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Gravity Anomaly of Polyhedral Bodies Having a Polynomial Density Contrast

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Abstract We analytically evaluate the gravity anomaly associated with a polyhedral body having an arbitrary geometrical shape and a polynomial density contrast in both the orizontal and vertical directions. The gravity anomaly is evaluated at an arbitrary point that does not necessarily coincide with the origin of the reference frame in which the density function is assigned. Density contrast is assumed to be a third-order polynomial as a maximum but the general approach exploited in the paper can be easily extended to higher-order polynomial functions. Invoking recent results of potential theory, the solution derived in the paper is shown to be singularity-free and is expressed as sum of algebraic quantities that only depend upon the 3D coordinates of the polyhedron vertices and upon the polynomial density function. The accuracy, robustness and effectiveness of the proposed approach is illustrated by numerical comparisons with examples derived from the existing literature.

Keywords Gravity anomaly · Polyhedral bodies · Polynomial density contrast · Singularity

1 Introduction

Gravity is an economic tool for exploring and discovering natural resources (Jacoby and Smilde, 2009). In this respect density is one of the most diagnostic physical property of a mineral deposit, and is also fundamental to oil and gas exploration. To date, density has been one of the most difficult property to measure and infer.

During the last decade, there has been significant development in gravity survey, particularly with the advent of GPS and gravity gradiometry. In conventional gravity survey, Earth's gravity acceleration is measured using gravimeter whereas in gravity gradiometer survey, the gravity gradient or how the gravitational acceleration changes over distance (or in some cases time) is measured.

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Recent reviews (LaFehr, 1980; Paterson and Reeves, 1985; Hansen, 2001) document the continuous evolution of instruments, field operations, data-processing techniques, and methods of interpretation. A steady progression in instrumentation (torsion balance, gravimeters based on land or underwater, in boreholes or on board satellites, aircraft or marine vessels, modern versions of absolute gravimeters, and gravity gradiometers) has enabled the acquisition of gravity data in nearly all environments, see, e.g., Nabighian (2005) for a quite recent historical account.

Despite being eclipsed by seismology, it is impressive to realize that about 40 different commercial gravity sensors and gravity gradiometers are available (Chapin, 2008) and about 30 different gravity sensor and gravity gradiometers designs have either been proposed or developed. In particular, gravity gradiometry is still used in exploration (Dransfield, 2007) and for regional gravity mapping (Jekeli, 2006).

Gravity data sets are effectively used to estimate locations and shapes of bodies, embedded in Earth, exhibiting anomalous mass density with respect to a constant reference value (Zhang et al., 2014). More refined Earth models can be obtained by inverting gravity data (Li and Oldenburg, 1998; Zhdanov, 2002) in conjuction with seismic and electro-magnetic induction data (Moorkamp et al., 2011; Aydemir et al., 2014; Roberts et al., 2016).

Recent improvements in gravimeter efficiency and inversion algorithms have increased the possibility of collecting and inverting huge data sets over extended areas in order to derive 3D density models (Kamm et al., 2015). In particular, gravity methods are extensively used in geoid determination (Bajracharya and Sideris, 2004) and mineral exploration (Beiki and Pedersen, 2010; Martinez et al., 2013; Abtahi et al., 2016).

In conclusion it is of paramount importance to efficiently evaluate the gravity anomaly associated with a body characterized by complex density distributions since this represents an important task in forward modelling and inversion.

Due to the mathematical complexity of the problem, the gravity anomaly of an irregular body whose density contrast is spatially variable has been first computed by approximating the body as a collection of vertical rectangular parallelepipeds (prisms) in which the density is assumed to be constant.

Numerical computations were first carried out by Talwani et al. (1959) and Bott (1960). Closed form expressions of the gravity anomaly were subsequently derived by Nagy (1966), Banerjee and Das Gupta (1977), Cady (1980), Nagy et al. (2000), Tsoulis (2000), Jiancheng and Wenbin (2010), D'Urso (2012), see also Plouff (1975, 1976), Won and Bevis (1987), Montana et al. (1992) for computer codes. The case of spheroidal shell has been addressed by Johnson and Litehiser (1972). Analytical expressions of the gravity anomaly for prisms have been derived by D'Urso (2016), for a linearly varying density, by Rao (1985, 1986, 1990), Rao et al. (1994), Gallardo-Delgado et al. (2003) for a quadratic density contrast, by García-Abdeslem (1992, 2005), for a cubic density variation with depth. A good collection of earlier references for 3D prisms can be found in Li and Chouteau (1998) who name, among others, a formula contributed in Sorokin (1951).

Non-polynomial density-contrast models for 3D bodies have been considered by Cordell (1973), Chai and Hinze (1988), Litinsky (1989), Rao et al. (1990), Chakravarthi et al. (2002), Silva et al. (2006), Chakravarthi and Sundararajan (2007), Chappell and Kusznir (2008), Zhou (2009b) and, for 2D bodies, by Gendzwill (1970), Murthy and Rao (1979), Pan (1989), Guspí (1990), Ruotoistenmäki (1992), Martín-Atienza and García-Abdeslem (1999), Zhang et al. (2001), Zhou (2008, 2009a, 2010). For more complicated forms of the density contrast, see, e.g., Cai and Wang (2005) and Mostafa (2008).

Alternative to the use of prisms, characterized by complicated functions describing density contrast, is the case of polyhedrons endowed with a a simple description of density contrast. Analytical formulas for the gravimetric analysis of polyhedra having constant density have been contributed by Paul (1974), Barnett (1976), Strakhov (1978), Okabe (1979), Waldvogel (1979), Golizdra (1981), Strakhov et al. (1986), Götze and Lahmeyer (1988), Pohanka (1988), Murthy et al. (1989), Kwok (1991b), Werner (1994), Holstein and Ketteridge (1996), Petrović (1996), Werner and Scheeres (1997), Li and Chouteau (1998), Tsoulis (2012), D'Urso (2013a, 2014a), Conway (2015), Werner (2017). Subsequent advancements have been only concerned with a linear density variation, (Pohanka, 1998; Hansen, 1999; Holstein, 2003; Hamayun et al., 2009; D'Urso, 2014b); actually, handling more complex density functions in conjunction with polyhedral models considerably increases the difficulties of the treatment, especially if analytical solutions are looked for.

For 2D bodies having density contrast depending only on depth, Zhou (2008) converted the original domain integral for gravity anomaly to a Line Integral (LI) by using Stokes theorem. In particular he derived two types of LIs for computing the gravity anomaly of bodies. In a subsequent paper (Zhou, 2009a) the author extended his method to account for density contrast functions which depended not only on depth but also on horizontal or, jointly, on horizontal and vertical directions. The gravity anomaly at observation points different from the origin has been evaluated in Zhou (2010) since, historically, gravity anomaly was computed only at the origin of the reference frame. In the same paper, Zhou dealt with the singularity of the gravity anomaly arising where the observation point is coincident with the vertices of the integration domain, an issue already discussed in Kwok (1991a), for prism-based modelling, and Tsoulis and Petrović (2001) for polyhedra.

The first approach for evaluating the gravity anomaly of bodies characterized by a complicated density contrast, even in presence of two-dimensional domains, has been either numerical or of semi-analytical nature based on the use of prisms, (Murthy and Rao, 1979; Rao et al., 1990; Chakravarthi et al., 2002; Chakravarthi and Sundararajan, 2007; Zhou, 2009b), or with 2D geometrical shapes, (Gendzwill, 1970; Murthy and Rao, 1979; Pan, 1989; Guspí, 1990; Ruotoistenmäki, 1992; Martín-Atienza and García-Abdeslem, 1999; Zhang et al., 2001; Zhou, 2008, 2009a, 2010). Actually, this last geometrical assumption, which can be used to model domains extending towards infinity in one direction, significantly simplifies the mathematical treatment of the problem.

Nevertheless, starting from the first researches on the subject (Hubbert, 1948), all authors have systematically transformed the original domain integrals into integrals of lower dimension in order to simplify the adoption of quadrature rules for the numerical evaluation of the gravity anomaly.

The derivation of analytical expressions for the gravity anomaly of polygonal bodies has been achieved only recently (D'Urso, 2015c) by exploiting the generalized Gauss theorem first presented in D'Urso (2012, 2013a), and subsequently applied to several problems ranging from geodesy (D'Urso, 2014a,b; D'Urso and Trotta, 2015b; D'Urso, 2016), to geomechanics (D'Urso and Marmo, 2009; Sessa and D'Urso, 2013; D'Urso and Marmo, 2015a), to geophysics (D'Urso and Marmo, 2013b), elasticity (Marmo and Rosati, 2016; Marmo et al., 2016a,b, 2017; Trotta et al., 2016a,b) and to heat transfer (Rosati and Marmo, 2014).

The methodology outlined in D'Urso (2015c) is here generalized in order to derive an analytical expression of the gravity anomaly for polyhedral bodies having density contrast expressed as a polynomial function of arbitrary degree in both the horizontal and vertical directions, an issue recently addressed in Ren et al. (2017). The result is obtained by first reducing the original domain integral to a 2D boundary integral by virtue of the generalized Gauss theorem. Remarkably, this also allows one to prove that the boundary integral expression of the gravity anomaly is singularity free whatever is the position of the observation point with respect to the body.

Being Ω polyhedral, the 2D expression of the gravity anomaly is written as finite sum of 2D integrals extended to the faces of Ω. By a further application of the generalized Gauss theorem each face integral is reduced to the sum of 1D integrals extended to the edges of the face. Such 1D integrals are analytically evaluated as products between the position vectors of the end vertices of each edge and scalar coefficients providing the analytical value of integrals of real variable.

Although these last integrals may exhibit a singularity when the projection of the observation point onto a face belongs to an edge, it is proved that such a singularity produces a null contribution of the i-th edge to the general expression of gravity anomaly; hence, one infers that the derived expression is singularity-free.

By exploiting a suitable change of variables, we also derive an enhanced algebraic formula which expresses the gravity anomaly at an arbitrary point P and specializes to the ordinary one when $P = O$. Remarkably, the enhanced expression of the gravity anomaly has been derived without any modification of the density contrast function since this is still defined in the original reference frame. The enhanced formula has been implemented in a MATLAB code, and its accuracy and robustness has been assessed by numerical comparisons with examples derived from the literature.

2 Gravity Anomaly of Polyhedral Bodies at the Origin O of the Reference Frame

Let us consider a Cartesian reference frame having origin at an arbitrary point O and a polyhedral body Ω. We shall assume that the density $\Delta\rho$ of the body, usually denominated density contrast, is a function of the generic point whose position with respect to O is defined by the vector $\mathbf{r}$. The symbol $\Delta\rho$ emphasizes the fact that the density of Ω is a variation with respect to that of the surrounding medium.

Denoting by G the gravitational constant, we shall first evaluate the gravity anomaly at O; it is defined by

$$\Delta\mathbf{g}(O) = G\int_{\Omega} \frac{\Delta\rho(\mathbf{r})\,\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,dV \tag{1}$$

and the integrand function represents the magnitude of attraction on a unit mass at O arising from the infinitesimal mass $\Delta\rho dV$.

We remark that the denomination of gravity anomaly adopted to denote equation (1), though not strictly correct, is based on a common practice in the specialized literature. Actually, equation (1) is a formula for the gravitational attraction of a mass body and may be approximatively seen as the formula for the influence of a mass body on the gravity anomaly since, for small bodies, the effect on gravity is the dominant part of the effect on the gravity anomaly.

An in-depth discussion on this topic is reported in Vaníček et al. (2004) where the interested reader can find an example of how the effect of a mass body on the gravity anomaly can be formulated in a theoretically consistent manner.

The vertical component of the gravity anomaly at O is provided by

$$\Delta g_z(O) = G\int_{\Omega} \frac{\Delta\rho(\mathbf{r})\mathbf{r}\cdot\mathbf{k}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,dV\,, \tag{2}$$

$\mathbf{k}$ being the unit vector directed along the vertical axis. The evaluation of Δg_z at an arbitrary point P will be addressed in section 3 since a considerably more elaborate expression is arrived at.

It is usually of interest to dispose of a procedure to actually compute Δg_z since most gravimeters can only measure the vertical component of the gravity field. Nevertheless the procedure detailed in the paper can be equally applied to all components of (1) and to physical problems governed by the Poisson equation (Blakely, 2010).

The computation of the integral in (2) is a hard task since the density contrast function $\Delta\rho$ does usually have a very complicated expression for the necessity of modelling 3D anomalies of Earth. For simplicity this can be modeled as an ensemble of 3D anomalies in a layered medium or a sequence of strata with horizontally undulated interfaces, e.g., sedimentary basins and underlying bedrock. In each layer mass density typically exhibits depth-dependent variations (García-Abdeslem, 1992).

However geological processes of exogenetic (fluvial, coastal, glacial,...) and endogenetic (rock diagenesis, plate tectonics, volcano eruptions, earthquakes,...) nature can induce both horizontal and vertical variations in mass density (Martín-Atienza and García-Abdeslem, 1999). Thus, a suitable expression of the density variation can allow for potentially faithful representations of the Earth subsurface with a relatively smaller amount of computations and parameters. Additionally, disposing of analytical expressions of the gravity anomaly associated with complicated expressions $\Delta\rho$ can be useful for benchmarking numerical approaches.

A quite general expression for $\Delta\rho$, able to accommodate a large variety of geological formations, is given by a triple polynomial in x, y and z, (García-Abdeslem, 2005; Zhou, 2009b; Ren et al., 2017)

$$\Delta\rho(\mathbf{r}) = \theta(x,y,z) = \sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\sum_{k=0}^{N_z} c_{ijk}x^i y^j z^k \tag{3}$$

where N_x, N_y and N_z represent the maximum power of the polynomial density variation along x, y and z respectively. In the sequel we shall confine the treatment to the case

$$N_x + N_y + N_z = 3 \tag{4}$$

since this will suffice to address the majority of the practical applications and, at the same time, to present our formulation at a degree of generality sufficient to be generalized to the cases $N_x + N_y + N_z > 3$.

Thus, under the assumption (4), equation (3) specializes to

$$\begin{aligned}\theta(\mathbf{r}) = c_{000} &+ c_{100}x + c_{010}y + c_{001}z + \\ &+ c_{200}x^2 + c_{020}y^2 + c_{002}z^2 + c_{110}xy + c_{011}yz + c_{101}xz + \\ &+ c_{300}x^3 + c_{030}y^3 + c_{003}z^3 + c_{210}x^2y + c_{021}y^2z + c_{102}xz^2 + \\ &+ c_{120}xy^2 + c_{012}yz^2 + c_{201}x^2z + c_{111}xyz\,.\end{aligned} \tag{5}$$

The scalars c_{ijk} represent the coefficients of the polynomial law; they can be estimated from the known data points by a least-square approach (Jacoby and Smilde, 2009).

Paralleling the analogous treatment developed in D'Urso (2015c), we first reformulate the general expression (3) of the density contrast by writing

$$\theta(\mathbf{r}) = \theta_{\mathbf{o}} + \mathbf{c}\cdot\mathbf{r} + \mathbf{C}\cdot\mathbf{D}_{\mathbf{rr}} + \mathbb{C}\cdot\mathbb{D}_{\mathbf{rrr}} \tag{6}$$

where θ_o is a scalar denoting the density at $\mathbf{o} = (0,0,0)$, $\mathbf{c}$ is a vector, $\mathbf{C}$ and $\mathbf{D_{rr}}$ are symmetric second-order tensors, $\mathbb{C}$ and $\mathbb{D}_{\mathbf{rrr}}$ are third-order tensors; furthermore, it has been set

$$\mathbf{D_{rr}} = \mathbf{r}\otimes\mathbf{r} \qquad \mathbb{D}_{\mathbf{rrr}} = \mathbf{r}\otimes\mathbf{r}\otimes\mathbf{r}\,. \tag{7}$$

The second-order (rank-two) tensor $\mathbf{r}\otimes\mathbf{r}$ has the following matrix representation

$$[\mathbf{r}\otimes\mathbf{r}] = \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix}, \tag{8}$$

so that, being:

$$\mathbf{C}\cdot(\mathbf{r}\otimes\mathbf{r}) = C_{11}x^2 + 2C_{12}xy + 2C_{13}xz + C_{22}y^2 + 2C_{23}yz + C_{33}z^2\,, \tag{9}$$

a quadratic distribution of density can be assigned by suitably defining the coefficients of the symmetric tensor $\mathbf{C}$. Analogously, the third-order tensors $\mathbb{C}$ and $\mathbf{r}\otimes\mathbf{r}\otimes\mathbf{r}$, are represented in matrix form as:

$$\mathbb{C} = \begin{bmatrix} \mathbb{C}_{111} & \mathbb{C}_{112} & \mathbb{C}_{113} \\ \mathbb{C}_{121} & \mathbb{C}_{122} & \mathbb{C}_{123} \\ \mathbb{C}_{131} & \mathbb{C}_{132} & \mathbb{C}_{133} \\ \hdashline \mathbb{C}_{211} & \mathbb{C}_{212} & \mathbb{C}_{213} \\ \mathbb{C}_{221} & \mathbb{C}_{222} & \mathbb{C}_{223} \\ \mathbb{C}_{231} & \mathbb{C}_{232} & \mathbb{C}_{233} \\ \hdashline \mathbb{C}_{311} & \mathbb{C}_{312} & \mathbb{C}_{313} \\ \mathbb{C}_{321} & \mathbb{C}_{322} & \mathbb{C}_{323} \\ \mathbb{C}_{331} & \mathbb{C}_{332} & \mathbb{C}_{333} \end{bmatrix} \qquad \mathbf{r}\otimes(\mathbf{r}\otimes\mathbf{r}) = \begin{bmatrix} x\begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix} \\ \hdashline y\begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix} \\ \hdashline z\begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix} \end{bmatrix}, \tag{10}$$

i.e. as vectors of rank-two tensors. Being

$$\begin{aligned} \mathbb{C}\cdot(\mathbf{r}\otimes\mathbf{r}\otimes\mathbf{r}) = {} & \mathbb{C}_{111}x^3 + \mathbb{C}_{222}y^3 + \mathbb{C}_{333}z^3 + \\ & + \left(\mathbb{C}_{112} + \mathbb{C}_{121} + \mathbb{C}_{211}\right)x^2y + \left(\mathbb{C}_{113} + \mathbb{C}_{131} + \mathbb{C}_{311}\right)x^2z + \\ & + \left(\mathbb{C}_{223} + \mathbb{C}_{232} + \mathbb{C}_{322}\right)y^2z + \left(\mathbb{C}_{122} + \mathbb{C}_{221} + \mathbb{C}_{212}\right)xy^2 + \\ & + \left(\mathbb{C}_{133} + \mathbb{C}_{331} + \mathbb{C}_{313}\right)xz^2 + \left(\mathbb{C}_{233} + \mathbb{C}_{332} + \mathbb{C}_{323}\right)yz^2 + \\ & + \left(\mathbb{C}_{123} + \mathbb{C}_{132} + \mathbb{C}_{213} + \mathbb{C}_{231} + \mathbb{C}_{312} + \mathbb{C}_{321}\right)xyz\,, \end{aligned} \tag{11}$$

the representation (3) of the density contrast is recovered from (6) by setting

$$\begin{aligned} &\theta_0 = c_{000} \quad && c_1 = c_{100} \quad && c_2 = c_{010} \quad && c_3 = c_{001} \\ &C_{11} = c_{200} && C_{22} = c_{020} && C_{33} = c_{002} && \\ &C_{12} = c_{110}/2 && C_{13} = c_{101}/2 && C_{23} = c_{011}/2 && \end{aligned} \tag{12}$$

and

$$
\begin{aligned}
&\mathbb{C}_{111} = c_{300} && \mathbb{C}_{222} = c_{030} && \mathbb{C}_{333} = c_{003} \\
&\mathbb{C}_{112} = \mathbb{C}_{121} = \mathbb{C}_{211} = c_{210}/3 && \mathbb{C}_{113} = \mathbb{C}_{131} = \mathbb{C}_{311} = c_{201}/3 && \\
&\mathbb{C}_{223} = \mathbb{C}_{232} = \mathbb{C}_{322} = c_{021}/3 && \mathbb{C}_{122} = \mathbb{C}_{221} = \mathbb{C}_{212} = c_{120}/3 && \\
&\mathbb{C}_{133} = \mathbb{C}_{331} = \mathbb{C}_{313} = c_{102}/3 && \mathbb{C}_{233} = \mathbb{C}_{332} = \mathbb{C}_{323} = c_{012}/3 && \\
&\mathbb{C}_{123} = \mathbb{C}_{132} = \mathbb{C}_{213} = \mathbb{C}_{231} = \mathbb{C}_{312} = \mathbb{C}_{321} = c_{111}/6\,. && &&
\end{aligned}
\tag{13}
$$

In conclusion, we derive from (2) the following expression of the gravity anomaly

$$
\Delta g_z(\mathbf{o}) = G\Big[\theta_{\mathbf{o}} d^{\Omega}_{\mathbf{r}} + \mathbf{c}\cdot\mathbf{d}^{\Omega}_{\mathbf{r}} + \mathbf{C}\cdot\mathbf{D}^{\Omega}_{\mathbf{rr}} + \mathbb{C}\cdot\mathbb{D}^{\Omega}_{\mathbf{rrr}}\Big] \tag{14}
$$

where

$$
d^{\Omega}_{\mathbf{r}} = \int_{\Omega} \frac{\mathbf{r}\cdot\mathbf{k}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,dV \qquad \mathbf{d}^{\Omega}_{\mathbf{r}} = \int_{\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,dV \tag{15}
$$

and

$$
\mathbf{D}^{\Omega}_{\mathbf{rr}} = \int_{\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})\mathbf{r}\otimes\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,dV \qquad \mathbb{D}^{\Omega}_{\mathbf{rrr}} = \int_{\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})\mathbf{r}\otimes\mathbf{r}\otimes\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,dV\,. \tag{16}
$$

In order to transform the previous domain integrals into boundary integrals we apply Gauss theorem in the generalized form illustrated in D'Urso (2013a, 2014a) so as to correctly take into account the singularity at $\mathbf{r} = \mathbf{o} = (0,0,0)$.

This will be done in the following two subsections while in the subsequent ones the boundary integrals extended to the faces of Ω will be further reduced to 1D integrals extended to the edges of each face by means of a further application of Gauss theorem. These last integrals will be first expressed as function of the 2D coordinates of the vertices in the reference frame local to each face and then reformulated in terms of the 3D coordinates representing the basic geometric data defining the polyhedron.

