sum_{j:\|\nabla P(y_j(P,t))\| \geq \epsilon} \omega_j \|\nabla P(y_j(P,t))\| \geq \sum_{j=1}^{N} \omega_j \|\nabla P(y_j(P,t))\| - \epsilon.$$

Since $|y_j(P,t) - x_j| \leq t$, the partition $\mathcal{R}'' = \{R_1'', \ldots, R_N''\} \in \mathcal{P}(a, b, c_3, 13c_4)$ where $R_j'' = R_j \cup \{y_j(P,t)\}$ is a partition of $\mathbb{X}$ and $\mu_{\mathbb{X}}(R_j'') = \omega_j$ for all $j = 1, \ldots, N$. We can therefore now apply Propositions 20 and 21 to P and the new partition $\mathcal{R}''$:

$$\frac{d}{dt}\left(\sum_{j=1}^{N} \omega_j P(y_j(P,t))\right) \geq \sum_{j=1}^{N} \omega_j \|\nabla P(y_j(P,t))\| - \epsilon$$

$$\geq \int_{\mathbb{X}} \|\nabla P(y)\| d\mu_{\mathbb{X}}(y) - \left| \int_{\mathbb{X}} \|\nabla P(y)\| d\mu_{\mathbb{X}}(y) - \sum_{j=1}^{N} \omega_j \|\nabla P(y_j(P,t))\| \right| - \epsilon$$

$$\geq \frac{1}{2} \int_{\mathbb{X}} \|\nabla P(y)\| d\mu_{\mathbb{X}}(y) - \epsilon = \frac{1}{2} - \epsilon,$$

for every $P \in \partial\Omega$ and for every $t \in [0, 12c_4 b^{\frac{1}{d}} N^{-\frac{1}{d}}]$. In conclusion we obtain

$$\sum_{j=1}^{N} \omega_j P(x_j(P)) = \sum_{j=1}^{N} \omega_j P(x_j) + \int_0^{12c_4 b^{\frac{1}{d}} N^{-\frac{1}{d}}} \frac{d}{dt}\left(\sum_{j=1}^{N} \omega_j P(y_j(P,t))\right)$$

$$\geq \frac{12c_4 b^{\frac{1}{d}}}{N^{\frac{1}{d}}} \left(\frac{1}{2} - \epsilon\right) - \frac{3c_4 b^{\frac{1}{d}}}{N^{\frac{1}{d}}} = (3 - 12\epsilon) \frac{c_4 b^{\frac{1}{d}}}{N^{\frac{1}{d}}} > 0. \quad \square$$

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