2.1 Analytical Expression of the Gravity Anomaly at O in Terms of 2D Integrals

Let us now illustrate a general approach to express the 3D integrals in (14) as 2D integrals extended to the faces constituting the boundary of Ω. Generality lies in the fact that, owing to the symmetry of the integrals, application of Gauss theorem can be based upon a unique formula. Actually, we are going to prove the result

$$
\int_{\Omega} \frac{k_{\mathbf{r}}[\otimes\mathbf{r},m]}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,dV = \frac{1}{m+1}\int_{\partial\Omega} \frac{k_{\mathbf{r}}[\otimes\mathbf{r},m](\mathbf{r}\cdot\mathbf{n})}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,dA \qquad m = 0,1,\ldots \tag{17}
$$

where $k_{\mathbf{r}} = \mathbf{r}\cdot\mathbf{k}$, $\mathbf{n}$ is the 3D outward unit normal to the boundary $\partial\Omega$ of the polyhedral body and $[\otimes\mathbf{r},m]$ denotes a rank-m tensor defined by

$$
[\otimes\mathbf{r},m] = \begin{cases} 1 & \text{if} \quad m = 0 \\ \mathbf{r} & \text{if} \quad m = 1 \\ \mathbf{r}\otimes\mathbf{r} & \text{if} \quad m = 2 \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ \underbrace{\mathbf{r}\otimes\mathbf{r}\otimes\cdots\otimes\mathbf{r}}_{m\ times} & \text{if} \quad m > 2\,. \end{cases} \tag{18}
$$

To fix the ideas we shall prove the identity (17) for $m = 2$

$$\int_{\Omega} \frac{k_{\mathbf{r}}\mathbf{r}\otimes\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}dV = \frac{1}{3}\int_{\partial\Omega} \frac{k_{\mathbf{r}}(\mathbf{r}\otimes\mathbf{r})(\mathbf{r}\cdot\mathbf{n})}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}dA \tag{19}$$

since it allows us to illustrate our approach to a degree of generality sufficient to extend the final result to all integrals in (14) and to the additional ones, not reported in (14), containing tensors of rank superior to three, i.e. tensors of the kind $[\otimes\mathbf{r}, m]$ where $m > 3$.

Recalling the identity proved in the appendix of D'Urso (2015c)

$$\begin{aligned}\operatorname{div}[\psi(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c})] &= (\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c})\operatorname{grad}\psi + \psi[(\operatorname{grad}\mathbf{a})\mathbf{c}]\otimes\mathbf{b}+ \\ &+\psi\mathbf{a}\otimes[(\operatorname{grad}\mathbf{b})\mathbf{c}] + \psi(\mathbf{a}\otimes\mathbf{b})\operatorname{div}\mathbf{c}\end{aligned} \tag{20}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ (ψ) are vector (scalar) differentiable fields, we have

$$\begin{aligned}\operatorname{div}\Big[k_{\mathbf{r}}(\mathbf{r}\otimes\mathbf{r})\otimes\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\Big] &= \Big[(\mathbf{r}\otimes\mathbf{r})\otimes\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\Big]\operatorname{grad}k_{\mathbf{r}} + k_{\mathbf{r}}\Big[(\operatorname{grad}\mathbf{r})\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\Big]\otimes\mathbf{r}+ \\ &+k_{\mathbf{r}}\mathbf{r}\otimes\Big[(\operatorname{grad}\mathbf{r})\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\Big] + k_{\mathbf{r}}(\mathbf{r}\otimes\mathbf{r})\operatorname{div}\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,.\end{aligned} \tag{21}$$

Applying the further identity proved in the appendix of D'Urso (2015c)

$$\operatorname{grad}(\mathbf{a}\cdot\mathbf{b}) = [\operatorname{grad}\mathbf{a}]^T\mathbf{b} + [\operatorname{grad}\mathbf{b}]^T\mathbf{a} \tag{22}$$

where $(\cdot)^T$ stands for transpose, one gets

$$\operatorname{grad}k_{\mathbf{r}} = \operatorname{grad}(\mathbf{r}\cdot\mathbf{k}) = (\operatorname{grad}\mathbf{r})\mathbf{k} = \mathbf{k} \tag{23}$$

since $\mathbf{k}$ is a constant vector field and $\operatorname{grad}\mathbf{r} = \mathbf{I}$, being $\mathbf{I}$ the rank-two identity tensor. Substituting the previous relation in (21) one obtains

$$\begin{aligned}\operatorname{div}\Big[k_{\mathbf{r}}(\mathbf{r}\otimes\mathbf{r})\otimes\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\Big] &= \Big[(\mathbf{r}\otimes\mathbf{r})\otimes\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\Big]\mathbf{k} + k_{\mathbf{r}}\Big[\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\otimes\mathbf{r} + \mathbf{r}\otimes\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\Big]+ \\ &+k_{\mathbf{r}}(\mathbf{r}\otimes\mathbf{r})\operatorname{div}\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}} = \\ &= 3k_{\mathbf{r}}\frac{\mathbf{r}\otimes\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}} + k_{\mathbf{r}}(\mathbf{r}\otimes\mathbf{r})\operatorname{div}\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,.\end{aligned} \tag{24}$$

Finally, integrating the previous identity over Ω yields

$$\int_{\Omega} k_{\mathbf{r}}\frac{\mathbf{r}\otimes\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}dV = \frac{1}{3}\int_{\Omega}\operatorname{div}\Big[k_{\mathbf{r}}(\mathbf{r}\otimes\mathbf{r})\otimes\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\Big]dV - \frac{1}{3}\int_{\Omega}k_{\mathbf{r}}(\mathbf{r}\otimes\mathbf{r})\operatorname{div}\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}dV\,. \tag{25}$$

The second integral on the right-hand side can be computed by means of the general result (Tang, 2006)

$$\int_{\Omega}\varphi(\mathbf{r})\operatorname{div}\left[\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\right]dV = \begin{cases} 0 & \text{if} \quad \mathbf{o}\notin\Omega \\ \alpha_V(\mathbf{o})\varphi(\mathbf{o}) & \text{if} \quad \mathbf{o}\in\Omega \end{cases} \tag{26}$$

where φ is a continuous scalar field and the quantity α_V represents the angular measure, expressed in steradians, of the intersection between Ω and a spherical neighbourhood of the singularity point $\mathbf{r} = \mathbf{o}$, see D'Urso (2012, 2013a, 2014a) for additional details.

The previous expression can be extended to arbitrary tensors by applying it to each scalar component of the tensor.

On account of (26) one infers that the second integral on the right-hand side of (25) is the null rank-two tensor **O** since

$$\int_{\Omega} k_{\mathbf{r}}(\mathbf{r}\otimes\mathbf{r})\,\mathrm{div}\,\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,dV = \begin{cases} \mathbf{O} & \text{if} \quad \mathbf{o}\notin\Omega \\ [k_{\mathbf{r}}\mathbf{r}\otimes\mathbf{r}]_{\mathbf{r}=\mathbf{o}}\,\alpha_V(\mathbf{o}) & \text{if} \quad \mathbf{o}\in\Omega\,. \end{cases} \tag{27}$$

However, the expression $[k_{\mathbf{r}}(\mathbf{r}\otimes\mathbf{r})]_{\mathbf{r}=\mathbf{o}}$ amounts to evaluating the quantity $k_{\mathbf{r}}(\mathbf{r}\otimes\mathbf{r})$ at the singularity point $\mathbf{r}=\mathbf{o}$, what yields trivially the null tensor **O**. Hence, according to (27), the last integral in (25) is always the null tensor, independently from the position of singularity point $\mathbf{r}=\mathbf{o}$ with respect to the domain Ω of integration.

In conclusion, upon application of Gauss theorem to the second integral in (25), we finally infer the identity (19). Remarkably, the derivation of this identity has also allowed us to prove that the singularity at $\mathbf{r}=\mathbf{o}$, of the integrand function appearing on the left-hand side of (19), can be actually ignored.

Furthermore, it is not difficult to rephrase the path of reasoning detailed in formulas (21)-(27) so as to prove the more general formula (17). Hence, defining

$$d_{\mathbf{r}}^{\partial\Omega} = \int_{\partial\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})(\mathbf{r}\cdot\mathbf{n})}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,dA \qquad \mathbf{d}_{\mathbf{r}}^{\partial\Omega} = \int_{\partial\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})\,\mathbf{r}\,(\mathbf{r}\cdot\mathbf{n})}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,dA \tag{28}$$

$$\mathbf{D}_{\mathbf{rr}}^{\partial\Omega} = \int_{\partial\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})\,\mathbf{r}\otimes\mathbf{r}\,(\mathbf{r}\cdot\mathbf{n})}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,dA \qquad \mathbb{D}_{\mathbf{rrr}}^{\partial\Omega} = \int_{\partial\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})\,\mathbf{r}\otimes\mathbf{r}\otimes\mathbf{r}\,(\mathbf{r}\cdot\mathbf{n})}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\,dA\,, \tag{29}$$

one has, recalling definitions (15) and (16)

$$d_{\mathbf{r}}^{\Omega} = d_{\mathbf{r}}^{\partial\Omega} \qquad \mathbf{d}_{\mathbf{r}}^{\Omega} = \frac{\mathbf{d}_{\mathbf{r}}^{\partial\Omega}}{2} \qquad \mathbf{D}_{\mathbf{rr}}^{\Omega} = \frac{\mathbf{D}_{\mathbf{rr}}^{\partial\Omega}}{3} \qquad \mathbb{D}_{\mathbf{rrr}}^{\Omega} = \frac{\mathbb{D}_{\mathbf{rrr}}^{\partial\Omega}}{4}\,. \tag{30}$$

In conclusion, application of formula (17) allows us to rewrite formula (14) as follows

$$\Delta g_z(\mathbf{o}) = G\Big[\theta_{\mathbf{o}} d_{\mathbf{r}}^{\partial\Omega} + \frac{\mathbf{c}\cdot\mathbf{d}_{\mathbf{r}}^{\partial\Omega}}{2} + \frac{\mathbf{C}\cdot\mathbf{D}_{\mathbf{rr}}^{\partial\Omega}}{3} + \frac{\mathbb{C}\cdot\mathbb{D}_{\mathbf{rrr}}^{\partial\Omega}}{4}\Big], \tag{31}$$

an expression that will be further elaborated in the next subsection by transforming the 2D integrals (28), (29) in 1D integrals.

2.2 Analytical Expression of the Gravity Anomaly at O in terms of Face Integrals

In order to derive an expression suitable for programming, we specialize formula (31) to polyhedral domains since this is by far the most general case in the gravity inversion problems.

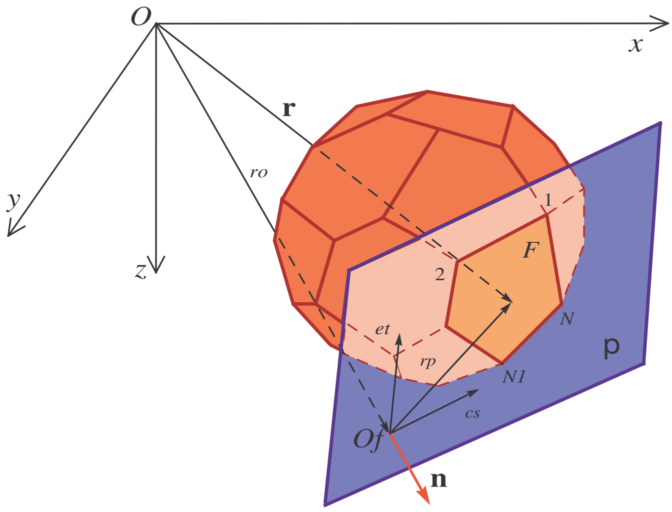


Fig. 1 Polyhedral domain Ω and decomposition of the position vector of a point on a face.

For a polyhedral body characterized by N_F faces, the integrals in (28)-(29) can be written as

$$
\begin{aligned}
d_{\mathbf{r}}^{\partial\Omega} &= \sum_{i=1}^{N_F} \int_{F_i} \frac{(\mathbf{r}_i \cdot \mathbf{k})(\mathbf{r}_i \cdot \mathbf{n}_i)}{(\mathbf{r}_i \cdot \mathbf{r}_i)^{3/2}} dA_i = \sum_{i=1}^{N_F} d_i \int_{F_i} \frac{\mathbf{r}_i \cdot \mathbf{k}}{(\mathbf{r}_i \cdot \mathbf{r}_i)^{3/2}} dA_i \\
\mathbf{d}_{\mathbf{r}}^{\partial\Omega} &= \sum_{i=1}^{N_F} \int_{F_i} \frac{(\mathbf{r}_i \cdot \mathbf{k})\,\mathbf{r}_i\,(\mathbf{r}_i \cdot \mathbf{n}_i)}{(\mathbf{r}_i \cdot \mathbf{r}_i)^{3/2}} dA_i = \sum_{i=1}^{N_F} d_i \int_{F_i} \frac{(\mathbf{r}_i \cdot \mathbf{k})\mathbf{r}_i}{(\mathbf{r}_i \cdot \mathbf{r}_i)^{3/2}} dA_i \\
\mathbf{D}_{\mathbf{rr}}^{\partial\Omega} &= \sum_{i=1}^{N_F} \int_{F_i} \frac{(\mathbf{r}_i \cdot \mathbf{k})\,(\mathbf{r}_i \otimes \mathbf{r}_i)\,(\mathbf{r}_i \cdot \mathbf{n}_i)}{(\mathbf{r}_i \cdot \mathbf{r}_i)^{3/2}} dA_i = \sum_{i=1}^{N_F} d_i \int_{F_i} \frac{(\mathbf{r}_i \cdot \mathbf{k})\,\mathbf{r}_i \otimes \mathbf{r}_i}{(\mathbf{r}_i \cdot \mathbf{r}_i)^{3/2}} dA_i \\
\mathbb{D}_{\mathbf{rrr}}^{\partial\Omega} &= \sum_{i=1}^{N_F} \int_{F_i} \frac{(\mathbf{r}_i \cdot \mathbf{k})\,(\mathbf{r}_i \otimes \mathbf{r}_i \otimes \mathbf{r}_i)\,(\mathbf{r}_i \cdot \mathbf{n}_i)}{(\mathbf{r}_i \cdot \mathbf{r}_i)^{3/2}} dA_i = \sum_{i=1}^{N_F} d_i \int_{F_i} \frac{(\mathbf{r}_i \cdot \mathbf{k})\,\mathbf{r}_i \otimes \mathbf{r}_i \otimes \mathbf{r}_i}{(\mathbf{r}_i \cdot \mathbf{r}_i)^{3/2}} dA_i
\end{aligned}
\tag{32}
$$

where the second equality in each formula above stems from the fact that the vector $\mathbf{r}_i$ spanning the i-th face, see, e.g., fig. 1, can be decomposed as follows

$$
\mathbf{r}_i = \mathbf{r}_i^{\perp} + \mathbf{r}_i^{\parallel}, \tag{33}
$$

i.e. as sum of a vector $\mathbf{r}_i^{\perp}$ orthogonal to F_i and a vector $\mathbf{r}_i^{\parallel}$ parallel to the face. Accordingly, denoting by $\mathbf{n}_i$ the unit vector pointing outwards Ω, one can set $\mathbf{r}_i \cdot \mathbf{n}_i = \mathbf{r}_i^{\perp} \cdot \mathbf{n}_i = d_i$, since d_i represents the signed distance between the origin and the i-th face F_i measured orthogonally to this last one.

The 2D integrals above can be transformed to a line integral by a further application of Gauss theorem. To this end we denote by O_i the orthogonal projection on F_i of the observation point O and assume O_i as origin of a 2D reference frame local to the face.

Furthermore, we express formula (33) in the alternative form

$$\mathbf{r}_i = \mathbf{r}_i^{\perp} + \mathbf{r}_i^{\parallel} = (\mathbf{r}_i \cdot \mathbf{n}_i)\mathbf{n}_i + \mathbf{r}_i^{\parallel} = d_i \mathbf{n}_i + \mathbf{T}_{F_i} \boldsymbol{\rho}_i \tag{34}$$

where the vector $\boldsymbol{\rho}_i = (\xi_i, \eta_i)$ represents the position vector of a generic point of the i-th face with respect to O_i and

$$\mathbf{T}_{F_i} = \begin{bmatrix} \mathbf{u}_{i1} & \mathbf{v}_{i1} \\ \mathbf{u}_{i2} & \mathbf{v}_{i2} \\ \mathbf{u}_{i3} & \mathbf{v}_{i3} \end{bmatrix} \tag{35}$$

is the linear operator mapping the 2D vector $\boldsymbol{\rho}_i$ to the 3D one $\mathbf{r}_i^{\parallel}$. In turn $\mathbf{u}_i$ and $\mathbf{v}_i$ represent two distinct, yet arbitrary, 3D unit vectors parallel to F_i.

We emphasize the use of roman and greek letters in (34) to denote, respectively, 3D and 2D vectors. The same notational distinction will be adopted throughout the paper.

Setting

$$\mathbf{r}_i \cdot \mathbf{k} = d_i \mathbf{n}_i \cdot \mathbf{k} + \mathbf{T}_{F_i} \boldsymbol{\rho}_i \cdot \mathbf{k} = d_i n_{i3} + \boldsymbol{\rho}_i \cdot \mathbf{T}_{F_i}^T \mathbf{k} = d_i n_{i3} + \boldsymbol{\rho}_i \cdot \boldsymbol{\kappa}_i \,, \tag{36}$$

the first two integrals in (32) become

$$d_{\mathbf{r}}^{\partial\Omega} = \sum_{i=1}^{N_F} d_i \left\{ d_i n_{i3} \int_{F_i} \frac{dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} + \boldsymbol{\kappa}_i \cdot \int_{F_i} \frac{\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \right\} \tag{37}$$

$$\begin{aligned} \mathbf{d}_{\mathbf{r}}^{\partial\Omega} = \sum_{i=1}^{N_F} d_i \Bigg\{ & d_i^2 n_{i3} \mathbf{n}_i \int_{F_i} \frac{dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} + d_i n_{i3} \int_{F_i} \frac{\mathbf{T}_{F_i} \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i + \\ & + d_i \mathbf{n}_i \left[\int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \cdot \boldsymbol{\kappa}_i \right] + \int_{F_i} \frac{\mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \boldsymbol{\kappa}_i \Bigg\} . \end{aligned} \tag{38}$$

Thus, defining

$$\varphi_{F_i} = \int_{F_i} \frac{dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \quad \boldsymbol{\varphi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \quad \boldsymbol{\Phi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \,, \tag{39}$$

one finally has

$$d_{\mathbf{r}}^{\partial\Omega} = \sum_{i=1}^{N_F} d_i \left\{ d_i n_{i3} \varphi_{F_i} + \boldsymbol{\kappa}_i \cdot \boldsymbol{\varphi}_{F_i} \right\} \tag{40}$$

and

$$\mathbf{d}_{\mathbf{r}}^{\partial\Omega} = \sum_{i=1}^{N_F} d_i \left\{ d_i^2 n_{i3} \varphi_{F_i} \mathbf{n}_i + d_i n_{i3} \mathbf{T}_{F_i} \boldsymbol{\varphi}_{F_i} + d_i \mathbf{n}_i \left(\boldsymbol{\kappa}_i \cdot \boldsymbol{\varphi}_{F_i} \right) + \mathbf{T}_{F_i} \boldsymbol{\Phi}_{F_i} \boldsymbol{\kappa}_i \right\} . \tag{41}$$

To suitably shorten the expression of the last two integrals in (32) we set

$$\mathfrak{C}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \qquad \mathfrak{D}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \tag{42}$$

$$\mathfrak{C}_{F_i}\boldsymbol{\kappa}_i = \int_{F_i} \frac{(\boldsymbol{\rho}_i \cdot \boldsymbol{\kappa}_i)\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \qquad \mathfrak{D}_{F_i}\boldsymbol{\kappa}_i = \int_{F_i} \frac{(\boldsymbol{\rho}_i \cdot \boldsymbol{\kappa}_i)\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \tag{43}$$

and introduce the formal operator $\mathbb{T}_{F_i}^{b...b}$ where the symbol $b...b$ denotes an arbitrary sequence of 0 and 1. In particular

$$\mathbb{T}_{F_i}^{11}\boldsymbol{\Phi}_{F_i} = \mathbb{T}_{F_i}^{11}\int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i = \mathbf{T}_{F_i}\boldsymbol{\Phi}_{F_i}\mathbf{T}_{F_i}^T, \tag{44}$$

$$\mathbb{T}_{F_i}^{111}\mathfrak{C}_{F_i} = \mathbb{T}_{F_i}^{111}\int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \tag{45}$$

and

$$\mathbb{T}_{F_i}^{1010}\mathfrak{D}_{F_i} = \mathbb{T}_{F_i}^{1010}\int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \tag{46}$$

since the suffix 1 (0) of $\mathbb{T}_{F_i}$ indicates that the operator $\mathbf{T}_{F_i}$ has (not) to be applied to the vector $\boldsymbol{\rho}_i$.

Accordingly, the third integral in (32) becomes

$$\begin{aligned}\mathbf{D}_{\mathbf{rr}}^{\partial\Omega} = \sum_{i=1}^{N_F} d_i \Big\{ & d_i n_{i3}\Big[d_i^2 \varphi_{F_i}\mathbf{n}_i \otimes \mathbf{n}_i + d_i\Big(\mathbf{n}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\varphi}_{F_i} + \mathbf{T}_{F_i}\boldsymbol{\varphi}_{F_i} \otimes \mathbf{n}_i\Big) + \mathbf{T}_{F_i}\boldsymbol{\Phi}_{F_i}\mathbf{T}_{F_i}^T\Big] + \\ & + d_i^2 \mathbf{n}_i \otimes \mathbf{n}_i\Big(\boldsymbol{\kappa}_i \cdot \boldsymbol{\varphi}_{F_i}\Big) + d_i\Big[\mathbf{n}_i \otimes \mathbf{T}_{F_i}\Big(\boldsymbol{\Phi}_{F_i}\boldsymbol{\kappa}_i\Big) + \mathbf{T}_{F_i}\Big(\boldsymbol{\Phi}_{F_i}\boldsymbol{\kappa}_i\Big) \otimes \mathbf{n}_i\Big] + \mathbf{H}_i \Big\}\end{aligned} \tag{47}$$

where

$$\mathbf{H}_i = \mathbf{T}_{F_i}\Big(\mathfrak{C}_{F_i}\boldsymbol{\kappa}_i\Big)\mathbf{T}_{F_i}^T. \tag{48}$$

Furthermore, setting

$$\begin{aligned}\boldsymbol{\Phi}_{F_i} \wedge \mathbf{n}_i &= \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \qquad \boldsymbol{\Phi}_{F_i} \wedge \Big(\mathbf{n}_i \otimes \mathbf{n}_i\Big) = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \mathbf{n}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \\ \mathfrak{C}_{F_i} \wedge \mathbf{n}_i &= \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \qquad \mathfrak{C}_{F_i} \vee \mathbf{n}_i = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i,\end{aligned} \tag{49}$$

it turns out to be

$$
\begin{aligned}
\mathbb{D}^{\partial\Omega}_{\mathbf{rrr}} = \sum_{i=1}^{N_F} d_i \Bigg\{ & d_i n_{i3} \Big[d_i^3 \varphi_{F_i} \mathbf{n}_i \otimes \mathbf{n}_i \otimes \mathbf{n}_i + d_i^2 \Big(\mathbf{n}_i \otimes \mathbf{n}_i \otimes \mathbf{T}_{F_i} \boldsymbol{\varphi}_{F_i} + \mathbf{n}_i \otimes \mathbf{T}_{F_i} \boldsymbol{\varphi}_{F_i} \otimes \mathbf{n}_i + \\
& + \mathbf{T}_{F_i} \boldsymbol{\varphi}_{F_i} \otimes \mathbf{n}_i \otimes \mathbf{n}_i \Big) + d_i \mathbf{n}_i \otimes \mathbb{T}^{11}_{F_i} \mathbf{\Phi}_{F_i} + d_i \mathbb{T}^{101}_{F_i} \Big(\mathbf{\Phi}_{F_i} \wedge \mathbf{n}_i \Big) + \\
& + d_i \mathbb{T}^{11}_{F_i} \mathbf{\Phi}_{F_i} \otimes \mathbf{n}_i + \mathbb{T}^{111}_{F_i} \mathfrak{C}_{F_i} \Big] + d_i^3 \mathbf{n}_i \otimes \mathbf{n}_i \otimes \mathbf{n}_i \Big(\boldsymbol{\kappa}_i \cdot \boldsymbol{\varphi}_{F_i} \Big) + \\
& + d_i^2 \Big[\mathbf{n}_i \otimes \mathbf{n}_i \otimes \mathbf{T}_{F_i} \Big(\mathbf{\Phi}_{F_i} \boldsymbol{\kappa}_i \Big) + \mathbf{n}_i \otimes \mathbb{T}^{101}_{F_i} \Big(\mathbf{\Phi}_{F_i} \wedge \mathbf{n}_i \Big) \boldsymbol{\kappa}_i + \\
& + \mathbb{T}^{1000}_{F_i} \mathbf{\Phi}_{F_i} \wedge \Big(\mathbf{n}_i \otimes \mathbf{n}_i \Big) \boldsymbol{\kappa}_i \Big] + d_i \Big[\mathbf{n}_i \otimes \mathbb{T}^{110}_{F_i} \mathfrak{C}_{F_i} \boldsymbol{\kappa}_i + \mathbb{T}^{1010}_{F_i} \Big(\mathfrak{C}_{F_i} \wedge \mathbf{n}_i \Big) \boldsymbol{\kappa}_i + \\
& + \mathbb{T}^{1100}_{F_i} \Big(\mathfrak{C}_{F_i} \vee \mathbf{n}_i \Big) \boldsymbol{\kappa}_i \Big] + \mathbb{T}^{1110}_{F_i} \mathfrak{D}_{F_i} \boldsymbol{\kappa}_i \Bigg\}
\end{aligned}
\tag{50}
$$

being

$$
\mathbb{T}^{101}_{F_i} \Big(\mathbf{\Phi}_{F_i} \wedge \mathbf{n}_i \Big) = \int_{F_i} \frac{\mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \mathbf{T}_{F_i} \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \,, \tag{51}
$$

$$
\mathbb{T}^{1000}_{F_i} \mathbf{\Phi}_{F_i} \wedge \Big(\mathbf{n}_i \otimes \mathbf{n}_i \Big) = \int_{F_i} \frac{\mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \mathbf{n}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \boldsymbol{\kappa}_i \,, \tag{52}
$$

$$
\mathbb{T}^{110}_{F_i} \mathfrak{C}_{F_i} \boldsymbol{\kappa}_i = \int_{F_i} \frac{\mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \boldsymbol{\kappa}_i = \int_{F_i} \frac{(\boldsymbol{\rho}_i \cdot \boldsymbol{\kappa}_i) \mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i} \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \,, \tag{53}
$$

$$
\mathbb{T}^{1010}_{F_i} \Big(\mathfrak{C}_{F_i} \wedge \mathbf{n}_i \Big) = \int_{F_i} \frac{\mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \,, \tag{54}
$$

$$
\mathbb{T}^{1100}_{F_i} \Big(\mathfrak{C}_{F_i} \vee \mathbf{n}_i \Big) = \int_{F_i} \frac{\mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \,, \tag{55}
$$

$$
\mathbb{T}^{1110}_{F_i} \mathfrak{D}_{F_i} \boldsymbol{\kappa}_i = \int_{F_i} \frac{\mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \boldsymbol{\kappa}_i = \int_{F_i} \frac{(\boldsymbol{\rho}_i \cdot \boldsymbol{\kappa}_i) \mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i} \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \,. \tag{56}
$$

Notice that the symbols in (49), as well as the ones in (50), are purely formal since they involve the tensor product of 2D and 3D vectors. They have been deliberately introduced to focus the reader's attention on the main issues involved in the evaluation of the quantities $d^{\partial\Omega}_{\mathbf{r}}$, $\mathbf{d}^{\partial\Omega}_{\mathbf{r}}$, $\mathbf{D}^{\partial\Omega}_{\mathbf{rr}}$, and $\mathbb{D}^{\partial\Omega}_{\mathbf{rrr}}$. Actually, one first evaluates the integrals

$$
\int_{F_i} \frac{[\otimes \boldsymbol{\rho}_i, m]}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \qquad m \in [0, 4] \tag{57}
$$

as tensor product of 2D vectors, see, e.g., Appendix 1 and 2. Only subsequently the resulting formula is combined with the 2D vector $\boldsymbol{\kappa}_i$ and expressed in terms of 3D vectors, by means of the operator $\mathbf{T}_{F_i}$, or suitably combined with the 3D vector $\mathbf{n}_i$ to evaluate the integrals in (50).

The simultaneous presence in (57) of the quantity d_i and of the exponent $3/2$ in the denominator makes the evaluation of the integrals in (57) by far more diffult than the analogous ones addressed in D'Urso (2015c) for polygonal bodies. Actually the case $d_i = 0$, meaning that the observation point O belongs to the face F_i, or equivalently that $O_i \equiv O$, needs to be properly addressed since the integrals can become singular.

For the same reason we shall not consider the fact that the integrals in (57) need to be composed with the vector $\boldsymbol{\kappa}_i$ producing

$$\left[\int_{F_i} \frac{[\otimes\boldsymbol{\rho}_i, m]}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i\right]\boldsymbol{\kappa}_i = \int_{F_i} \frac{[\otimes\boldsymbol{\rho}_i, m-1](\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \quad m \in [1,4]\,, \tag{58}$$

since this would require to consider separately these cases in the discussion of the singularities of the algebraic expressions resulting from (57); instead, we shall perform the combination after the integration. Moreover, due to the presence of the exponent $3/2$, the definite integrals that need to be computed to transform the integrals (57) into their algebraic counterparts do not exhibit anymore the useful recurrence property invoked in the appendix of D'Urso (2015c) so that it is more convenient to evaluate the integrals in (57) prior to their composition with $\boldsymbol{\kappa}_i$.

Last, but not least, most of the integrals in (57) have been already computed in D'Urso (2013a, 2014a,b) so that we include in the Appendix 1 only the explicit evaluation of the new ones.

2.3 Analytical Expression of Face Integrals in terms of 1D Integrals

It has been emphasized in the previous subsection that the main burden associated with the evaluation of the expressions (37), (38), (47) and (50) is the evaluation of the integrals (57). Similarly to the integrals (15) and (16), they can be transformed into simpler 1D integrals by a further application of the generalized Gauss theorem (Tang, 2006).

For some of them, namely the ones in (57) defined by $m = 0$, $m = 1$, and $m = 2$, this has been done in previous papers (D'Urso, 2013a, 2014a,b); for $m = 3$ and $m = 4$ this has been carried out in Appendix 1. For sake of clarity their expressions are collected hereafter for increasing values of m.

• Integral (57) for $m = 0$

$$\varphi_{F_i} = \int_{F_i} \frac{dA_i}{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{3/2}} = \frac{\alpha_i}{|d_i|} - \int_{\partial F_i} \frac{\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\nu}(s_i)}{\left[\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\rho}_i(s_i)\right]\left[\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2}} ds_i\,. \tag{59}$$

where s_i is the curvilinear abscissa along the boundary ∂F_i of the face F_i, $\boldsymbol{\nu}$ is the outward unit normal to F_i and α_i is a scalar, defined in Appendix 2, representing the measure, expressed in radians, of the intersection between F_i and a circular neighbourhood of the singularity point $\boldsymbol{\rho} = \boldsymbol{o}$ when $d_i = 0$.

• Integral (57) for $m = 1$

$$\boldsymbol{\varphi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{3/2}} = -\int_{\partial F_i} \frac{\boldsymbol{\nu}(s_i)}{\left[\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2}} ds_i\,. \tag{60}$$

• Integral (57) for $m = 2$

$$\mathbf{\Phi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}} = -\int_{\partial F_i} \frac{\boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\nu}(s_i)}{\left[\boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2}} ds_i + \psi_{F_i} \mathbf{I}_{2D} \tag{61}$$

where $\mathbf{I}_{2D}$ is the rank-two two-dimensional identity tensor,

$$\psi_{F_i} = \int_{F_i} \frac{dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} = \int_{\partial F_i} \frac{\left[\boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2} \left[\boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\nu}(s_i)\right]}{\boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\rho}_i(s_i)} ds_i - \alpha_i |d_i| \tag{62}$$

and α_i has been introduced just before formula (60).

• Integral (57) for $m = 3$

$$\mathfrak{C}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}} = -\int_{\partial F_i} \frac{\boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\nu}(s_i)}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} ds_i + \mathbf{I}_{2D} \otimes_{23} \boldsymbol{\psi}_{F_i} + \boldsymbol{\psi}_{F_i} \otimes \mathbf{I}_{2D} \tag{63}$$

where the symbol $\otimes_{23}$ denotes the tensor product obtained by interchanging the second and third index of the rank-three tensor $\mathbf{I}_{2D} \otimes \boldsymbol{\psi}_{F_i}$ and

$$\boldsymbol{\psi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} = \int_{\partial F_i} \left[\boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2} \boldsymbol{\nu}(s_i) ds_i \,. \tag{64}$$

• Integral (57) for $m = 4$

$$\begin{aligned} \mathfrak{D}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}} &= -\int_{\partial F_i} \frac{\boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\nu}(s_i)}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} ds_i + \\ &\quad + \mathbf{I}_{2D} \otimes_{24} \boldsymbol{\Psi}_{F_i} + \boldsymbol{\Psi}_{F_i} \otimes_{23} \mathbf{I}_{2D} + \boldsymbol{\Psi}_{F_i} \otimes \mathbf{I}_{2D} \end{aligned} \tag{65}$$

where the symbol $\otimes_{24}$ denotes the tensor product obtained by interchanging the second and fourth index of the rank-four tensor $\mathbf{I}_{2D} \otimes \boldsymbol{\Psi}_{F_i}$ and

$$\begin{aligned} \boldsymbol{\Psi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} &= -\int_{\partial F_i} \left[\boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2} \boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\nu}(s_i) ds_i - \\ &\quad - \frac{\mathbf{I}_{2D}}{3} \left\{ \int_{\partial F_i} \left[\boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2} \boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\nu}(s_i) ds_i - d_i^2 \psi_{F_i} \right\} . \end{aligned} \tag{66}$$

Since each face is polygonal the previous line integrals can be further expressed as sums extended to the N_{E_i} edges that define the boundary ∂F_i. For the j-th edge a suitable parameterization allows one to transform each 1D integral into an integral of a real variable; this is scaled by a suitable combination of the vectors $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$ that define the position vectors of the end vertices of the edge in the 2D reference frame local to F_i.

In particular we set

$$\hat{\boldsymbol{\rho}}_i(\lambda_j) = \boldsymbol{\rho}_j + \lambda_j(\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j) = \boldsymbol{\rho}_j + \lambda_j \Delta\boldsymbol{\rho}_j \tag{67}$$

where the function $\hat{\boldsymbol{\rho}}_i$ associates with each value of the adimensional abscissa

$$\lambda_j = s_j/l_j, \tag{68}$$

the position vector spanning the j-th edge. The quantity s_j, $s_j \in [0,\ l_j]$, is the curvilinear abscissa along the j-th edge and $l_j = |\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j|$ is the edge length. The position vector spanning the j-th edge of F_i can also be expressed as function of s_j and a new function $\boldsymbol{\rho}_i$, fulfilling the condition $\boldsymbol{\rho}_i(s_i) = \hat{\boldsymbol{\rho}}_i(\lambda_j)$. Hence

$$\boldsymbol{\rho}_i(s_j)\cdot\boldsymbol{\rho}_i(s_j) = \hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j) = p_j\lambda_j^2 + 2q_j\lambda_j + u_j = P_u(\lambda_j) \tag{69}$$

where, according to (67)

$$p_j = \Delta\boldsymbol{\rho}_j\cdot\Delta\boldsymbol{\rho}_j \qquad q_j = \boldsymbol{\rho}_j\cdot\Delta\boldsymbol{\rho}_j \qquad u_j = \boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_j\,. \tag{70}$$

Furthermore

$$\boldsymbol{\rho}(s_j)\cdot\boldsymbol{\rho}(s_j) + d_i^2 = p_j\lambda_j^2 + 2q_j\lambda_j + v_j \tag{71}$$

where $v_j = u_j + d_i^2$. We shall also set $P_v(\lambda_j) = P_u(\lambda_j) + d_i^2$.

2.4 Algebraic expression of face integrals in terms of 2D vectors

Refering to the Appendices 1 and 2 for further details we hereby report the algebraic counterparts of the integrals (57) for m=0,..,4.

• Integral (57) for $m = 0$

$$\varphi_{F_i} = \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}}\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right)\int_0^1 \frac{d\lambda_j}{P_u(\lambda_j)\left[P_v(\lambda_j)\right]^{1/2}} = \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}}\varphi_j\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right) \tag{72}$$

where φ_j is defined in (221). The symbol $(\cdot)^{\perp}$ denotes a clockwise rotation of the 2D vector $(\cdot)$ necessary to express the outward unit normal $\boldsymbol{\nu}_j$ to the j-th edge according to the formula

$$\boldsymbol{\nu}_j = \frac{\left(\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j\right)^{\perp}}{l_j} = \frac{\Delta\boldsymbol{\rho}_j^{\perp}}{l_j}\,. \tag{73}$$

The clockwise rotation indicated by the symbol $(\cdot)^{\perp}$ depends on the convention adopted to circulate along the boundary ∂F_i. In particular, we have assumed that the vertices of each face have been numbered consecutively by circulating along ∂F_i in a counter-clockwise sense with respect to the normal $\mathbf{n}_i$ to the face. Thus

$$\Delta\boldsymbol{\rho}_j = \begin{bmatrix}\Delta\xi_j\\ \Delta\eta_j\end{bmatrix} \Rightarrow \Delta\boldsymbol{\rho}_j^{\perp} = \begin{bmatrix}-\Delta\eta_j\\ \Delta\xi_j\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\Delta\boldsymbol{\rho}_j\,. \tag{74}$$

• Integral (57) for $m = 1$

$$\boldsymbol{\varphi}_{F_i} = -\sum_{j=1}^{N_{E_i}}\Delta\boldsymbol{\rho}_j^{\perp}\int_0^1 \frac{d\lambda_j}{\left[P_v(\lambda_j)\right]^{1/2}} = -\sum_{j=1}^{N_{E_i}} I_{0j}\,\Delta\boldsymbol{\rho}_j^{\perp} \tag{75}$$

where the scalar I_{0j} is defined in (211).

• Integral (57) for $m = 2$

$$\begin{aligned}\boldsymbol{\Phi}_{F_i} &= -\sum_{j=1}^{N_{E_i}} \int_0^1 \frac{\hat{\boldsymbol{\rho}}_i(\lambda_j)}{\left[P_v(\lambda_j)\right]^{1/2}} d\lambda_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} + \psi_{F_i}\mathbf{I}_{2D} = \\ &= -\sum_{j=1}^{N_{E_i}} \left[I_{0j}\,\boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} + I_{1j}\,\Delta\boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} \right] + \psi_{F_i}\mathbf{I}_{2D}\end{aligned} \tag{76}$$

where I_{0j} is defined in (211), I_{1j} in (212) while ψ_{F_i} is provided by

$$\psi_{F_i} = \sum_{j=1}^{N_{E_i}} \int_0^1 \frac{\left[P_v(\lambda_j)\right]^{1/2}}{\left[P_u(\lambda_j)\right]} d\lambda_j = \sum_{j=1}^{N_{E_i}} \psi_j^i \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) - |d_i|\alpha_i \tag{77}$$

and ψ_j^i is defined in (219).

• Integral (57) for $m = 3$

$$\begin{aligned}\mathfrak{C}_{F_i} &= -\sum_{j=1}^{N_{E_i}} \int_0^1 \frac{\hat{\boldsymbol{\rho}}_i(\lambda_j) \otimes \hat{\boldsymbol{\rho}}_i(\lambda_j) d\lambda_j}{\left[P_v(\lambda_j)\right]^{1/2}} + \mathbf{I}_{2D} \otimes_{23} \boldsymbol{\psi}_{F_i} + \boldsymbol{\psi}_{F_i} \otimes \mathbf{I}_{2D} = \\ &= -\sum_{j=1}^{N_{E_i}} \left[I_{0j}\,\mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j}\,\mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{2j}\,\mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} \right] \otimes \Delta\boldsymbol{\rho}_j^{\perp} + \mathbf{I}_{2D} \otimes_{23} \boldsymbol{\psi}_{F_i} + \boldsymbol{\psi}_{F_i} \otimes \mathbf{I}_{2D}\end{aligned} \tag{78}$$

where I_{0j}, I_{1j}, I_{2j} are defined in (211), (212) and (213) respectively, $\mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j}$, $\mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}$ and $\mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}$ are defined in (180) and

$$\boldsymbol{\psi}_{F_i} = \sum_{j=1}^{N_{E_i}} l_j \boldsymbol{\nu}_j \int_0^1 \left[P_v(\lambda_j)\right]^{1/2} d\lambda_j = \sum_{j=1}^{N_{E_i}} I_{4j}\,\Delta\boldsymbol{\rho}_j^{\perp}\,, \tag{79}$$

the scalar I_{4j} being defined in (215).

• Integral (57) for $m = 4$

$$\begin{aligned}\mathfrak{D}_{F_i} &= -\sum_{j=1}^{N_{E_i}} \left\{ \int_0^1 \frac{\hat{\boldsymbol{\rho}}_i(\lambda_j) \otimes \hat{\boldsymbol{\rho}}_i(\lambda_j) \otimes \hat{\boldsymbol{\rho}}_i(\lambda_j) d\lambda_j}{\left[P_v(\lambda_j)\right]^{1/2}} \otimes \Delta\boldsymbol{\rho}_j^{\perp} \right\} + \mathbf{I}_{2D} \otimes_{24} \boldsymbol{\Psi}_{F_i} + \boldsymbol{\Psi}_{F_i} \otimes_{23} \mathbf{I}_{2D} + \boldsymbol{\Psi}_{F_i} \otimes \mathbf{I}_{2D} = \\ &= -\sum_{j=1}^{N_{E_i}} \left[I_{0j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + \mathbb{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{3j}\,\mathbb{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} \right] \otimes \Delta\boldsymbol{\rho}_j^{\perp} + \\ &\quad + \mathbf{I}_{2D} \otimes_{24} \boldsymbol{\Psi}_{F_i} + \boldsymbol{\Psi}_{F_i} \otimes_{23} \mathbf{I}_{2D} + \boldsymbol{\Psi}_{F_i} \otimes \mathbf{I}_{2D}\end{aligned} \tag{80}$$

where I_{0j}, I_{1j}, I_{2j}, I_{3j} are defined in (211), (212), (213) and (214) respectively, $\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\boldsymbol{\rho}_j}$, $\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}$, $\mathbb{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}$ and $\mathbb{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}$ are defined in (191), (192) and (193) and

$$\begin{aligned}\boldsymbol{\Psi}_{F_i} = & \sum_{j=1}^{N_{E_i}} \left\{ \left[\int_0^1 \left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j) + d_i^2 \right]^{1/2} \left(\boldsymbol{\rho}_j + \lambda_j \Delta\boldsymbol{\rho}_j \right) d\lambda_j \right] \otimes \Delta\boldsymbol{\rho}_j^{\perp} - \right.\\ & \left. - \frac{\mathbf{I}_{2D}}{3} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp} \right) \int_0^1 \left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j) + d_i^2 \right]^{1/2} d\lambda_j \right\} + \frac{d_i^2}{3}\left(\psi_i - |d_i|\alpha_i \right) = \\ = & \sum_{j=1}^{N_{E_i}} \left[\left(I_{4j}\boldsymbol{\rho}_j + I_{5j}\Delta\boldsymbol{\rho}_j \right) \otimes \Delta\boldsymbol{\rho}_j^{\perp} - \frac{\mathbf{I}_{2D}}{3} (\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}) I_{4j} \right] + \frac{d_i^2}{3}\left(\psi_i - |d_i|\alpha_i \right),\end{aligned} \tag{81}$$

I_{4j}, I_{5j}, and ψ_i being defined in (215), (216) and (219) respectively.

For future reference we also include the algebraic expressions of the integrals in formula (43).

$$\mathfrak{C}_{F_i\boldsymbol{\kappa}_i} = -\sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\kappa}_i \cdot \Delta\boldsymbol{\rho}_j^{\perp} \right)\left(I_{0j}\,\mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j}\,\mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{2j}\,\mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} \right) + \boldsymbol{\kappa}_i \otimes \boldsymbol{\psi}_{F_i} + \boldsymbol{\psi}_{F_i} \otimes \boldsymbol{\kappa}_i \tag{82}$$

$$\begin{aligned}\mathfrak{D}_{F_i\boldsymbol{\kappa}_i} = & -\sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\kappa}_i \cdot \Delta\boldsymbol{\rho}_j^{\perp} \right)\left(I_{0j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{2j}\,\mathbb{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + \right.\\ & \left. + I_{3j}\,\mathbb{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} \right) + \boldsymbol{\Psi}_{F_i} \otimes \boldsymbol{\kappa}_i + \boldsymbol{\Psi}_{F_i} \otimes_{23} \boldsymbol{\kappa}_i + \boldsymbol{\kappa}_i \otimes \boldsymbol{\Psi}_{F_i}\,.\end{aligned} \tag{83}$$

All the previous quantities are expressed in terms of 2D vectors representing the coordinates of the end vertices of each edge in the reference frame local to each face F_i. Conversely, all tensors appearing in (37), (38), (47) and (50) have to expressed in terms of the 3D position vectors defining the vertices of the polyhedron Ω since these represent the basic geometric entities that define it. This task will be accomplished in the following subsection.

2.5 Algebraic expression of the integrals in terms of 3D vectors

The aim of this subsection is the show how the algebraic expressions derived in the previous subsection can be expressed in terms of 3D vectors in order to apply formula (31), what is fully accounted for in the next subsection. This is done by inverting (34) so as to express 2D coordinates of each vertex as function of the relevant 3D ones. In particular, premultiplying relation (34) by $\mathbf{T}_{F_i}^T$, where $(\cdot)^T$ stands for transpose, one obtains

$$\boldsymbol{\rho}_j = \mathbf{T}_{F_i}^T(\mathbf{r}_j - d_i\mathbf{n}_i) \tag{84}$$

since it is easy to check that $\mathbf{T}_{F_i}^T\mathbf{T}_{F_i} = \mathbf{I}_{2D}$.

Additional quantities that need to be expressed in terms of 3D vectors are

$$\mathbf{T}_{F_i}\Delta\boldsymbol{\rho}_j = \mathbf{r}_{j+1} - \mathbf{r}_i = \Delta\mathbf{r}_j \tag{85}$$

and

$$\mathbf{T}_{F_i}\Delta\boldsymbol{\rho}_j^{\perp} = \mathbf{T}_{F_i}\left[\mathbf{T}_{F_i}^T\Delta\mathbf{r}_j\right]^{\perp}. \tag{86}$$

We also set

$$\mathbf{f}_i = \mathbf{T}_{F_i}\boldsymbol{\varphi}_{F_i} = -\sum_{j=1}^{N_{E_i}} I_{0j}\mathbf{T}_{F_i}\Delta\boldsymbol{\rho}_j^{\perp} \tag{87}$$

according to (75) and

$$\mathbf{g}_i = \mathbf{T}_{F_i}\boldsymbol{\Phi}_{F_i}\boldsymbol{\kappa}_i = -\sum_{j=1}^{N_{E_i}}\left(\Delta\boldsymbol{\rho}_j^{\perp}\cdot\boldsymbol{\kappa}_i\right)\left[I_{0j}\mathbf{r}_j + I_{1j}\Delta\mathbf{r}_j\right] + \psi_{F_i}\mathbf{T}_{F_i}\mathbf{T}_{F_i}^T\mathbf{k} \tag{88}$$

according to (36) and (76); furthermore, we set

$$\mathbf{G}_i = \mathbf{T}_{F_i}\boldsymbol{\Phi}_{F_i}\mathbf{T}_{F_i}^T \tag{89}$$

see, e.g., formula (44).

Finally, recalling (44), (46), (48) and (49) it turns out to be

$$\mathbb{T}_{F_i}^{101}\left(\boldsymbol{\Phi}_{F_i}\wedge\mathbf{n}_i\right) = \int_{F_i}\frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{n}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i = \mathbf{G}_i\otimes_{23}\mathbf{n}_i\,, \tag{90}$$

$$\mathbb{T}_{F_i}^{110}\boldsymbol{\Phi}_{F_i}\otimes\mathbf{n}_i = \int_{F_i}\frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\otimes\mathbf{n}_i = \mathbf{G}_i\otimes\mathbf{n}_i\,, \tag{91}$$

$$\mathbb{G}_i = \mathbb{T}_{F_i}^{111}\mathfrak{C}_{F_i} = \int_{F_i}\frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\,, \tag{92}$$

$$\begin{aligned}\mathbb{T}_{F_i}^{101}\left(\boldsymbol{\Phi}_{F_i}\wedge\mathbf{n}_i\right)\boldsymbol{\kappa}_i &= \mathbf{T}_{F_i}\int_{F_i}\frac{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\right)\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\otimes\mathbf{n}_i = \mathbf{T}_{F_i}\int_{F_i}\frac{\left(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\right)dA_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\boldsymbol{\kappa}_i\otimes\mathbf{n}_i = \\ &= \mathbf{T}_{F_i}\boldsymbol{\Phi}_{F_i}\boldsymbol{\kappa}_i\otimes\mathbf{n}_i = \mathbf{g}_i\otimes\mathbf{n}_i\,,\end{aligned} \tag{93}$$

$$\begin{aligned}\mathbb{T}_{F_i}^{100}\boldsymbol{\Phi}_{F_i}\wedge\left(\mathbf{n}_i\otimes\mathbf{n}_i\right)\boldsymbol{\kappa}_i &= \int_{F_i}\frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{n}_i\otimes\mathbf{n}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\boldsymbol{\kappa}_i = \mathbf{T}_{F_i}\int_{F_i}\frac{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\right)\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\otimes\mathbf{n}_i\otimes\mathbf{n}_i = \\ &= \mathbf{T}_{F_i}\boldsymbol{\Phi}_{F_i}\boldsymbol{\kappa}_i\otimes\mathbf{n}_i\otimes\mathbf{n}_i = \mathbf{g}_i\otimes\mathbf{n}_i\otimes\mathbf{n}_i\,,\end{aligned} \tag{94}$$

$$\begin{aligned}\mathbb{T}_{F_i}^{110}\mathfrak{C}_{F_i}\boldsymbol{\kappa}_i &= \int_{F_i}\frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\boldsymbol{\kappa}_i = \int_{F_i}\frac{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\right)\left(\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\right)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i = \\ &= \mathbf{T}_{F_i}\int_{F_i}\frac{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\right)\left(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\right)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\mathbf{T}_{F_i}^T = \mathbf{T}_{F_i}\left[\int_{F_i}\frac{\left(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\right)dA_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\boldsymbol{\kappa}_i\right]\mathbf{T}_{F_i}^T = \\ &= \mathbf{T}_{F_i}\left(\mathfrak{C}_{F_i}\boldsymbol{\kappa}_i\right)\mathbf{T}_{F_i}^T = \mathbf{H}_i\,,\end{aligned} \tag{95}$$

$$\mathbb{T}_{F_i}^{1010}\big(\mathfrak{C}_{F_i}\wedge\mathbf{n}_i\big)\boldsymbol{\kappa}_i=\int_{F_i}\frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{n}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\boldsymbol{\kappa}_i=\int_{F_i}\frac{\big(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\big)\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{n}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i=$$
$$=\int_{F_i}\frac{\big(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\big)\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\otimes_{23}\mathbf{n}_i=\mathbf{H}_i\otimes_{23}\mathbf{n}_i\,,\tag{96}$$

$$\mathbb{T}_{F_i}^{1100}\big(\mathfrak{C}_{F_i}\vee\mathbf{n}_i\big)=\int_{F_i}\frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{n}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\boldsymbol{\kappa}_i=\int_{F_i}\frac{\big(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\big)\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{n}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i=$$
$$=\mathbf{T}_{F_i}\left[\int_{F_i}\frac{\big(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\big)\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\right]\mathbf{T}_{F_i}^T\otimes\mathbf{n}_i=\left[\mathbf{T}_{F_i}\big(\mathfrak{C}_{F_i}\boldsymbol{\kappa}_i\big)\mathbf{T}_{F_i}^T\right]\otimes\mathbf{n}_i=\mathbf{H}_i\otimes\mathbf{n}_i\,,\tag{97}$$

$$\begin{aligned}\mathbb{H}_i=\mathbb{T}_{F_i}^{1110}\mathfrak{D}_{F_i}\boldsymbol{\kappa}_i&=\int_{F_i}\frac{(\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\boldsymbol{\kappa}_i=\\&=\int_{F_i}\frac{(\boldsymbol{\kappa}_i\cdot\boldsymbol{\rho}_i)\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\,.\end{aligned}\tag{98}$$

The explicit evaluation of the last integral will be dealt with in the next subsection together with further considerations on actual evaluation of all third-order tensors appearing in (50).

2.6 Algebraic expression of the gravity anomaly at O

In order to make the reader fully acquainted with the operative steps required to compute the gravity anomaly at O, it is instructive to further comment on the formulas derived in the previous subsections in order to apply formula (31). As a matter of fact the evaluation of $d_{\mathbf{r}}^{\partial_i\Omega}$, $\mathbf{d}_{\mathbf{r}}^{\partial_i\Omega}$, $\mathbf{D}_{\mathbf{rr}}^{\partial_i\Omega}$, provided by formulas (37), (38) and (47), respectively, is trivial since they can be obtained by standard matrix operations.

More difficult is the evaluation of the third-order tensors appearing in (50), by taking also into account the fact that they have to first expressed in terms of 2D vectors and only subsequently, as specified in the previous subsection, reformulated in terms of 3D vectors.

To fix the ideas, let us start from the last addend in (50) that has been further detailed in (98). By means of formula (83), we actually dispose of an expression that can be written more concisely as

$$\int_{F_i}\frac{(\boldsymbol{\kappa}_i\cdot\boldsymbol{\rho}_i)\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i=\sum_{j=1}^{N_{E_i}}\Big[\alpha_j\mathbb{D}_{\rho\rho\rho}^{(j)}+\boldsymbol{\Lambda}_{\rho\rho}\otimes\boldsymbol{\beta}+\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}+\boldsymbol{\beta}\otimes\boldsymbol{\Lambda}_{\rho\rho}\Big]\tag{99}$$

where the right-hand side is a symbolic representation of the linear combination between third-order tensors $\mathbb{D}_{\rho\rho\rho}^{(j)}$, such as $\mathbb{D}_{\rho_j\rho_j\rho_j}$, $\mathbb{D}_{\rho_j\rho_j\Delta\rho_j}$, $\mathbb{D}_{\rho_j\Delta\rho_j\Delta\rho_j}$, $\mathbb{D}_{\Delta\rho_j\Delta\rho_j\Delta\rho_j}$, and tensor products between 2D vectors $\boldsymbol{\beta}$ and rank-two tensors $\boldsymbol{\Lambda}_{\rho\rho}$, this last one expressed as tensor product of 2D vectors.

Hence, to evaluate the left-hand side of (98) starting from (99) we have to transform the rank-three tensors on the right-hand side of (99) defined in terms of 2D vectors by applying the formal operator $\mathbb{T}_{F_i}^{111}$ to get,

$$\int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \boldsymbol{\kappa}_i = \mathbb{T}_{F_i}^{111} \sum_{j=1}^{N_{E_i}} \Big[\alpha_j \mathbb{D}_{\rho\rho\rho}^{(j)} + \boldsymbol{\Lambda}_{\rho\rho} \otimes \boldsymbol{\beta} + \boldsymbol{\Lambda}_{\rho\rho} \otimes_{23} \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \boldsymbol{\Lambda}_{\rho\rho} \Big]. \tag{100}$$

This is trivial for the rank-three tensor $\mathbb{D}_{\rho\rho\rho}^{(j)}$ since it is expressed as tensor product of three 2D vectors $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, $\boldsymbol{\varepsilon}$, so that

$$\mathbb{T}_{F_i}^{111} \mathbb{D}_{\rho\rho\rho}^{(j)} = \mathbb{T}_{F_i}^{111} (\boldsymbol{\gamma} \otimes \boldsymbol{\delta} \otimes \boldsymbol{\varepsilon}) = \mathbf{T}_{F_i}\boldsymbol{\gamma} \otimes \mathbf{T}_{F_i}\boldsymbol{\delta} \otimes \mathbf{T}_{F_i}\boldsymbol{\varepsilon} = \mathbf{t} \otimes \mathbf{v} \otimes \mathbf{w} \tag{101}$$

and the last tensor product between 3D vectors can be expressed in matrix form according to the rule which one adopts to define the matrix associated with a rank-three tensor, a rule that usually depends upon the adopted programming language.

For istance, extending the rule defined in (10) to three arbitrary 3D vectors one has

$$\Big[\mathbf{t} \otimes (\mathbf{v} \otimes \mathbf{w})\Big] = \left[t_1 \begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{pmatrix} \Bigg| \; t_2 \begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{pmatrix} \Bigg| \; t_3 \begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{pmatrix} \right]^T \tag{102}$$

where, for typographical reasons, we have represented the matrix associated with $\mathbf{t} \otimes (\mathbf{v} \otimes \mathbf{w})$ as a row rather than as a column.

Let us now apply the formal operator $\mathbb{T}_{F_i}^{111}$, already exploited in (101), to the last three addends in (100). Differently from $\mathbb{D}_{\rho\rho\rho}^{(j)}$, that is computed recursively as function of the j-th edge of F_i, the rank-two tensor $\boldsymbol{\Lambda}_{\rho\rho}$ is already available as a whole since it has been evaluated elsewhere, e.g. in a different subroutine. Hence, we already dispose of

$$\mathbb{T}_{F_i}^{11} \boldsymbol{\Lambda}_{\rho\rho} = \mathbf{T}_{F_i} \boldsymbol{\Lambda}_{\rho\rho} \mathbf{T}_{F_i}^T = \mathbf{L}_{\rho\rho} \tag{103}$$

where the roman letter $\mathbf{L}$ has been adopted to emphasize that the matrix associated with $\mathbf{L}_{\rho\rho}$ is 3×3. Accordingly

$$\mathbb{T}_{F_i}^{111} \Big(\boldsymbol{\Lambda}_{\rho\rho} \otimes \boldsymbol{\beta} \Big) = \mathbf{L}_{\rho\rho} \otimes \mathbf{T}_{F_i} \boldsymbol{\beta} = \mathbf{L}_{\rho\rho} \otimes \mathbf{b} \tag{104}$$

where $\mathbf{b}$ is a 3D vector.

Thus, we can exploit the general scheme in (102) by writing

$$\Big[\mathbf{L} \otimes \mathbf{b}\Big] = \Big[\Big(\mathbf{L} \otimes \mathbf{b}\Big)_1, \quad \Big(\mathbf{L} \otimes \mathbf{b}\Big)_2, \quad \Big(\mathbf{L} \otimes \mathbf{b}\Big)_3 \Big]^T . \tag{105}$$

where

$$\Big[\Big(\mathbf{L} \otimes \mathbf{b}\Big)_1\Big] = \begin{bmatrix} (\mathbf{L}_{\rho\rho})_{11} b_1 & (\mathbf{L}_{\rho\rho})_{11} b_2 & (\mathbf{L}_{\rho\rho})_{11} b_3 \\ (\mathbf{L}_{\rho\rho})_{12} b_1 & (\mathbf{L}_{\rho\rho})_{12} b_2 & (\mathbf{L}_{\rho\rho})_{12} b_3 \\ (\mathbf{L}_{\rho\rho})_{13} b_1 & (\mathbf{L}_{\rho\rho})_{13} b_2 & (\mathbf{L}_{\rho\rho})_{13} b_3 \end{bmatrix}, \tag{106}$$

$$\Big[\Big(\mathbf{L} \otimes \mathbf{b}\Big)_2\Big] = \begin{bmatrix} (\mathbf{L}_{\rho\rho})_{21} b_1 & (\mathbf{L}_{\rho\rho})_{21} b_2 & (\mathbf{L}_{\rho\rho})_{21} b_3 \\ (\mathbf{L}_{\rho\rho})_{22} b_1 & (\mathbf{L}_{\rho\rho})_{22} b_2 & (\mathbf{L}_{\rho\rho})_{22} b_3 \\ (\mathbf{L}_{\rho\rho})_{23} b_1 & (\mathbf{L}_{\rho\rho})_{23} b_2 & (\mathbf{L}_{\rho\rho})_{23} b_3 \end{bmatrix}, \tag{107}$$

$$\left[\left(\mathbf{L}\otimes\mathbf{b}\right)_3\right]=\begin{bmatrix}(\mathbf{L}_{\rho\rho})_{31}b_1 & (\mathbf{L}_{\rho\rho})_{31}b_2 & (\mathbf{L}_{\rho\rho})_{31}b_3\\(\mathbf{L}_{\rho\rho})_{32}b_1 & (\mathbf{L}_{\rho\rho})_{32}b_2 & (\mathbf{L}_{\rho\rho})_{32}b_3\\(\mathbf{L}_{\rho\rho})_{33}b_1 & (\mathbf{L}_{\rho\rho})_{33}b_2 & (\mathbf{L}_{\rho\rho})_{33}b_3\end{bmatrix}. \tag{108}$$

Analogously one has

$$\mathbb{T}^{111}_{F_i}\left(\boldsymbol{\beta}\otimes\boldsymbol{\Lambda}_{\rho\rho}\right)=\mathbf{T}_{F_i}\boldsymbol{\beta}\otimes\mathbf{L}_{\rho\rho}=\mathbf{b}\otimes\mathbf{L}_{\rho\rho} \tag{109}$$

so that the associated matrix is

$$\left[\mathbf{b}\otimes\mathbf{L}\right]=\left[\left(\mathbf{b}\otimes\mathbf{L}\right)_1,\quad\left(\mathbf{b}\otimes\mathbf{L}\right)_2,\quad\left(\mathbf{b}\otimes\mathbf{L}\right)_3\right]^T \tag{110}$$

where

$$\left[\left(\mathbf{b}\otimes\mathbf{L}\right)_1\right]=\left[b_1\begin{pmatrix}(\mathbf{L}_{\rho\rho})_{11} & (\mathbf{L}_{\rho\rho})_{12} & (\mathbf{L}_{\rho\rho})_{13}\\(\mathbf{L}_{\rho\rho})_{21} & (\mathbf{L}_{\rho\rho})_{22} & (\mathbf{L}_{\rho\rho})_{23}\\(\mathbf{L}_{\rho\rho})_{31} & (\mathbf{L}_{\rho\rho})_{32} & (\mathbf{L}_{\rho\rho})_{33}\end{pmatrix}\right], \tag{111}$$

$$\left[\left(\mathbf{b}\otimes\mathbf{L}\right)_2\right]=\left[b_2\begin{pmatrix}(\mathbf{L}_{\rho\rho})_{11} & (\mathbf{L}_{\rho\rho})_{12} & (\mathbf{L}_{\rho\rho})_{13}\\(\mathbf{L}_{\rho\rho})_{21} & (\mathbf{L}_{\rho\rho})_{22} & (\mathbf{L}_{\rho\rho})_{23}\\(\mathbf{L}_{\rho\rho})_{31} & (\mathbf{L}_{\rho\rho})_{32} & (\mathbf{L}_{\rho\rho})_{33}\end{pmatrix}\right], \tag{112}$$

$$\left[\left(\mathbf{b}\otimes\mathbf{L}\right)_3\right]=\left[b_3\begin{pmatrix}(\mathbf{L}_{\rho\rho})_{11} & (\mathbf{L}_{\rho\rho})_{12} & (\mathbf{L}_{\rho\rho})_{13}\\(\mathbf{L}_{\rho\rho})_{21} & (\mathbf{L}_{\rho\rho})_{22} & (\mathbf{L}_{\rho\rho})_{23}\\(\mathbf{L}_{\rho\rho})_{31} & (\mathbf{L}_{\rho\rho})_{32} & (\mathbf{L}_{\rho\rho})_{33}\end{pmatrix}\right]. \tag{113}$$

A little bit more akward is how to address the tensor product $\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}$. This case has been deliberately left at last since constructing the matrix associated with the rank-three tensor $\mathbb{T}^{111}_{F_i}\left(\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}\right)$ allows us to solve the problem concerning the tensor in (90).

Actually, if we could split the tensor $\boldsymbol{\Lambda}_{\rho\rho}$ as tensor product of two 2D vectors in the form $\boldsymbol{\Lambda}_{\rho\rho}=\boldsymbol{\gamma}\otimes\boldsymbol{\delta}$ we would trivially have

$$\mathbb{T}^{111}_{F_i}\left(\boldsymbol{\Lambda}_{\rho\rho}\otimes\boldsymbol{\beta}\right)=\mathbb{T}^{111}_{F_i}\left(\boldsymbol{\gamma}\otimes\boldsymbol{\delta}\otimes_{23}\boldsymbol{\beta}\right)=\mathbb{T}^{111}_{F_i}\left(\boldsymbol{\gamma}\otimes\boldsymbol{\beta}\otimes\boldsymbol{\delta}\right)=\mathbf{t}\otimes\mathbf{b}\otimes\mathbf{v} \tag{114}$$

and exploit the general scheme in (102) to construct the relevant matrix. Unfortunately we directly dispose of the matrix $\mathbf{L}_{\rho\rho}$ whose entries have to appear as first and third entries in the previous, purely illustrative, scheme.

This does not represent a real problem since, coherently with the matrix representation (102), we can define the matrix associated with

$$\mathbb{T}^{111}_{F_i}\left(\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}\right)=\mathbf{L}_{\mathbf{rbr}} \tag{115}$$

as

$$\left[\mathbf{L}_{\mathbf{rbr}}\right]=\left[\left(\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}\right)_1,\quad\left(\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}\right)_2,\quad\left(\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}\right)_3\right]^T \tag{116}$$

where

$$\left[\left(\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}\right)_1\right]=\begin{bmatrix}b_1(\mathbf{L}_{\rho\rho})_{11} & b_1(\mathbf{L}_{\rho\rho})_{12} & b_1(\mathbf{L}_{\rho\rho})_{13}\\b_2(\mathbf{L}_{\rho\rho})_{11} & b_2(\mathbf{L}_{\rho\rho})_{12} & b_2(\mathbf{L}_{\rho\rho})_{13}\\b_3(\mathbf{L}_{\rho\rho})_{11} & b_3(\mathbf{L}_{\rho\rho})_{12} & b_3(\mathbf{L}_{\rho\rho})_{13}\end{bmatrix}, \tag{117}$$

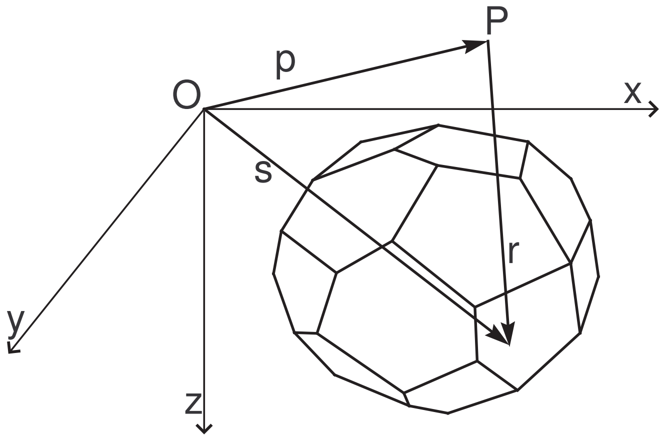


Fig. 2 Representation of geometric quantities used to assign density contrast (**s**) and define the position of Ω with respect to an arbitray point P.

$$\left[\left(\boldsymbol{\Lambda}_{\rho\rho} \otimes_{23} \boldsymbol{\beta}\right)_2\right] = \begin{bmatrix} b_1(\mathbf{L}_{\rho\rho})_{21} & b_1(\mathbf{L}_{\rho\rho})_{22} & b_1(\mathbf{L}_{\rho\rho})_{23} \\ b_2(\mathbf{L}_{\rho\rho})_{21} & b_2(\mathbf{L}_{\rho\rho})_{22} & b_2(\mathbf{L}_{\rho\rho})_{23} \\ b_3(\mathbf{L}_{\rho\rho})_{21} & b_3(\mathbf{L}_{\rho\rho})_{22} & b_3(\mathbf{L}_{\rho\rho})_{23} \end{bmatrix}, \tag{118}$$

$$\left[\left(\boldsymbol{\Lambda}_{\rho\rho} \otimes_{23} \boldsymbol{\beta}\right)_3\right] = \begin{bmatrix} b_1(\mathbf{L}_{\rho\rho})_{31} & b_1(\mathbf{L}_{\rho\rho})_{32} & b_1(\mathbf{L}_{\rho\rho})_{33} \\ b_2(\mathbf{L}_{\rho\rho})_{31} & b_2(\mathbf{L}_{\rho\rho})_{32} & b_2(\mathbf{L}_{\rho\rho})_{33} \\ b_3(\mathbf{L}_{\rho\rho})_{31} & b_3(\mathbf{L}_{\rho\rho})_{32} & b_3(\mathbf{L}_{\rho\rho})_{33} \end{bmatrix}, \tag{119}$$

and $\mathbf{L}_{\rho\rho}$ is obtained from (103) and $\mathbf{b} = \mathbf{T}_{F_i}\boldsymbol{\beta}$.

Remarkably, the same notational scheme as in the previous formula can be exploited for the tensor in (90) since $\mathbf{G}_i$ can be obtained from (44) by standard matrix operations.

Furthermore, setting $\mathbb{M} = \mathbf{G}_i \otimes_{23} \mathbf{n}_i$, the matrix $[\mathbb{M}]$ can be obtained analogously to (116). Stated equivalently, to construct the matrix associated with the rank-three tensor $\mathbb{M}$, one has to first evaluate $\boldsymbol{\Phi}_{F_i}$, transform it as in (44) to get $\mathbf{G}_i$, and exploit the notational scheme (116) by replacing $\mathbf{L}_{\rho\rho}$ with $\mathbf{G}_i$.

The notational schemes detailed in (101)-(102), (104)-(105), (109)-(110) and (115)-(116) can be suitably exploited to evaluate the tensors in (91)-(97) and, hence, the tensor $\mathbb{D}^{\partial\Omega}_{\mathbf{rrr}}$ in (50). Namely, the tensors $\mathbf{G}_i \otimes \mathbf{n}_i$ in (91) and $\mathbf{H}_i \otimes \mathbf{n}_i$ in (97) can be evaluated by applying the scheme (105), the tensor $\mathbb{G}_i$ in (92) by applying the scheme (101)-(102) and the tensor $\mathbf{H}_i \otimes_{23} \mathbf{n}_i$ in (96) by applying the scheme (115)-(116). Finally, the tensors in (93) and (95) are rank-two tensors and the tensor in (94) can be evaluated as in (102).

3 Gravity anomaly of polyhedral bodies at an arbitrary point P

In the previous sections it has been assumed that the observation point P would coincide with the origin of the reference frame in which the anomalous density of a body is assigned.

This has allowed us to set the stage and to define the most problematic issues to address, both from the analytical and numerical point of view.

However when gravity measures are carried out at several points and/or when multiple bodies are taken into account it is by far more convenient to fix an arbitrary reference frame in which both the coordinates of each observation point and the density of all bodies are simultaneously assigned.

To suitably extend the formulas contributed in the previous section, one can exploit a coordinate transformation (Zhou, 2010) by translating the origin of the reference frame to the observation point and modifying in accordance the expression of the density contrast by expressing the coefficients of the polynomial law in the new reference frame.

Alternatively, one can follow the approach outlined in D'Urso (2015c) and define the position vector $\mathbf{r}$ entering the definition of the gravity anomaly as follows

$$\mathbf{r} = \mathbf{s} - \mathbf{p} \tag{120}$$

where $\mathbf{p}$ is the position vector of the observation point and $\mathbf{s}$ is the position vector of an arbitrary point belonging to Ω, see e.g., fig. 2. In this way we can leave the expression (6) unchanged by writing

$$\Delta\rho(\mathbf{s}) = \theta(x,y,z) = \theta_{\mathbf{o}} + \mathbf{c}\cdot\mathbf{s} + \mathbf{C}\cdot\mathbf{D_{ss}} + \mathbb{C}\cdot\mathbb{D}_{\mathbf{sss}} \tag{121}$$

where $\mathbf{D_{ss}}$ and $\mathbb{D}_{\mathbf{sss}}$ are defined as in (7) and write

$$\Delta g_z(P) = G\int_{\Omega} \frac{\Delta\rho(\mathbf{s})\mathbf{r}\cdot\mathbf{k}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}dV\,. \tag{122}$$

Clearly in the case of multiple observation points P_i and/or bodies one can simply write

$$\Delta g_z(P_i) = G\sum_{j=1}^{N_B}\int_{\Omega_j} \frac{\Delta\rho(\mathbf{s}_j)\mathbf{r}_j\cdot\mathbf{k}}{(\mathbf{r}_j\cdot\mathbf{r}_j)^{3/2}}dV \tag{123}$$

where Ω_j is the domain of the j-th body, N_B is the number of bodies to analyze and $\mathbf{r}_j = \mathbf{s}_j - \mathbf{p}_i$, $\mathbf{p}_i$ being the position vector of P_i with respect to the assigned reference frame having origin at an arbitrary point O. However, being mainly interested to illustrate the rationale of our approach, we shall make reference in the sequel to the case of a single observation point and a single body.

To exploit the results illustrated in the previous section, it is convenient to express $\mathbf{s}$ as function of $\mathbf{r}$ by means of (120). For brevity this is detailed only for the rank-three tensor $\mathbb{D}_{\mathbf{sss}}$ since it is the more cumbersome to handle. In particular, we infer from (120)

$$\mathbb{D}_{\mathbf{sss}} = \mathbf{s}\otimes\mathbf{s}\otimes\mathbf{s} = (\mathbf{r}+\mathbf{p})\otimes(\mathbf{r}+\mathbf{p})\otimes(\mathbf{r}+\mathbf{p}) = \mathbb{D}_{\mathbf{rrr}} + \mathbb{D}_{\mathbf{rrp}} + \mathbb{D}_{\mathbf{ppr}} + \mathbb{D}_{\mathbf{ppp}} \tag{124}$$

where $\mathbb{D}_{\mathbf{ppp}} = \mathbf{p}\otimes\mathbf{p}\otimes\mathbf{p}$,

$$\mathbb{D}_{\mathbf{rrp}} = \mathbf{r}\otimes\mathbf{r}\otimes\mathbf{p} + \mathbf{r}\otimes\mathbf{p}\otimes\mathbf{r} + \mathbf{p}\otimes\mathbf{r}\otimes\mathbf{r} \tag{125}$$

and

$$\mathbb{D}_{\mathbf{ppr}} = \mathbf{p}\otimes\mathbf{p}\otimes\mathbf{r} + \mathbf{p}\otimes\mathbf{r}\otimes\mathbf{p} + \mathbf{r}\otimes\mathbf{p}\otimes\mathbf{p} = \mathbf{D_{pp}}\otimes\mathbf{r} + \mathbf{p}\otimes\mathbf{r}\otimes\mathbf{p} + \mathbf{r}\otimes\mathbf{D_{pp}}\,. \tag{126}$$

Hence, the expression (122) for the gravity anomaly becomes

$$\begin{aligned}\Delta g_z(\mathbf{p}) = G\Big\{&[\theta_{\mathbf{o}} + \mathbf{c}\cdot\mathbf{p} + \mathbf{C}\cdot\mathbf{D_{pp}} + \mathbb{C}\cdot\mathbb{D}_{\mathbf{ppp}}]d^{\Omega}_{\mathbf{r}} + \mathbf{c}\cdot\mathbf{d}^{\Omega}_{\mathbf{r}} + \\ &+\mathbf{C}\cdot[\mathbf{d}^{\Omega}_{\mathbf{r}}\otimes\mathbf{p} + \mathbf{p}\otimes\mathbf{d}^{\Omega}_{\mathbf{r}} + \mathbf{D}^{\Omega}_{\mathbf{rr}}] + \mathbb{C}\cdot[\mathbf{D_{pp}}\otimes\mathbf{d}^{\Omega}_{\mathbf{r}} + \mathbf{p}\otimes\mathbf{d}^{\Omega}_{\mathbf{r}}\otimes\mathbf{p} + \mathbf{d}^{\Omega}_{\mathbf{r}}\otimes\mathbf{D_{pp}}] + \\ &+\mathbb{C}\cdot[\mathbf{D}^{\Omega}_{\mathbf{rr}}\otimes\mathbf{p} + \mathbf{d}^{\Omega}_{\mathbf{r}}\otimes\mathbf{p}\otimes\mathbf{d}^{\Omega}_{\mathbf{r}} + \mathbf{p}\otimes\mathbf{D}^{\Omega}_{\mathbf{rr}}] + \mathbb{C}\cdot\mathbb{D}^{\Omega}_{\mathbf{rrr}}\Big\},\end{aligned} \tag{127}$$

which represents the generalization of (14) to the case $\mathbf{p} \neq \mathbf{o}$.

Special attention has to be paid to the symbol $\mathbf{d}^{\Omega}_{\mathbf{r}}\otimes\mathbf{p}\otimes\mathbf{d}^{\Omega}_{\mathbf{r}}$ which is a shorthand to denote the third-order tensor

$$\mathbf{d}^{\Omega}_{\mathbf{r}}\otimes\mathbf{p}\otimes\mathbf{d}^{\Omega}_{\mathbf{r}} = \int_{\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})\mathbf{r}\otimes\mathbf{p}\otimes\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}dV = \mathbf{D}^{\Omega}_{\mathbf{rr}}\otimes_{23}\mathbf{p}\,. \tag{128}$$

In spite of its symbol, which has been adopted to emphasize its symmetric expression, the tensor above cannot be obtained as triple tensor product of the vectors $\mathbf{d}^{\Omega}_{\mathbf{r}}$ and $\mathbf{p}$. Rather, it is conveniently computed starting from the rank-two tensor $\mathbf{D}^{\Omega}_{\mathbf{rr}}$, after having computed its algebraic expression, as detailed in subsection 2.6.

Although $\mathbf{r}$ is now defined from (120) it can be shown that formula (17) holds as well. Thus, recalling (30) and setting

$$\theta_{\mathbf{p}} = \mathbf{c}\cdot\mathbf{p} + \mathbf{C}\cdot\mathbf{D_{pp}} + \mathbb{C}\cdot\mathbb{D}_{\mathbf{ppp}}\,, \tag{129}$$

formula (127) specializes to

$$\begin{aligned}\Delta g_z(\mathbf{p}) = G\Bigg\{&(\theta_{\mathbf{o}} + \theta_{\mathbf{p}})d^{\partial\Omega}_{\mathbf{r}} + \frac{\mathbf{c}\cdot\mathbf{d}^{\partial\Omega}_{\mathbf{r}}}{2} + \mathbf{C}\cdot\Big[\frac{\mathbf{d}^{\partial\Omega}_{\mathbf{r}}}{2}\otimes\mathbf{p} + \mathbf{p}\otimes\frac{\mathbf{d}^{\partial\Omega}_{\mathbf{r}}}{2} + \frac{\mathbf{D}^{\partial\Omega}_{\mathbf{rr}}}{3}\Big] + \\ &+\mathbb{C}\cdot\Big[\frac{1}{2}\Big(\mathbf{D_{pp}}\otimes\mathbf{d}^{\partial\Omega}_{\mathbf{r}} + \mathbf{p}\otimes\mathbf{d}^{\partial\Omega}_{\mathbf{r}}\otimes\mathbf{p} + \mathbf{d}^{\partial\Omega}_{\mathbf{r}}\otimes\mathbf{D_{pp}}\Big) + \\ &+\frac{1}{3}\Big(\mathbf{D}^{\partial\Omega}_{\mathbf{rr}}\otimes\mathbf{p} + \mathbf{d}^{\partial\Omega}_{\mathbf{r}}\otimes\mathbf{p}\otimes\mathbf{d}^{\partial\Omega}_{\mathbf{r}} + \mathbf{p}\otimes\mathbf{D}^{\partial\Omega}_{\mathbf{rr}}\Big) + \frac{\mathbb{D}^{\partial\Omega}_{\mathbf{rrr}}}{4}\Big]\Bigg\}.\end{aligned} \tag{130}$$

Obviously, (130) coincides with (31) when $\mathbf{p} = \mathbf{o}$.

Formula (130) can be operatively evaluated for a a polyhedral body by considering formulas (37), (38), (47) and (50) for $d^{\Omega}_{\mathbf{r}}$, $\mathbf{d}^{\Omega}_{\mathbf{r}}$, $\mathbf{D}^{\Omega}_{\mathbf{rr}}$ and $\mathbb{D}^{\Omega}_{\mathbf{rrr}}$, respectively, and the procedures detailed in subsection 2.3-2.6 to express them in terms of 3D vectors. In particular the third order tensor $\mathbf{d}^{\partial\Omega}_{\mathbf{r}}\otimes\mathbf{p}\otimes\mathbf{d}^{\partial\Omega}_{\mathbf{r}}$ is obtained by applying the notational scheme (115)-(116) and replacing $\mathbf{L}_{\rho\rho}$ with $\mathbf{D}^{\Omega}_{\mathbf{rr}}$ and $\mathbf{b}$ with $\mathbf{p}$, respectively.

4 Eliminable Singularities of the Algebraic Expressions of the Gravity Anomaly

It has already been shown that the analytical expression (31) of the gravity anomaly is singularity-free in the sense that its expression holds rigorously whatever is the position of the point O with respect to Ω. The same property holds true for the expression (130) referred to an arbitrary point P. However their algebraic counterparts, being expressed by means of the quantities detailed in subsection 2.4, do include further singularities.

They are associated with the expression of the line integrals provided in the Appendices since they become singular when the generic face F_i contains the observation point, either O or P, and this belongs to the line containing the j-th edge of the boundary ∂F_i.

However, we are going to prove analytically that the contribution of the singular line integral to the domain integral in which its computation is required is zero. Hence, from the computational point of view, the singularity of the j-th line integral does not have any practical effect and it can be simply ignored when computing the associated domain integral.

As shown in Appendix 2, some of the 2D domain integrals required in the present context, have already been computed in previous papers D'Urso (2013a, 2014a,b) so that the discussion on their singularity-free nature can be found in the quoted reference. Nevertheless we shall systematically prove this property also for these last integrals, namely the ones having $\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}$ in the denominator, since we are going to use new and simpler arguments; the same arguments will be exploited to prove the singularity-free nature of the integrals having $\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}$ in the denominator.

4.1 Eliminable singularity of the integral ψ_{F_i}

We know from formulas (218) and (219) that

$$\begin{aligned}\psi_{F_i} &= \int_{F_i} \frac{dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} = \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) \int_0^1 \frac{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j) \cdot \hat{\boldsymbol{\rho}}_i(\lambda_j) + d_i^2\right]^{1/2}}{\hat{\boldsymbol{\rho}}_i(\lambda_j) \cdot \hat{\boldsymbol{\rho}}_i(\lambda_j)} d\lambda_j - \alpha_i |d_i| = \\ &= \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) \int_0^1 \frac{\left(p_j \lambda_j^2 + 2 q_j \lambda_j + v_j\right)^{1/2}}{p_j \lambda_j^2 + 2 q_j \lambda_j + u_j} d\lambda_j - \alpha_i |d_i| = \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} - \alpha_i |d_i|\end{aligned} \tag{131}$$

where, see also (70), we have set

$$p_j = \Delta\boldsymbol{\rho}_j \cdot \Delta\boldsymbol{\rho}_j = l_j^2 \quad q_j = \boldsymbol{\rho}_j \cdot \Delta\boldsymbol{\rho}_j \quad u_j = \boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j \quad v_j = u_j + d_i^2 = |\mathbf{r}_j|^2 \,. \tag{132}$$

Useful in the sequel are also the quantities (D'Urso, 2013a, 2014a,b)

$$p_j + q_j = \boldsymbol{\rho}_{j+1} \cdot \Delta\boldsymbol{\rho}_j \quad p_j + 2q_j + v_j = \boldsymbol{\rho}_{j+1} \cdot \boldsymbol{\rho}_{j+1} + d_i^2 = |\mathbf{r}_{j+1}|^2 \tag{133}$$

and the discriminant $\Delta_j = q_j^2 - p_j u_j$ of the denominator in (131). In particular, it turns out to be

$$-\Delta_j = p_j u_j - q_j^2 = \left(\boldsymbol{\rho}_{j+1} \cdot \boldsymbol{\rho}_{j+1}\right) \cdot \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j\right) - \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}\right)^2 \geq 0 \tag{134}$$

by virtue of the Cauchy-Schwartz inequality (Tang, 2006).

Clearly, our main concern is when $\Delta_j = 0$. In particular, setting $\boldsymbol{o} = (0,0)$, it is apparent from the previous expression that the denominator of the j-th integral on the right-hand side of (131) can become singular if $\boldsymbol{\rho}_j = \boldsymbol{o}$, $\boldsymbol{\rho}_{j+1} = \boldsymbol{o}$ or $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$ are parallel and point in opposite directions, i.e. if the projection of the observation point onto F_i belongs to the segment $[\boldsymbol{\rho}_j, \boldsymbol{\rho}_{j+1}]$. In turn this may happen independently from the value of d_i, i.e. whether or not the i-th face of the polyhedron Ω does contain the observation point.

In both cases, $d_i \neq 0$ or $d_i = 0$, we are going to prove by mathematical arguments that the contribution of such an edge to ψ_{F_i} is zero so that its computation can be skipped. Let us first consider the case $d_i \neq 0$.

As shown in D'Urso (2013a, 2014a) the evaluation of the line integral on the right-hand side of (131) is carried out by setting $t = \lambda_j + q_j/p_j$; this yields

$$I_{6j} = \int_0^1 \frac{\left(p_j\lambda_j^2 + 2q_j\lambda_j + v_j\right)^{1/2}}{p_j\lambda_j^2 + 2q_j\lambda_j + u_j} d\lambda_j = \frac{1}{\sqrt{p_j}} \int_{q_j/p_j}^{1+q_j/p_j} \frac{\sqrt{t^2 + B_j}}{t^2 + A_j} dt \tag{135}$$

where

$$A_j = -\frac{\Delta_j}{p_j^2} = \frac{p_j u_j - q_j^2}{p_j^2} \qquad B_j = \frac{p_j v_j - q_j^2}{p_j^2} = A_j + \frac{d_i^2}{p_j} = A_j + \frac{d_i^2}{l_j^2}\,. \tag{136}$$

Notice that the denominator in (135) is positive if $-\Delta_j = p_j^2 A_j > 0$. In this case the primitive of the integrand on the right-hand side of (135) becomes

$$I_{6j} = \frac{1}{\sqrt{p_j}} \left\{ \sqrt{\frac{B_j - A_j}{A_j}} \arctan \frac{\sqrt{B_j - A_j}}{\sqrt{A_j}\sqrt{B_j + t^2}} + \ln\left(t + \sqrt{B_j + t^2}\right) \right\}_{q_j/p_j}^{1+q_j/p_j} \tag{137}$$

or equivalently

$$I_{6j} = \left\{ \frac{|d_i|}{\sqrt{-\Delta_j}} \arctan \frac{|d_i|}{\sqrt{-\Delta_j}\sqrt{B_j + t^2}} + \frac{\ln\left(t + \sqrt{B_j + t^2}\right)}{\sqrt{p_j}} \right\}_{q_j/p_j}^{1+q_j/p_j}\,. \tag{138}$$

Conversely, should it be $\Delta_j = 0$, and hence $A_j = 0$, the integrand on the right-hand side of (135) becomes singular at one point belonging to the interval $\left[q_j/p_j,\ 1 + q_j/p_j\right]$. Actually, we infer from (134) and the properties of the Cauchy-Schwartz inequality that $\Delta_j = 0$ if and only if $\boldsymbol{\rho}_j = \boldsymbol{o}$, $\boldsymbol{\rho}_{j+1} = \boldsymbol{o}$ or the segment $[\boldsymbol{\rho}_j,\ \boldsymbol{\rho}_{j+1}]$ contains the null vector in its interior.

Actually if $\boldsymbol{\rho}_j = \boldsymbol{o}$ $\left(\boldsymbol{\rho}_{j+1} = \boldsymbol{o}\right)$, it turns out to be $q_j/p_j = 0$ $\left(1 + q_j/p_j = 0\right)$; hence the denominator in (135) becomes singular since $t^2 + A_j = \boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j/p_j$ $\left(\boldsymbol{\rho}_{j+1} \cdot \boldsymbol{\rho}_{j+1}/p_j\right) = 0$ at the left (right) extreme of the integration integral.

Furthermore, should the projection of the observation point fall within the segment $[\boldsymbol{\rho}_j,\ \boldsymbol{\rho}_{j+1}]$, one has $\boldsymbol{\rho}_{j+1} = \beta_j \boldsymbol{\rho}_j$ $(\beta_j < 0)$ where $q_j/p_j = (\beta_j - 1)\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j/p_j < 0$ and $1 + q_j/p_j = \beta_j(\beta_j - 1)\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j/p_j > 0$. Accordingly, the integration interval in (135) splits in two intervals having 0 as right (left) extreme. At that point, however, $t = 0$ and $A_j = -\Delta_j/p_j^2 = 0$ by assumption so that the integrand in (135) becomes singular.

However, we are going to prove that, in the previous three cases, the singularity is eliminable and that the integral attains a finite value. Let us discuss separately the three cases, namely $\boldsymbol{\rho}_j = \boldsymbol{o}$, $\boldsymbol{\rho}_{j+1} = \boldsymbol{o}$ and $\boldsymbol{\rho}_{j+1} = \beta_j \boldsymbol{\rho}_j$ $(\beta_j < 0)$.

In this first case, $\boldsymbol{\rho}_j = \boldsymbol{o}$, the integration interval is $[0,\ 1]$ and we have singularity of the integrand in (135) at the left extreme while the argument of the logarithm is positive. Thus, recalling (131) and (138), the contribution of the integral I_{6j} to ψ_{F_i} is provided by

$$\left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = \boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp} \left[\frac{|d_i|}{\sqrt{-\Delta_j}} \arctan \frac{|d_i|}{\sqrt{-\Delta_j}\sqrt{B_j + t^2}} + \frac{\ln\left(t + \sqrt{B_j + t^2}\right)}{\sqrt{p_j}} \right]_0^1\,. \tag{139}$$

Setting $\boldsymbol{\rho}_j = |\boldsymbol{\rho}_j|\mathbf{e} = \varepsilon\,\mathbf{e}$ and observing that, on account of (134),

$$-\Delta_j = \left(\boldsymbol{\rho}_{j+1}\cdot\boldsymbol{\rho}_{j+1}\right)|\boldsymbol{\rho}_j|^2 - \left(|\boldsymbol{\rho}_j|\mathbf{e}\cdot\boldsymbol{\rho}_{j+1}\right)^2 = \varepsilon^2\left[\boldsymbol{\rho}_{j+1}\cdot\boldsymbol{\rho}_{j+1} - \left(\mathbf{e}\cdot\boldsymbol{\rho}_{j+1}\right)^2\right], \tag{140}$$

we infer that $\sqrt{-\Delta_j}$ is infinitesimal of the same order as $\varepsilon = |\boldsymbol{\rho}_j|$ when $\varepsilon \to 0$, a property we state by writing $\sqrt{-\Delta_j} = O(\varepsilon)$. Hence (139) becomes

$$\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = \lim_{\varepsilon\to 0}\varepsilon\left\{\left[\frac{|d_i|}{\sqrt{-\Delta_j(\varepsilon)}}\arctan\frac{|d_i|}{\sqrt{-\Delta_j(\varepsilon)}\sqrt{B_j+t^2}}\right]_{\varepsilon}^{1} + \frac{1}{\sqrt{p_j}}\left[\ln\left(t+\sqrt{B_j+t^2}\right)\right]_0^1\right\} \tag{141}$$

since the $\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp} = O(\varepsilon)$ if $\varepsilon \to 0$.

Since the arctan function is finite at $t = 1$ and the same does occur for the ln function at $t = 0$ and $t = 1$, we finally have

$$\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = -|d_i|\lim_{\varepsilon\to 0}\frac{\varepsilon}{\sqrt{-\Delta_j(\varepsilon)}}\arctan\frac{|d_i|}{\sqrt{-\Delta_j(\varepsilon)}\sqrt{B_j+\varepsilon^2}} = -\frac{\pi}{2}|d_i|\,. \tag{142}$$

However if $\boldsymbol{\rho}_j = \boldsymbol{o}$ for the j-th edge, it will turn out to be $\boldsymbol{\rho}_{j+1} = \boldsymbol{o}$ for the $(j-1)$-th edge. Hence the arctan function in (138) will be evaluated in the interval $[-1, \varepsilon]$, with $\varepsilon \to 0$, and one has $\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = \pi|d_i|/2$.

To conclude the total contribution provided to φ_{F_i} by the two edges for which it simultaneously happen that $\boldsymbol{\rho}_j = \boldsymbol{o}$ for the j-th edge and $\boldsymbol{\rho}_{j+1} = \boldsymbol{o}$ for the $(j-1)$-th edge is zero.

A null contribution to φ_{F_i} is also provided by edges for which the projection of the observation point is internal to the edge. In this case $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$ are parallel so that the product $\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}$ is zero. Accordingly, both $\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}$ and $\sqrt{-\Delta_j}$ are $O(\varepsilon)$, that is both of them are infinitesimal of order ε as $\varepsilon \to 0$. In conclusion (139) yields

$$\begin{aligned}\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} &= |d_i|\lim\nolimits_{\varepsilon\to 0}\left\{\frac{\varepsilon}{\sqrt{-\Delta_j(\varepsilon)}}\left[\arctan\frac{|d_i|}{\sqrt{-\Delta_j(\varepsilon)}\sqrt{B_j+t^2}}\right]_{-1}^{0} +\right.\\ &\left. + \frac{\varepsilon}{\sqrt{-\Delta_j(\varepsilon)}}\left[\arctan\frac{|d_i|}{\sqrt{-\Delta_j(\varepsilon)}\sqrt{B_j+t^2}}\right]_0^1 + \frac{\varepsilon}{\sqrt{p_j}}\left[\ln\left(t+\sqrt{B_j+t^2}\right)\right]_0^1\right\} = 0\,.\end{aligned} \tag{143}$$

Actually, the ln function is finite both at $t = 0$ and $t = 1$. Furthermore, by repeating the arguments exploited in (142), the arctan function attains finite and opposite values both at $t = 0$ and $t \pm 1$.

In conclusion we have proved that, when $d_i \neq 0$ and the projection of the observation point does belong to the closed interval having $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$ as extremes, the contribution of the relevant edge can be skipped since the overall contribution to φ_{F_i} associated with such a singular case is lumped within the addend $\alpha_i|d_i|$.

Let us now prove that the same result is obtained if $|d_i| = 0$, i.e. if the face F_i does contain the observation point. In this case the integral in (131) can be expressed as follows

$$\psi_{F_i} = \sum_{j=1}^{N_{E_i}}\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = \sum_{j=1}^{N_{E_i}}\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right)\int_0^1 \frac{d\lambda_j}{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j)\right]^{1/2}} - \alpha_i|d_i|\,. \tag{144}$$

Also in this case, the j-th edge characterized by $\boldsymbol{\rho}_j = \boldsymbol{o}$ or $\boldsymbol{\rho}_{j+1} = \boldsymbol{o}$ or $\boldsymbol{\rho}_{j+1} = \beta_j \boldsymbol{\rho}_j$ ($\beta_j < 0$) does not give any contribution to φ_{F_i}. Let us examine separately the three cases

• $\boldsymbol{\rho}_j = \boldsymbol{o}$

In this case the parameterization (67) yields $\hat{\boldsymbol{\rho}}_i(\lambda_j) = \lambda_j \boldsymbol{\rho}_{j+1}$ so that the j-th integral in (144) becomes

$$I_{6j} = \int_0^1 \frac{d\lambda_j}{\lambda_j \left(\boldsymbol{\rho}_{j+1} \cdot \boldsymbol{\rho}_{j+1}\right)^{1/2}} = \frac{1}{\sqrt{p_j}} \int_0^1 \frac{d\lambda_j}{\lambda_j} . \tag{145}$$

Setting $\varepsilon = |\boldsymbol{\rho}_j|$ and being $\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}$ infinitesimal of order ε, it turns out to be

$$\left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = \frac{1}{\sqrt{p_j}} \lim_{\varepsilon \to 0} \varepsilon \Big[\ln \lambda_j \Big]_{\varepsilon}^{1} = 0 \tag{146}$$

since the logarithm tends to infinite with an arbitrarily low degree.

• $\boldsymbol{\rho}_{j+1} = \boldsymbol{o}$

Setting $\hat{\boldsymbol{\rho}}_i(\lambda_j) = (1-\lambda_j)\boldsymbol{\rho}_j$ the integral in (144) can be written

$$I_{6j} = \frac{1}{\sqrt{u_j}} \int_0^1 \frac{d\lambda_j}{1-\lambda_j} = -\frac{1}{\sqrt{u_j}} \int_1^0 \frac{d\eta_j}{\eta_j} \tag{147}$$

where $\eta_j = 1 - \lambda_j$. Hence, setting $\varepsilon = |\boldsymbol{\rho}_{j+1}|$, one has

$$\left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = -\frac{1}{\sqrt{u_j}} \lim_{\varepsilon \to 0} \varepsilon \Big[\ln \eta_j \Big]_{1}^{\varepsilon} = 0 \tag{148}$$

due to the behavior of the logarithm at infinity.

• $\boldsymbol{\rho}_{j+1}$ parallel to $\boldsymbol{\rho}_j$

We are considering the case in which the observation point is projected onto the face F_i inside the j-th edge $\left[\boldsymbol{\rho}_j, \boldsymbol{\rho}_{j+1}\right]$. Hence we can set $\boldsymbol{\rho}_{j+1} = \beta_j \boldsymbol{\rho}_j, \beta_j < 0$, since $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$ point in opposite directions. Setting

$$\boldsymbol{\rho}_j(\lambda_j) = \left[1 + \lambda_j(\beta_j - 1)\right]\boldsymbol{\rho}_j = \tau_j \boldsymbol{\rho}_j , \tag{149}$$

the integral in (144) becomes

$$\begin{aligned} I_{6j} &= \frac{1}{\sqrt{u_j}} \int_0^1 \frac{d\lambda_j}{|1 + \lambda_j(\beta_j - 1)|} = \frac{1}{(\beta_j - 1)\sqrt{u_j}} \int_1^{\beta_j} \frac{d\tau_j}{|\tau_j|} = \frac{1}{(1-\beta_j)\sqrt{u_j}} \int_{\beta_j}^{1} \frac{d\tau_j}{|\tau_j|} = \\ &= \frac{1}{(1-\beta_j)\sqrt{u_j}} \left[\int_{\beta_j}^{0} \frac{d\tau_j}{|\tau_j|} + \int_0^1 \frac{d\tau_j}{|\tau_j|} \right] = \\ &= \frac{1}{(1-\beta_j)\sqrt{u_j}} \left\{ \Big[\ln \tau_j \Big]_0^{|\beta_j|} + \Big[\ln \tau_j \Big]_0^1 \right\}. \end{aligned} \tag{150}$$

Being $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$ parallel, $\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp} = 0$. Hence, setting $\varepsilon = |\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}|$

$$\left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = \frac{1}{(1-\beta_j)\sqrt{u_j}} \lim_{\varepsilon \to 0} \varepsilon \Big[\ln|\beta_j| - 2\ln\varepsilon \Big] = 0 \tag{151}$$

similarly to (146).

4.2 Eliminable singularity of the integral $\boldsymbol{\psi}_{F_i}$

The expression (220) of the integral

$$\begin{aligned}\boldsymbol{\psi}_{F_i} &= \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} = \sum_{j=1}^{N_{E_i}} I_{4j} \Delta\boldsymbol{\rho}_j^{\perp} = \\ &= \sum_{j=1}^{N_{E_i}} \frac{1}{2\sqrt{p_j}} \left\{ \frac{p_j v_j - q_j^2}{p_j} LN_j + \frac{1}{\sqrt{p_j}} \left[\left(p_j + q_j\right) \sqrt{p_j + 2q_j + v_j} - q_j \sqrt{v_j} \right] \right\} \Delta\boldsymbol{\rho}_j^{\perp}\end{aligned} \tag{152}$$

is composed of two addends. The second one is well-defined, according to (132) and (133), whatever is the value of d_i and the position of j-th edge with respect to the observation point.

The first addend in (152) is well defined for $d_i \neq 0$ since

$$LN_j = \ln k_j = \ln \frac{\boldsymbol{\rho}_{j+1} \cdot \left(\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j\right) + l_j |\mathbf{r}_{j+1}|}{\boldsymbol{\rho}_j \cdot \left(\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j\right) + l_j |\mathbf{r}_j|} \tag{153}$$

on the basis of formula (73) in D'Urso (2014b).

Conversely, should it be $d_i = 0$ and $\boldsymbol{\rho}_i = \boldsymbol{o}$ or $\boldsymbol{\rho}_j = \boldsymbol{o}$ or $\boldsymbol{\rho}_{j+1} = \beta_j \boldsymbol{\rho}_j$ ($\beta_j < 0$), one has

$$\frac{p_j v_j - q_j^2}{p_j} LN_j = \frac{-\Delta_j}{p_j} LN_j = \lim_{\varepsilon \to 0} \frac{-\Delta_j(\varepsilon^2) LN_j(\varepsilon)}{2p_j} = 0 \tag{154}$$

since $-\Delta_j$ tends to zero quadratically and LN_j tends to infinite with an arbitrary low degree.

In conclusion edges characterized by singularities of the relevant integral I_{4j} give no contribution to $\boldsymbol{\psi}_{F_i}$.

4.3 Eliminable singularity of the integral $\boldsymbol{\Psi}_{F_i}$

The expression (208) of the integral

$$\boldsymbol{\Psi}_{F_i} = \sum_{j=1}^{N_{E_i}} \left[\left(I_{4j} \boldsymbol{\rho}_j + I_{5j} \Delta\boldsymbol{\rho}_j \right) \otimes \Delta\boldsymbol{\rho}_j^{\perp} - \frac{\mathbf{I}_{2D}}{3} (\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}) I_{4j} \right] + \frac{d_i^2}{3} \left(\psi_i - |d_i| \alpha_i \right) \tag{155}$$

depends upon the integrals ψ_i, I_{4j} and I_{5j}. The discussion on the well-posedness on ψ_i has already been detailed in subsection 4.1.

Conversely, the integrals I_{4j} and I_{5j} are composed, according to their expressions (215) and (216), of the quantities

$$\sqrt{v_j} \qquad \sqrt{p_j + 2q_j + v_j} \tag{156}$$

and of the additional integral I_{0j}. On the basis of the definition (132) and (134) the radicals in (156) are well-defined whater is value of d_i and the position of the j-th edge with respect to the observation point.

The dependence of the integrals I_{4j} and I_{5j} upon I_{0j} does not give any problem since its expression, according to (211), depends upon LN_j. Differently form (152) the quantity LN_j is not scaled by $p_j v_j - q_j^2$, so that we can not invoke the result (154). However the integral

$\boldsymbol{\Psi}_{F_i}$, and hence LN_j, is required for computing the integrals $\mathfrak{C}_{F_i}$ and $\mathfrak{D}_{F_i}$ in (42) that, in turn, are scaled by d_i in the expressions (47) and (50).

Hence, when d_i is zero, what makes LN_j undefined, we can invoke a result similar to (154) by writing

$$d_i LN_j = \lim_{\varepsilon \to 0} d_i(\varepsilon) LN_j(\varepsilon) = 0\,. \tag{157}$$

Stated equivalently, when $d_i = 0$ the contribution to the integral $\boldsymbol{\Psi}_{F_i}$ provided by the face F_i can be skipped.

4.4 Eliminable singularity of the integral φ_{F_i}

The expression provided in (221) for the integral

$$\varphi_{F_i} = \int_{F_i} \frac{dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} = \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}} \left[\frac{\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}}{|d_i| \sqrt{p_j u_j - q_j^2}} (ATN1_j - AT2N_j) \right] \tag{158}$$

is well-defined whatever is the value of d_i and the position of the j-th edge with respect to the observation point.

Also the case $d_i = 0$ does not represent a problem since φ_{F_i} is premultiplied by d_i in the formulas (37), (38) (47) and (50) for $d_{\mathbf{r}}^{\Omega}$, $\mathbf{d}_{\mathbf{r}}^{\Omega}$, $\mathbf{D}_{\mathbf{rr}}^{\Omega}$ and $\mathbb{D}_{\mathbf{rrr}}^{\Omega}$ respectively. Furhermore the discussion on the well-posedness of the quantity

$$\frac{\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}}{\sqrt{p_j v_j - q_j^2}} \big(ATN1_j - ATN2_j \big) \tag{159}$$

when $d_i = 0$ and the projection of the observation point lies within the segment $\left[\boldsymbol{\rho}_j, \boldsymbol{\rho}_{j+1}\right]$ is completely similar to that reported in subsection 4.1

4.5 Eliminable singularity of the integral $\boldsymbol{\varphi}_{F_i}$

We know from formula (222) that

$$\boldsymbol{\varphi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} = -\sum_{j=1}^{N_{E_i}} I_{0j}\, \Delta\boldsymbol{\rho}_j^{\perp} \tag{160}$$

where I_{0j} is provided by (211). Hence, the discussion on its well-posedness can be carried out similarly to (157) when $d_i = 0$ and the j-th edge does contain the observation point in its interior.

Actually the integral $\boldsymbol{\varphi}_{F_i}$ in the expression (37), (38) (47) and (50) for $d_{\mathbf{r}}^{\Omega}$, $\mathbf{d}_{\mathbf{r}}^{\Omega}$, $\mathbf{D}_{\mathbf{rr}}^{\Omega}$ and $\mathbb{D}_{\mathbf{rrr}}^{\Omega}$ is always scaled by d_i.

4.6 Eliminable singularity of the integral $\boldsymbol{\Phi}_{F_i}$

Recalling the expression (223)

$$\boldsymbol{\Phi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} = -\sum_{j=1}^{N_{E_i}} \left[LN_j \boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} + I_{1j} \Delta\boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} \right] + \psi_{F_i} \mathbf{I}_{2D} \,, \tag{161}$$

we infer that $\boldsymbol{\Phi}_{F_i}$ is well defined whatever is the value of d_i and the position of the observation point with respect to the j-th edge of the face F_i. This is trivial if $d_i \neq 0$ since LN_j, I_{1j} and ψ_{F_i} in the previous expression are well defined.

To discuss the well-posedness of $\boldsymbol{\Phi}_{F_i}$ in the case $d_i = 0$ and when the projection of the observation point onto F_i does belong to the segment $\left[\boldsymbol{\rho}_j, \boldsymbol{\rho}_{j+1}\right]$ we remind that $\boldsymbol{\Phi}_{F_i}$, as well as φ_{F_i} and $\boldsymbol{\varphi}_{F_i}$, is scaled by d_i in the expressions (47) and (50) for $\mathbf{D}^{\Omega}_{\mathbf{rr}}$ and $\mathbb{D}^{\Omega}_{\mathbf{rrr}}$. Hence the well-posedness of $d_i LN_j$ can be assessed as in (157), while that of ψ_{F_i} has been already proved in subsection 4.1.

Finally, according to formula (212), the well-posedness of I_{1j} depends upon that of I_{0j}; in turn this last one depends upon the product $d_i LN_j$ discussed above.

In conclusion we have proved that the gravity anomaly at an arbitrary point P can be computed effectively whatever is its position with respect to the polyhedron Ω. Actually the potential singularity of the integrals involved in the formulas (37), (38), (47) and (50) for $d^{\Omega}_{\mathbf{r}}$, $\mathbf{d}^{\Omega}_{\mathbf{r}}$, $\mathbf{D}^{\Omega}_{\mathbf{rr}}$ and $\mathbb{D}^{\Omega}_{\mathbf{rrr}}$ gives no contribution to the gravity anomaly.

5 Numerical examples

The formulas developed in the previous sections have been coded in a Matlab program in order to check their correctness and robustness. They have been applied to model tests and case studies derived from the specialized literature by assuming the density contrast to vary separately along the horizontal and the vertical directions or along both of them. In all examples the density contrast is expressed in units kilograms per cubic meter while distances are expressed in kilometers; the value of the gravitational constant G is $6,67259\, 10^{-11} m^3 kg^{-1} s^{-2}$.

Results obtained by the proposed approach have been carefully checked by comparing them whith those resulting from a numerical integration of the integrals involved in the computation of the gravity anomaly. They can be useful to allow for a comparison with computations carried out by using different methods or with more complex modellings, e.g. those reqired to evaluate the gravitational effects of an arbitrary volumetric mass layer in which a laterally varying radial density change has been assumed (Kingdon et al., 2009; Tenzer et al., 2012). To give an idea of the computational burden required in both approaches we have included the computing time (CT) obtained by running the Matlab code on a INTEL CORE2 PC with 16Gb of RAM and a i7-4700HQ CPU having clock speed of 2,40 GHz.

The first test has been taken from (García-Abdeslem, 2005) and refers to a prism extending along x and y between 10 and 20 km and delimited by the planes z=0 and z=8 km. Density contrast is expressed by the function

$$\Delta\rho(z) = -747.7 + 203.435z - 26.764z^2 + 1.4247z^3 = p + qz + rz^2 + sz^3 \tag{162}$$

where the density is expressed in kg/m^3 and z in kilometers.

In order to compare our results with those reported in (García-Abdeslem, 2005), the gravity anomaly has been computed at points P having y=15 km, z=-0.15 m and x ranging from 0 to 30 km . In particular the observer location was taken by García-Abdeslem (2005) -15 cm of the top of the prism to avoid a singularity in the analytic solution occurring when the observation and the source coordinates coincide.

Although our approach is singularity-free, as proved in section 4, we have deliberately repeated the computations made by García-Abdeslem (2005) to draw the reader's attention on the uncorrect values reported in fig. 3 of the quoted paper.

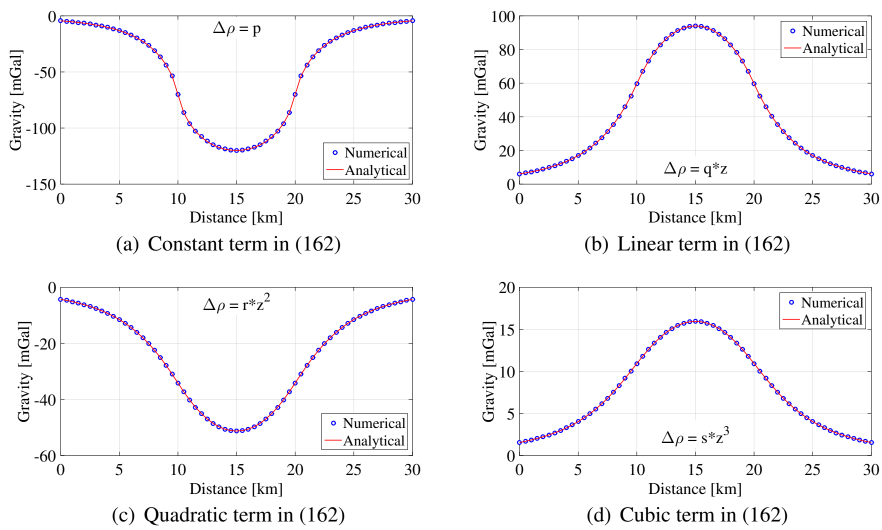
As a matter of fact all mathematical formulas in (García-Abdeslem, 2005) are correct but, for some reasons, the values of the gravity anomaly plotted in fig. 3 have been calculated by assuming wrong integration limits in formula (8) of his paper, namely x_1, y_1, z_1, x_2, y_2, z_2 (lowercase letters) instead of the correct X_1, Y_1, Z_1, X_2, Y_2, Z_2 (capital letters).

In other words formula (8) in (García-Abdeslem, 2005), reported herewith for completeness

$$I_k = \int_{X_1}^{X_2} dX \int_{Y_1}^{Y_2} dY \int_{Z_1}^{Z_2} dZ \left\{ \rho_k \frac{Z^k}{R^3} \right\} \qquad k = 1,2,3,4 \tag{163}$$

is correct but the result plotted in fig. 3 of the quoted paper have been obtained by considering x_1 instead X_1, y_1 instead Y_1 ... and so on. Please notice that, apart ρ_k, the notation in (163) is taken from the original paper so that the observation point is defined by the coordinates $P=(x_0, y_0, z_0)$ and (x,y,z) denote the source coordinates. According to García-Abdeslem (2005) the prism is bounded by the planes $x=x_1$, $y=y_1$, $z=z_1$, $x=x_2$, $y=y_2$, $z=z_2$ and it has been set $X=x-x_0$, $Y=y-y_0$, $Z=z-z_0$.

In conclusion, the correct values of the gravity anomaly at $x_0 \in [0, 30]$ km, $y_0 = 15$ km and $z_0 = -15$ cm, where we have used the notation of (García-Abdeslem, 2005), are reported in figs. 3a, 3b, 3c and 3d respectively for the separate cases of $\Delta\rho = p = \rho_1$, $\Delta\rho = qz = \rho_2$, $\Delta\rho = rz^2 = \rho_3$, $\Delta\rho = sz^3 = \rho_4$,



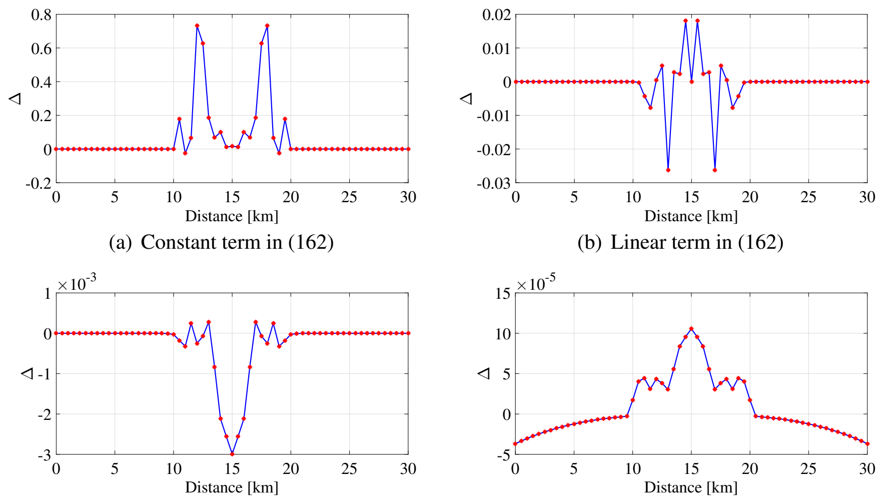
(a) Constant term in (162)

(b) Linear term in (162)

(c) Quadratic term in (162)

(d) Cubic term in (162)

Fig. 3 Gravitational attraction at P=[0,30]×15×(-0.00015) associated with the prism $\Omega \equiv [10, 20]\times[10, 20]\times[0, 8]$ (dimensions in kilometers) and density contrast given by (162).



(a) Constant term in (162)

(b) Linear term in (162)

(c) Quadratic term in (162)

(d) Cubic term in (162)

Fig. 4 Differences Δ between the analytical and numerical values plotted in fig. 3

The correctness of the values reported in fig. 3 has been checked by numerically integrating formula (162) with the aid of the adaptive quadrature procedure implemented in Matlab and by setting X_1=10-x_0, Y_1=10-y_0, Z_1=0.00015, X_2=20-x_0, Y_2=20-y_0, Z_2=8-0.00015. For completeness the differences between the analytical and numerical values reported in fig. 3 are plotted in fig. 4.

To fully test the correctness of the proposed formulation and the robustness of the relevant implementation, we have systematically carried out a comparison of the results associated with the analytical and the numerical evaluation of the integrals involved in the computation of the gravity anomaly. To emphasize the singularity-free nature of our solution, this has been done by considering the example in (García-Abdeslem, 2005) and evaluating the anomaly at z=0 and for several values of y, namely y=10, y=11 km, y=12.5 km and y=15 km.

The gravity anomaly has been evaluated for values of x ranging in the interval [0, 30] km and the relevant values are plotted in fig. 5. For completeness the analytical results are reported in table 1 together with those obtained by numerically evaluating the integrals in formula (163); for the reader's convenience the differences between the analytical and numerical values are plotted in fig. 6. The symbol NaN in table 1 for x=15 km, is due to the fact that the numerical procedure, adopted by Matlab to numerically evaluate the integrals in (163), failed to converge. Notice as well that the numerical procedure, besides being computationally more expensive, gives less precise results when the observation point belongs to Ω, i.e. y=10 km and y=15 km, and x moves towards the center of Ω; actually the numerical solution has only three significant digits at x=10 km and x=20 km.

To give a quick overlook of the symmetric nature of the solution with respect to the planes x=15 km and y=15 km we have reported in fig. 7a the contour plot of the gravity anomaly at z=0. The surface distribution of the gravity anomaly becomes unsymmetric, as shown in fig. 7b, by considering a density contrast depending upon an a horizontal direction

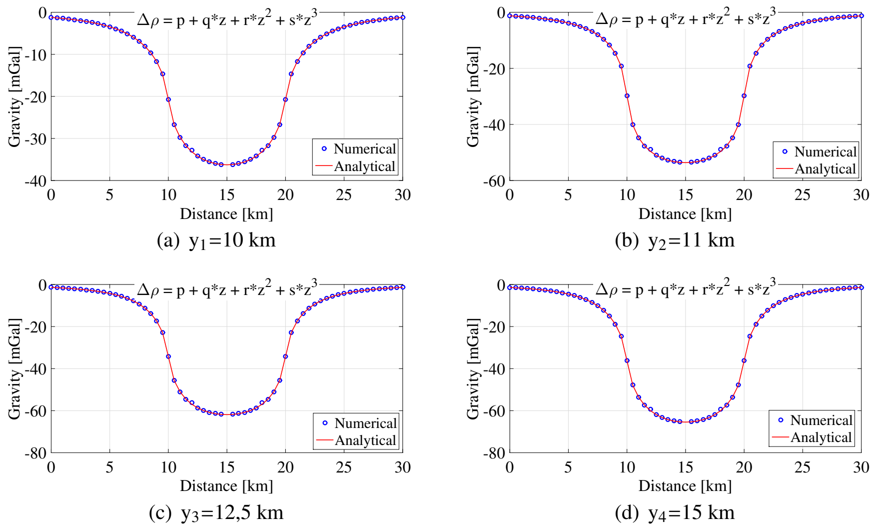


Fig. 5 Gravitational attraction at P=[0,30]×y_k×[0] (k=1,2,3,4) associated with the prism $\Omega \equiv [10, 20] \times [10, 20] \times [0, 8]$ (dimensions in kilometers) and density contrast given by (162).

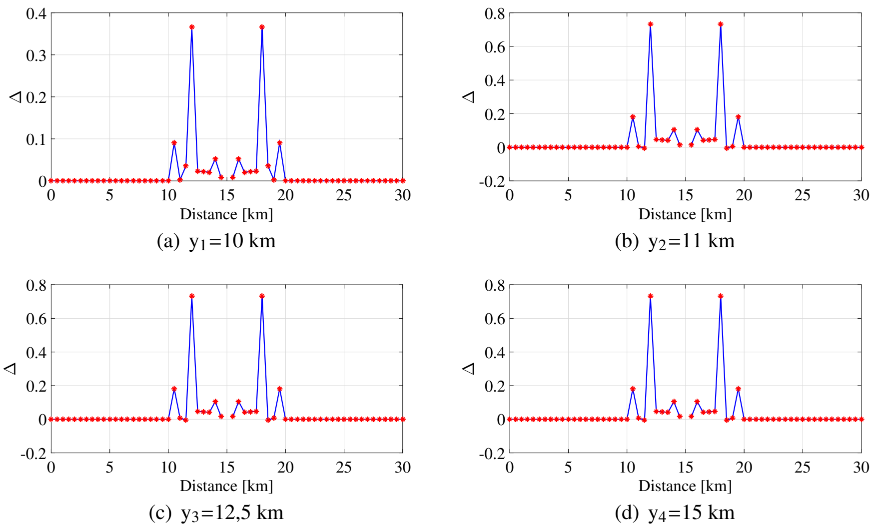


Fig. 6 Differences Δ between the analytical and numerical values plotted in fig. 5.

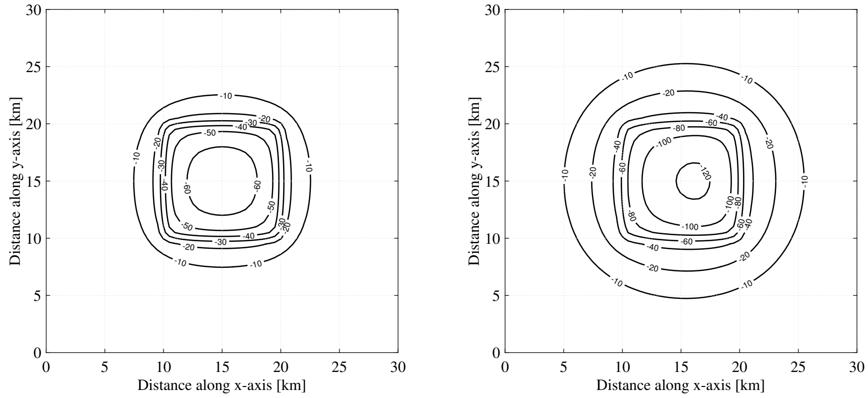
such as the expression considered in Zhou (2009b)

$$\Delta\rho(z) = -747.7 + 203.435z - 26.764z^2 + 1.4247z^3 - 23.205x\,. \tag{164}$$

To emphasize the dependence of the solution upon the monomials appearing in the expression of the density contrast we have plotted in fig. 8a and 8b the surface distribution of the gravity anomaly for the density contrast

$$\Delta\rho(z) = -747.7 + 203.435z - 26.764z^2 + 1.4247z^3 - 23.205y\,, \tag{165}$$

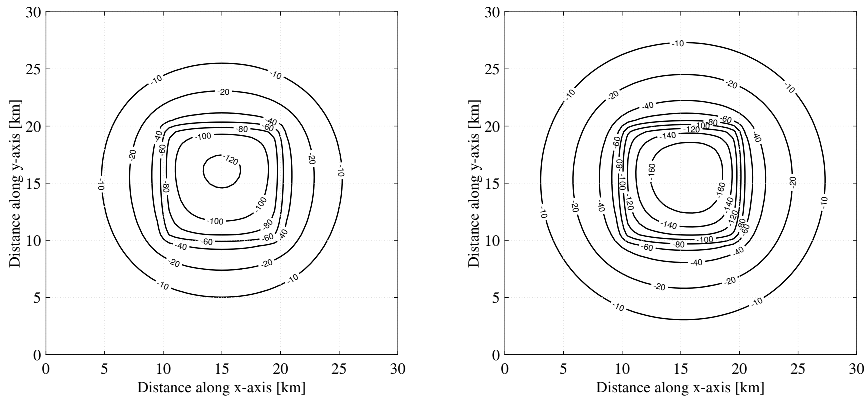
$$\Delta\rho(z) = -747.7 + 203.435z - 26.764z^2 + 1.4247z^3 - 23.205x - 23.205y\,. \tag{166}$$



(a) density contrast as in (162) (b) density contrast as in (164)

Fig. 7 Gravity anomaly distribution at z=0 associated with the prism $\Omega \equiv [10, 20] \times [10, 20] \times [0, 8]$ (dimensions in kilometers) and density contrast given by (162) (on the left) and (164) (on the right).

It is apparent from the last two plots that gravity anomaly vanishes less rapidly than in fig. 7a.



(a) density contrast as in (165) (b) density contrast as in (166)

Fig. 8 Gravity anomaly distribution at z=0 associated with the prism $\Omega \equiv [10, 20] \times [10, 20] \times [0, 8]$ (dimensions in kilometers) and density contrast given by (165) (on the left) and (166) (on the right).

x (km)	0,00	5,00	10,00	15,00	20,00	25,00	30,00	CT
a)	-1,22163576397609	-3,46372618679431	-20,7412785817980	-36,2650788733413	-20,7412785817980	-3,46372618679432	-1,22163576397614	1.9813
b)	-1,22163576397627	-3,46372618679431	-20,7413498102378	NaN	-20,7413498102377	-3,46372618679431	-1,22163576397627	143.4464
z=0 and	y=11 km							
x (km)	0,00	5,00	10,00	15,00	20,00	25,00	30,00	CT
a)	-1.28698607331256	-3.82357120782405	-29.72909079760424	-53.62521739346171	-29.72909079760428	-3.82357120782429	-1.28698607331263	1.8574
b)	-1.28698607331254	-3.82357120782415	-29.72928645482153	NaN	-29.72928645482145	-3.82357120782415	-1.28698607331254	154.6723
z=0 and	y=12,5 km							
x (km)	0,00	5,00	10,00	15,00	20,00	25,00	30,00	CT
a)	-1.36376684444623	-4.25957137389371	-34.23229607059629	-61.88280073665107	-34.23229607059632	-4.25957137389369	-1.36376684444629	1.894
b)	-1.36376684444609	-4.25957137389370	-34.23243794205016	NaN	-34.23243794205009	-4.25957137389370	-1.36376684444609	142.5479
z=0 and	y=15 km							
x (km)	0,00	5,00	10,00	15,00	20,00	25,00	30,00	CT
a)	-1,41650677516557	-4,56182411878455	-36,2650788733413	-65,4288804280923	-36,2650788733413	-4,56182411878455	-1,41650677516557	1.9127
b)	-1,41650677516342	-4,56182411878455	-36,2652685757159	NaN	-36,2652685757159	-4,56182411878455	-1,41650677516557	156.1096

6 Conclusions

The gravity anomaly at arbitrary points induced by a polyhedral body of arbitrary shape body whose shape is an arbitrary and characterized by polynomial density contrast has been obtained in closed form. It is expressed as sum of quantities that depend only upon the 3D coordinates of the vertices of the polyhedron and upon the parameters defining the density contrast. The solution procedure, based upon a generalized application of Gauss theorem, takes consistently into account the singularity intrinsic to the integrals to evaluate. In particular, by means of rigorous mathematical arguments, singularities are proved to give no contribution both to the analytical expression of the gravity anomaly and to its algebraic counterpart.

The formulation presented in the paper has been limited to polynomial density contrast varying with a cubic law as a maximum but it can be easily extended to polynomials of higher degree. The effectiveness of the proposed approach has been intensively tested by numerical comparisons, carried out by means of a Matlab code, with several example derived from the specialized literature. Future contributions will concern the cases of density contrast variable with exponential law for 2D and 3D domains.

7 Appendix 1 - Algebraic expression of integrals

We are going to show that the 2D integrals

$$\int_{F_i} \frac{[\otimes \boldsymbol{\rho}_i, m]}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \quad m \in [0,4] \tag{167}$$

can be evaluated analytically. As a matter of fact we only need to evaluate the integrals for $m = 3$ and $m = 4$

$$\mathfrak{C}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \qquad \mathfrak{D}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \,, \tag{168}$$

since the additional ones in (167) have been already computed in D'Urso (2013a, 2014a,b). For completeness these last ones are reported in Appendix 2.

A further integral, namely

$$\boldsymbol{\Psi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} dA_i \,, \tag{169}$$

required for the computation of the integrals (168), will be dealt with at the end of this Appendix.

The rationale for evaluating the integrals (168) is to first apply the generalized Gauss theorem D'Urso (2013a, 2014a) to transform them into 1D integrals and, subsequently, to compute such integrals by means of algebraic expressions depending upon the 2D coordinates of the vertices that define the face F_i.

In order to apply the Gauss theorem to the integrals in (168) let us first prove the identity

$$\mathrm{grad}\,[\varphi(\mathbf{a} \otimes \mathbf{b})] = (\mathbf{a} \otimes \mathbf{b}) \otimes \mathrm{grad}\varphi + \varphi\,\mathrm{grad}\mathbf{a} \otimes \mathbf{b} + \varphi\,\mathbf{a} \otimes \mathrm{grad}\mathbf{b} \,, \tag{170}$$

holding for scalar φ and vector $(\mathbf{a}, \mathbf{b})$ differentiable fields.

It can be easily verified by applying the chain rule to the ijk component of the third-order tensor on the left-hand side

$$\left\{\operatorname{grad}[\varphi(\mathbf{a}\otimes\mathbf{b})]\right\}_{jkq} = \left(\varphi a_j b_k\right)_{/q} = \varphi_{/q}\, a_j\, b_k + \varphi\, a_{j/q}\, b_k + \varphi\, a_j\, b_{k/q}\,. \tag{171}$$

In a similar fashion one can prove the further differential identity involving fourt-order tensors

$$\operatorname{grad}[\varphi(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c})] = (\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c})\operatorname{grad}\varphi + \varphi\operatorname{grad}\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c} + \varphi\,\mathbf{a}\otimes\operatorname{grad}\mathbf{b}\otimes\mathbf{c} + \varphi\,\mathbf{a}\otimes\mathbf{b}\otimes\operatorname{grad}\mathbf{c}\,. \tag{172}$$

Let us now apply the identity (171) as follows

$$\left[\operatorname{grad}\left(\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}\right)\right]_{jkq} = -\left[\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\right]_{jkq} + \frac{(\boldsymbol{\rho}_i)_{j/q}(\boldsymbol{\rho}_i)_k}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}} + \frac{(\boldsymbol{\rho}_i)_j(\boldsymbol{\rho}_i)_{k/q}}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}} \tag{173}$$

since

$$\operatorname{grad}\left[\frac{1}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}\right] = -\frac{\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\,. \tag{174}$$

Thus, being $(\boldsymbol{\rho}_i)_{j/q} = \delta_{jq}$ we infer from (173)

$$\operatorname{grad}\left(\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}\right) = -\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}} + \frac{\mathbf{I}_{2D}\otimes_{23}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}} + \frac{\boldsymbol{\rho}_i\otimes\mathbf{I}_{2D}}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}} \tag{175}$$

where $\mathbf{I}_{2D}$ is the 2D identity tensor and $\otimes_{23}$ denotes the tensor product obtained by interchanging the second and third index of the rank-three tensor $\mathbf{I}_{2D}\otimes\boldsymbol{\rho}_i$.

The integral over F_i of the first addend in the formula above can be transformed into a boundary integral by exploiting the differential identity (Bowen and Wang, 2006)

$$\int_{\Omega}\operatorname{grad}\mathbf{S}\,dV = \int_{\partial\Omega}\mathbf{S}\otimes\mathbf{n}\,dA \tag{176}$$

where $\mathbf{S}$ is a continuous tensor field.

Thus, integrating over F_i the previous relation and recalling the definition (64) one has

$$\int_{F_i}\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i = -\int_{\partial F_i}\frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}ds_i + \mathbf{I}_{2D}\otimes_{23}\boldsymbol{\psi}_{F_i} + \boldsymbol{\psi}_{F_i}\otimes\mathbf{I}_{2D} \tag{177}$$

where $\boldsymbol{\nu}$ is the unit normal pointing outwards the boundary ∂F_i of the i-th face F_i of the polyhedron.

Hence the first integral on the right-hand side of (177) becomes

$$\int_{\partial F_i}\frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}ds_i = \sum_{j=1}^{N_{E_i}}\int_0^{l_j}\frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)ds_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}\otimes\boldsymbol{\nu}_j \tag{178}$$

since $\boldsymbol{\nu}$ is constant on each of the N_{E_i} edges belonging to ∂F_i.

Recalling (68) and (73), formula (178) becomes

$$\int_{\partial F_i}\frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}ds_i = \sum_{j=1}^{N_{E_i}}\int_0^1\frac{\hat{\boldsymbol{\rho}}_i(\lambda_j)\otimes\hat{\boldsymbol{\rho}}_i(\lambda_j)d\lambda_j}{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j)+d_i^2\right]^{1/2}}\otimes\Delta\boldsymbol{\rho}_j^{\perp} \tag{179}$$

and the integral on the right-hand side can be further transformed by defining

$$\mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} = \boldsymbol{\rho}_j \otimes \boldsymbol{\rho}_j \quad \mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} = \boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j + \Delta\boldsymbol{\rho}_j \otimes \boldsymbol{\rho}_j \quad \mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} = \Delta\boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j . \tag{180}$$

Actually, recalling the parametrization (67) one has

$$\hat{\boldsymbol{\rho}}_i(\lambda_j) \otimes \hat{\boldsymbol{\rho}}_i(\lambda_j) = \mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + \lambda_j \mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + \lambda_j^2 \mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} , \tag{181}$$

$$\int_0^1 \frac{\hat{\boldsymbol{\rho}}_i(\lambda_j) \otimes \hat{\boldsymbol{\rho}}_i(\lambda_j) d\lambda_j}{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j) \cdot \hat{\boldsymbol{\rho}}_i(\lambda_j) + d_i^2\right]^{1/2}} = I_{0j} \mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j} \mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{2j} \mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} \tag{182}$$

where the explicit expression of the integrals

$$\begin{aligned} I_{0j} &= \int_0^1 \frac{d\lambda_j}{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j) \cdot \hat{\boldsymbol{\rho}}_i(\lambda_j) + d_i^2\right]^{1/2}} \qquad I_{1j} = \int_0^1 \frac{\lambda_j d\lambda_j}{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j) \cdot \hat{\boldsymbol{\rho}}_i(\lambda_j) + d_i^2\right]^{1/2}} \\ I_{2j} &= \int_0^1 \frac{\lambda_j^2 d\lambda_j}{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j) \cdot \hat{\boldsymbol{\rho}}_i(\lambda_j) + d_i^2\right]^{1/2}} \end{aligned} \tag{183}$$

is provided in Appendix 2.

In conclusion it turns out be

$$\int_{\partial F} \frac{\boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\nu}(s_i)}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} ds_i = \sum_{j=1}^{N_{E_i}} \left[I_{0j} \mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j} \mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{2j} \mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} \right] \otimes \Delta\boldsymbol{\rho}_j^{\perp} , \tag{184}$$

so that the integral of interest can be computed as fallows on account of (177)

$$\begin{aligned} \mathfrak{C}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i &= - \sum_{j=1}^{N_{E_i}} \left[I_{0j} \mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j} \mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{2j} \mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} \right] \otimes \Delta\boldsymbol{\rho}_j^{\perp} + \\ &+ \mathbf{I}_{2D} \otimes_{23} \boldsymbol{\psi}_{F_i} + \boldsymbol{\psi}_{F_i} \otimes \mathbf{I}_{2D} \end{aligned} \tag{185}$$

where the expression of $\boldsymbol{\psi}_{F_i}$ as explicit function of the position vectors defining the boundary of F_i is provided at the end of this Appendix.

Of interest is also the composition of the third-order tensor above with the vector $\boldsymbol{\kappa}_i$ since it appears in the expressions (47), (50) and (49). For this end let us first notice that

$$\begin{aligned} \left[\left(\mathbf{I}_{2D} \otimes_{23} \boldsymbol{\psi}_{F_i} \right) \boldsymbol{\kappa}_i \right]_{jk} &= \left(\mathbf{I}_{2D} \otimes_{23} \boldsymbol{\psi}_{F_i} \right)_{jkp} \left(\boldsymbol{\kappa}_i \right)_p = I_{jp} \left(\boldsymbol{\psi}_{F_i} \right)_k \left(\boldsymbol{\kappa}_i \right)_p = \\ &= \delta_{jp} \left(\boldsymbol{\kappa}_i \right)_p \left(\boldsymbol{\psi}_{F_i} \right)_k = \left(\boldsymbol{\kappa}_i \right)_j \left(\boldsymbol{\psi}_{F_i} \right)_k = \left(\boldsymbol{\kappa}_i \otimes \boldsymbol{\psi}_{F_i} \right)_{jk} . \end{aligned} \tag{186}$$

Hence

$$\begin{aligned} \mathfrak{C}_{F_i} \boldsymbol{\kappa}_i = \int_{F_i} \frac{(\boldsymbol{\rho}_i \cdot \boldsymbol{\kappa}_i)(\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i)}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i &= - \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\kappa}_i \cdot \Delta\boldsymbol{\rho}_j^{\perp} \right) \left(I_{0j} \mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j} \mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{2j} \mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} \right) + \\ &+ \boldsymbol{\kappa}_i \otimes \boldsymbol{\psi}_{F_i} + \boldsymbol{\psi}_{F_i} \otimes \boldsymbol{\kappa}_i \end{aligned} \tag{187}$$

so that the right-hand side fulfills the symmetry of the tensor on the left-hand side of the previous expression.

To evaluate analytically the second integral in (168) we exploit the identity (172) to get

$$\left[\operatorname{grad}\left(\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}\right)\right]_{jkpq}=-\left[\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\right]_{jkpq}+\frac{\delta_{jq}(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)_{kp}}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}+ \\ +\frac{\delta_{kq}(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)_{jp}}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}+\frac{\delta_{pq}(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)_{jk}}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}, \tag{188}$$

or equivalently

$$\operatorname{grad}\left(\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}\right)=-\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}+\frac{\mathbf{I}_{2D}\otimes_{24}(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}+ \\ +\frac{(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)\otimes_{23}\mathbf{I}_{2D}}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}+\frac{(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)\otimes\mathbf{I}_{2D}}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}} \tag{189}$$

where $\otimes_{24}$ denotes the tensor product obtained by interchanging the second and fourth index of the rank-four tensor $\mathbf{I}_{2D}\otimes(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)$.

Integrating the previous relation over F_i and applying Gauss theorem yields

$$\mathfrak{D}_{F_i}=\int_{F_i}\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i=-\int_{\partial F_i}\frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}ds_i+ \\ +\mathbf{I}_{2D}\otimes_{24}\boldsymbol{\Psi}_{F_i}+\boldsymbol{\Psi}_{F_i}\otimes_{23}\mathbf{I}_{2D}+\boldsymbol{\Psi}_{F_i}\otimes\mathbf{I}_{2D} \tag{190}$$

where $\boldsymbol{\Psi}_{F_i}$ is analytically evaluated in formula (208) of Appendix 2.

In view of the ensuing developments we further set

$$\mathbb{E}_{\rho_j\rho_j\rho_j}=\boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j \qquad \mathbb{E}_{\rho_j\rho_j\Delta\rho_j}=\boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j+\boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j+\Delta\boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j \tag{191}$$

$$\mathbb{E}_{\rho_j\Delta\rho_j\Delta\rho_j}=\boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j+\Delta\boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j+\Delta\boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j \tag{192}$$

$$\mathbb{E}_{\Delta\rho_j\Delta\rho_j\Delta\rho_j}=\Delta\boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j \tag{193}$$

yielding

$$\hat{\boldsymbol{\rho}}_i(\lambda_j)\otimes\hat{\boldsymbol{\rho}}_i(\lambda_j)\otimes\hat{\boldsymbol{\rho}}_i(\lambda_j)=\mathbb{E}_{\rho_j\rho_j\rho_j}+\lambda_j\mathbb{E}_{\rho_j\rho_j\Delta\rho_j}+\lambda_j^2\mathbb{E}_{\rho_j\Delta\rho_j\Delta\rho_j}+\lambda_j^3\mathbb{E}_{\Delta\rho_j\Delta\rho_j\Delta\rho_j}. \tag{194}$$

Accordingly, the integral on the right-hand side in (190) becomes

$$\int_{\partial F_i}\frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}ds_i=\sum_{j=1}^{N_{E_i}}\int_0^1\left\{\frac{\hat{\boldsymbol{\rho}}_i(\lambda_j)\otimes\hat{\boldsymbol{\rho}}_i(\lambda_j)\otimes\hat{\boldsymbol{\rho}}_i(\lambda_j)d\lambda_j}{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j)+d_i^2\right]^{1/2}}\otimes\Delta\boldsymbol{\rho}_j^{\perp}\right\}= \\ =-\sum_{j=1}^{N_{E_i}}\Big[I_{0j}\,\mathbb{E}_{\rho_j\rho_j\rho_j}+I_{1j}\,\mathbb{E}_{\rho_j\rho_j\Delta\rho_j}+ \\ +I_{2j}\,\mathbb{E}_{\rho_j\Delta\rho_j\Delta\rho_j}+I_{3j}\,\mathbb{E}_{\Delta\rho_j\Delta\rho_j\Delta\rho_j}\Big]\otimes\Delta\boldsymbol{\rho}_j^{\perp} \tag{195}$$

where the integrals I_{0j}, I_{1j}, I_{2j} and I_{3j} are explicitly evaluated in the Appendix 2.

In conclusion one has

$$\int_{\partial F_i} \frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}ds_i = \sum_{j=1}^{N_{E_i}}\Big[I_{0j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\boldsymbol{\rho}_j}+I_{1j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}+ + I_{2j}\,\mathbb{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}+I_{3j}\,\mathbb{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}\Big]\otimes\Delta\boldsymbol{\rho}_j^{\perp}+ + \mathbf{I}_{2D}\otimes_{24}\boldsymbol{\Psi}_{F_i}+\boldsymbol{\Psi}_{F_i}\otimes_{23}\mathbf{I}_{2D}+\boldsymbol{\Psi}_{F_i}\otimes\mathbf{I}_{2D}\,. \tag{196}$$

The composition of the previous integral with $\boldsymbol{\kappa}_i$, a quantity that is needed in (175) and (to be displayed), yields a third-order tensor. The contribution to the jkp component of this tensor provided by the tensor product $\boldsymbol{\Psi}_{F_i}\otimes_{23}\mathbf{I}_{2D}$ is given by

$$\begin{aligned}\Big[\big(\boldsymbol{\Psi}_{F_i}\otimes_{23}\mathbf{I}_{2D}\big)\,\boldsymbol{\kappa}_i\Big]_{jkp} &= \big(\boldsymbol{\Psi}_{F_i}\otimes_{23}\mathbf{I}_{2D}\big)_{jkpq}\big(\boldsymbol{\kappa}_i\big)_q = \big(\boldsymbol{\Psi}_{F_i}\big)_{jp}\big(\delta_{kq}\big)\big(\boldsymbol{\kappa}_i\big)_q = \\ &= \big(\boldsymbol{\Psi}_{F_i}\big)_{jp}\big(\boldsymbol{\kappa}_i\big)_k = \big(\boldsymbol{\Psi}_{F_i}\otimes_{23}\boldsymbol{\kappa}_i\big)_{jkp}\,.\end{aligned} \tag{197}$$

Analogously

$$\begin{aligned}\Big[\big(\mathbf{I}_{2D}\otimes_{24}\boldsymbol{\Psi}_{F_i}\big)\,\boldsymbol{\kappa}_i\Big]_{jkp} &= \big(\mathbf{I}_{2D}\otimes_{24}\boldsymbol{\Psi}_{F_i}\big)_{jkpq}\big(\boldsymbol{\kappa}_i\big)_q = \big(\delta_{jq}\big)\big(\boldsymbol{\Psi}_{F_i}\big)_{pk}\big(\boldsymbol{\kappa}_i\big)_q = \\ &= \big(\boldsymbol{\kappa}_i\big)_j\big(\boldsymbol{\Psi}_{F_i}\big)_{pk} = \big(\boldsymbol{\kappa}_i\big)_j\big(\boldsymbol{\Psi}_{F_i}\big)_{kp} = \big(\boldsymbol{\kappa}_i\otimes\boldsymbol{\Psi}_{F_i}\big)_{jkp}\end{aligned} \tag{198}$$

where the identity $\big(\boldsymbol{\Psi}_{F_i}\big)_{pk} = \big(\boldsymbol{\Psi}_{F_i}\big)_{kp}$ stems from the symmetry of $\boldsymbol{\Psi}_{F_i}$. Accordingly, we infer from (190) and (196)

$$\mathfrak{D}_{F_i}\boldsymbol{\kappa}_i = \int_{F_i}\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\boldsymbol{\kappa}_i = -\sum_{j=1}^{N_{E_i}}\big(\boldsymbol{\kappa}_i\cdot\Delta\boldsymbol{\rho}_j^{\perp}\big)\big(I_{0j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\boldsymbol{\rho}_j}+I_{1j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}+ + I_{2j}\,\mathbb{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}+I_{3j}\,\mathbb{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}\big)+ + \boldsymbol{\Psi}_{F_i}\otimes\boldsymbol{\kappa}_i+\boldsymbol{\Psi}_{F_i}\otimes_{23}\boldsymbol{\kappa}_i+\boldsymbol{\kappa}_i\otimes\boldsymbol{\Psi}_{F_i}\,. \tag{199}$$

The expression (185) for $\mathfrak{C}_{F_i}$ and (190) for $\mathfrak{D}_{F_i}$ require the computation of the integral $\boldsymbol{\Psi}_{F_i}$ defined in formula (169); it is evaluated analytically by invoking the differential identity

$$\text{grad}\big[\varphi\mathbf{a}\big] = \mathbf{a}\otimes\text{grad}\varphi+\varphi\,\text{grad}\mathbf{a} \tag{200}$$

holding for differentiable scalar (φ) and vector ($\mathbf{a}$) fields. Actually, applying the previous identity as follows

$$\text{grad}\big[(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}\boldsymbol{\rho}_i\big] = \frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}+(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}\mathbf{I}_{2D}\,, \tag{201}$$

integrating over F_i and setting

$$\iota_{F_i} = \int_{F_i}\big(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2\big)^{1/2}dA_i \tag{202}$$

one has

$$\boldsymbol{\Psi}_{F_i} = \int_{\partial F_i} \left[\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2} \boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}_i(s_i)ds_i - \iota_{F_i}\mathbf{I}_{2D}\,. \tag{203}$$

To compute the domain integral (202), we apply the differential identity

$$\text{div}\left[\varphi\mathbf{a}\right] = \text{grad}\varphi\cdot\mathbf{a} + \varphi\,\text{div}\mathbf{a} \tag{204}$$

to the vector field $\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2}\boldsymbol{\rho}_i$ to get

$$\text{div}\left[\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2}\boldsymbol{\rho}_i\right] = \frac{\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i}{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2}} + 2\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2}. \tag{205}$$

Adding and subtracting d_i^2 to the numerator yields

$$\text{div}\left[\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2}\boldsymbol{\rho}_i\right] = 3\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2} - \frac{d_i^2}{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2}}, \tag{206}$$

so that, upon integrating over F_i and applying Gauss theorem, one has

$$\iota_{F_i} = \frac{1}{3}\int_{\partial F_i} \left[\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2} \boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\nu}(s_i)ds_i - \frac{d_i^2}{3}\psi_{F_i}, \tag{207}$$

by recalling definition (62). In conclusion, we infer from (203) and the previous expression

$$\begin{aligned}
\boldsymbol{\Psi}_{F_i} &= \int_{\partial F_i} \left[\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2} \boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)ds_i - \\
&\quad - \frac{\mathbf{I}_{2D}}{3}\left\{\int_{\partial F_i} \left[\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2} \boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\nu}(s_i)ds_i - d_i^2\psi_{F_i}\right\} \\
&= \sum_{j=1}^{N_{E_i}}\left\{\left[\int_0^{l_j}\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2}\boldsymbol{\rho}_i ds_j\right]\otimes\boldsymbol{\nu}_j - \right. \\
&\qquad \left. - \frac{\mathbf{I}_{2D}}{3}\left[\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\nu}_j\right)\int_0^{l_j}\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{1/2}ds_j\right]\right\} + \frac{d_i^2}{3}\psi_{F_i} = \\
&= \sum_{j=1}^{N_{E_i}}\left\{\left[\int_0^{1}\left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j) + d_i^2\right]^{1/2}\left(\boldsymbol{\rho}_j + \lambda_j\Delta\boldsymbol{\rho}_j\right)d\lambda_j\right]\otimes\Delta\boldsymbol{\rho}_j^{\perp} - \right. \\
&\qquad \left. - \frac{\mathbf{I}_{2D}}{3}\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right)\int_0^{1}\left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j) + d_i^2\right]^{1/2}d\lambda_j\right\} + \frac{d_i^2}{3}\left(\psi_i - |d_i|\alpha_i\right) = \\
&= \sum_{j=1}^{N_{E_i}}\left[\left(I_{4j}\boldsymbol{\rho}_j + I_{5j}\Delta\boldsymbol{\rho}_j\right)\otimes\Delta\boldsymbol{\rho}_j^{\perp} - \frac{\mathbf{I}_{2D}}{3}(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp})I_{4j}\right] + \frac{d_i^2}{3}\left(\psi_i - |d_i|\alpha_i\right)
\end{aligned} \tag{208}$$

where ψ_i is defined in (219).

We have numerically verified that the sum over the N_{E_i} edges of the first addend on the right-hand side returns a symmetric rank-two tensor as the one the left-hand side.

8 Appendix 2 - Available expressions of integrals

We hereby collect some known formulas in order to allow the reader to implement the expression of the gravity anomaly contributed in the main body of the paper.

We first report the algebraic expression of some definite integrals that will be repeatedly referred to in the sequel; they have been computed elsewhere D'Urso (2013a, 2014a,b) though with a different denomination. Making reference to the quantities p_j, q_j, u_j, v_j introduced in formula (71), we set

$$ATN1_j = \arctan \frac{|d_i|(p_j + q_j)}{\sqrt{p_j u_j - q_j^2}\,\sqrt{p_j + 2q_j + v_j}}\,, \tag{209}$$

$$ATN2_j = \arctan \frac{|d_i| q_j}{\sqrt{p_j u_j - q_j^2}\,\sqrt{v_j}} \tag{210}$$

where the suffix $(\cdot)_j$ has been added to remind that they all refer to the j-th edge of the generic face F_i.

Of interest are also the following integrals

$$I_{0j} = \int_0^1 \frac{d\lambda_j}{\left[p_j\lambda^2 + 2q_j\lambda_j + v_j\right]^{1/2}} = \ln k_j = \ln \frac{p_j + q_j + \sqrt{p_j}\,\sqrt{p_j + 2q_j + v_j}}{q_j + \sqrt{p_j v_j}} = LN_j\,, \tag{211}$$

$$I_{1j} = \int_0^1 \frac{\lambda_j d\lambda_j}{\left[p_j\lambda^2 + 2q_j\lambda_j + v_j\right]^{1/2}} = \frac{1}{p_j}\left\{\sqrt{p_j + 2q_j + v_j} - \sqrt{v_j} - \frac{q_j}{\sqrt{p_j}} I_{0j}\right\}, \tag{212}$$

$$\begin{aligned} I_{2j} = \int_0^1 \frac{\lambda_j^2 d\lambda_j}{\left[p_j\lambda^2 + 2q_j\lambda_j + v_j\right]^{1/2}} &= \frac{1}{2p_j^2}\left[(p_j - 3q_j)\sqrt{p_j + 2q_j + v_j} + 3q_j\sqrt{v_j}\right] + \\ &+ \frac{3q_j^2 - p_j v_j}{2p_j^{5/2}} I_{0j}\,, \end{aligned} \tag{213}$$

$$\begin{aligned} I_{3j} = \int_0^1 \frac{\lambda_j^3 d\lambda_j}{\left[p_j\lambda^2 + 2q_j\lambda_j + v_j\right]^{1/2}} &= \frac{1}{6p_j^3}\Big[(2p_j^2 - 5p_j q_j - 4p_j v_j + 15q_j^2)\sqrt{p_j + 2q_j + v_j} + \\ &+ (4p_j v_j - 15q_j^2)\sqrt{v_j}\Big] + \frac{3p_j q_j v_j - 5q_j^3}{2p_j^{7/2}} I_{0j}\,, \end{aligned} \tag{214}$$

$$I_{4j} = \int_0^1 \left[p_j\lambda^2 + 2q_j\lambda_j + v_j\right]^{1/2} d\lambda_j = \frac{(p_j + q_j)\sqrt{p_j + 2q_j + v_j} - q_j\sqrt{v_j}}{2p_j} + \frac{p_j v_j - q_j^2}{2p_j^{3/2}} I_{0j}\,, \tag{215}$$

$$\begin{aligned} I_{5j} = \int_0^1 \lambda_j\left[p_j\lambda^2 + 2q_j\lambda_j + v_j\right]^{1/2} d\lambda_j &= \frac{1}{6p_j^2}\Big[(2p_j^2 + p_j q_j + 2p_j v_j - 3q_j^2)\sqrt{p_j + 2q_j + v_j} - \\ &- (2p_j v_j - 3q_j^2)\sqrt{v_j}\Big] + \frac{q_j^3 - p_j q_j v_j}{2p_j^{5/2}} I_{0j}\,, \end{aligned} \tag{216}$$

$$I_{6j} = \int_0^1 \frac{\left[p_j\lambda^2 + 2q_j\lambda_j + v_j\right]^{1/2}}{p_j\lambda^2 + 2q_j\lambda_j + u_j} d\lambda_j = \frac{|d_i|}{\sqrt{p_j u_j - q_j^2}}\left[ATN1_j - ATN2_j\right] + \frac{1}{\sqrt{p_j}} LN_j . \quad (217)$$

Let us now consider the evaluation of 2D integrals having either $(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}$ or $(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}$ in the denominator. The first domain integral to consider is

$$\psi_{F_i} = \int_{F_i} \frac{dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} = \psi_i - |d_i|\alpha_i \quad (218)$$

where

$$\psi_i = \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\nu}_j\right) \int_0^{l_j} \frac{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}}{\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i} ds_j = \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) \int_0^1 \frac{(p_j\lambda_j^2 + 2q_j\lambda_j + v_j)^{1/2}}{p_j\lambda_j^2 + 2q_j\lambda_j + u_j} d\lambda_j =$$
$$= \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) \left\{ \frac{|d_i|}{\sqrt{p_j u_j - q_j^2}} \left[ATN1_j - ATN2_j\right] + \frac{1}{\sqrt{p_j}} LN_j \right\} = \sum_{j=1}^{N_{E_i}} \psi_j^i \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right). \quad (219)$$

The derivation of the previous expression can be found, e.g., in formula (19) of D'Urso (2013a) and (23) of D'Urso (2014a).

The scalar α_i in (218) is the two-dimensional counterpart of the quantity α_V in (26) and accounts for the singularity of ψ_{F_i} when $d_i = 0$ and $\boldsymbol{\rho} = \boldsymbol{o}$ where $\boldsymbol{o} = (0, 0)$. Thus α_i represents the angular measure, expressed in radians, of the intersection between F_i and a circular neighbourhood of the singularity point $\boldsymbol{\rho} = \boldsymbol{o}$, see D'Urso (2013a, 2014a,b) for additional details. Although its computation is not required in the ensuing developments, we specify for completeness that α_i can be computed by means of the general algorithm detailed in D'Urso and Russo (2002).

Analogously formulas (19), (77) and (79) of D'Urso (2014b) yield

$$\boldsymbol{\psi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} = \sum_{j=1}^{N_{E_i}} \boldsymbol{\nu}_j \int_0^{l_j} (\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2} ds_i =$$
$$= \sum_{j=1}^{N_{E_i}} l_j \boldsymbol{\nu}_j \int_0^1 \left[p_j\lambda_j^2 + 2q_j\lambda_j + v_j\right]^{1/2} d\lambda_j = \sum_{j=1}^{N_{E_i}} I_{4j} \Delta\boldsymbol{\rho}_j^{\perp} \quad (220)$$

while formulas (37) and (81) of D'Urso (2014b)

$$\varphi_{F_i} = \int_{F_i} \frac{dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} = \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}} \left[\left(\boldsymbol{\rho}_j \cdot \boldsymbol{\nu}_j\right) \int_0^{l_j} \frac{ds_j}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i)(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} \right] =$$
$$= \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) \int_0^1 \frac{\lambda_j}{(p_j\lambda_j^2 + 2q_j\lambda_j + u_j)(p_j\lambda_j^2 + 2q_j\lambda_j + v_j)^{1/2}} = \quad (221)$$
$$= \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}} \left[\frac{\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}}{|d_i| \sqrt{p_j u_j - q_j^2}} (ATN1_j - ATN2_j) \right] = \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}} \varphi_j \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right).$$

Furthermore, on account of formulas (38) and (82) of D'Urso (2014b) it turns out to be

$$\begin{aligned} \boldsymbol{\varphi}_{F_i} &= \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} = -\sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\nu}_j \int_0^{l_j} \frac{ds_j}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} \right) = \\ &= -\sum_{j=1}^{N_{E_i}} \Delta\boldsymbol{\rho}_j^{\perp} \int_0^1 \frac{d\lambda_j}{(p_j \lambda_j^2 + 2q_j \lambda_j + v_j)^{1/2}} = -\sum_{j=1}^{N_{E_i}} I_{0j}\, \Delta\boldsymbol{\rho}_j^{\perp} \end{aligned} \tag{222}$$

while one infers from formulas (40) and (83) of D'Urso (2014b)

$$\begin{aligned} \boldsymbol{\Phi}_{F_i} &= \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i = \\ &= -\sum_{j=1}^{N_{E_i}} \int_0^{l_j} \frac{\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} ds_i \otimes \boldsymbol{\nu}_j + \psi_{F_i} \mathbf{I}_{2D} = \\ &= -\sum_{j=1}^{N_{E_i}} \int_0^1 \frac{\boldsymbol{\rho}_j + \lambda_j \Delta\boldsymbol{\rho}_j}{(p_j \lambda_j^2 + 2q_j \lambda_j + v_j)^{1/2}} d\lambda_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} + \psi_{F_i} \mathbf{I}_{2D} \\ &= -\sum_{j=1}^{N_{E_i}} \left[LN_j\, \boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} + I_{1j}\, \Delta\boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} \right] + \psi_{F_i} \mathbf{I}_{2D} \end{aligned} \tag{223}$$

where $\mathbf{I}_{2D}$ is the rank-two two-dimensional identity tensor.

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Answer to the Editor-in-Chief

prof. Michael Rycroft

Dear prof. Rycroft,

please find enclosed the revised version of the manuscript GEOP-S-16-00119R1:

Gravity Anomaly of Polyhedral Bodies Having a Polynomial Density Contrast

by

M.G. DõUrso, S. Trotta

which we have submitted for publication on Surveys in Geophyisics.

First of all let us thank You very much for your kindness as well as for useful and helpful comments. The paper has been revised according to the reviewersô suggestions and proposals.

Waiting from Your please receive
My best regards

Cassino, March 24th 2017

Maria Grazia DõUrso

Answers to reviewer # 2

The authors wish to thank the reviewer for careful review of the manuscript. According to the comments pointed out by the reviewer, the revised manuscript has been improved as follows:

- P1,first paragraph "most diagnostic" and "most difficult property". I do not think that the superlative formulations are correct and in my opinion would require proof. I recommend to relativize the statements by writing, e.g, "one of the most diagnostic".
- Your suggestion has been followed by writing "one of the most diagnostic" and "one of the most difficult".
- P2, first paragraph, spelling "historical".
- It has been corrected. Thank you.
- P2, second paragraph, "still used in exploriation" is sufficient, i.e., I suggest to omit "methods".
- The word "method" has been deleted.
- P2, third paragraph, spelling "electro-magnetic".
- It has been corrected. Thank you.
- P2, fourth paragraph, "improvements in gravimeter efficiency" (remove "s" from "gravimeters")..
- It has been corrected. Thank you.
- P2, fifth paragraph: I suggest to rephrase this, by clearly separating what is measured (anomaly, i.e., geophysical data, from real earth structures that can potentially be complex in their density distribution) and what is simulated/modelled (anomaly associated with a body with a known density distribution). The simulation is then an element of geophysical modelling inversion, but I would only call it basic if the bodies had particularly simple density distributions.
- We modified the sentence so as to (hopefully) explain that we were making reference to simulation, i.e. evaluation of gravity anomaly.
- P3, first paragraph. I recently saw a paper by Ren et al. (2016): *Gravity Anomalies of Arbitrary 3D Polyhedral Bodies with Horizontal and Vertical Mass Contrasts*, Surveys in Geophysics, which potentially has some overlap with the presented work. I personally did not study it, but I recommend to have a look. It might be worth citing after this paragraph.
- It has been quoted at the end of pag. 3.
- P4, below eq. 1: "represents the magnitude [...] from the infinitesimal mass" is strictly not correct as eq. 1 is the integral over the collection of all infinitesimal masses in Ω.

- We have modified the original sentence by writing: and the integrand function represents...
- P4, second last paragraph "governed by the Poisson equation" (i.e., add "the").
- It has been corrected. Thank you.
- P5, below eq. 3: "confine the treatment to the case" (i.e., add "the").
- It has been corrected. Thank you.
- P9, below eq. 30: spelling "as follows".
- It has been corrected. Thank you.
- P13, around eq. 58: "For the same rason we shall not consider [...] since this would require us to consider separately the cases [...] of the algebraic expressions resulting from (57)". Please consider adding for clarification e.g. "but instead perform the combination after the integration".
- We have added the sentence: instead we shall perform the combination after the integration.
- P13, below eq. 58: " ... do not exhibit anymore the useful recurrence property ...". Does this present a limitation or an additional difficulty for the extension of the presented approach to density contrasts of higher polynomial order than $N_x + N_y + N_z \leq 3$? If so, I recommend to mention it here.
- We have added a sentence to better explain our objective. It is not related to the generalization of the methods to the case $N_x + N_y + N_z > 3$ since this can be exploited provided that some further analytical and algebraic manipulations are carried out.
- P15, below eq. 66: "integral of a real variable" (i.e., add "a").
- It has been corrected. Thank you.
- P16, below eq. 76: I believe that "where LN_j is defined ..." should be "where I_{0j} is defined ...".
- It has been corrected. Thank you.
- P17, last paragraph, first sentence, please correct (e.g., "The aim of this subsection is to show how...").
- It has been corrected. Thank you.
- P18, below eq. 98: "will be dealt with" (i.e., add "with").
- It has been corrected. Thank you.
- P25, fourth paragraph, two times "at the denominator", change to "in the denominator".

– It has been corrected. Thank you.

– P28, below eq. 148: Replace “at infinite” with “at infinity”.

– It has been corrected. Thank you.

– The punctuation of equations is not always correct. Examples are eqs. 2,8, 10, 31, 33, 127 (comma missing), 5, 7, 13, 16, 18, 21, 24, 25, 27, 41, 56, 60, 64, 66, 73, 83, 100, 108, 113, 122, 126, 128, 130, 132, 136 (full-stop missing) and more.

– A comma has been added to equations 2, 8, 9, 10, 11, 29, 31, 32, 33, 36, 39, 44, 49, 51, 52, 53, 54, 55, 58, 79, 81, 90, 91, 92, 93, 94, 95, 96, 97, 106, 107, 111, 112, 117, 118, 119, 127, 140, 149, 161, 165, 168, 169, 170, 181, 184, 188, 201, 206, 207, 209, 211, 212, 213, 214, 215, 216.

  A full-stop has been added to equations 5, 7, 13, 16, 18, 21, 24, 25, 27, 30, 38, 41, 48, 55, 60, 66, 70, 73, 74, 83, 86, 98, 100, 105, 108, 122, 126, 128, 130, 132, 136, 138, 139, 142, 143, 144, 145, 150, 157, 164, 166, 171, 173, 174, 180, 186, 194, 196, 197, 199, 203, 204, 219, 221.

– There are possibly a few more of the minor grammatical mistakes like the ones pointed out above. I would recommend the authors to recheck carefully, or better yet, find a further pair of eyes to spot remaining mistakes in the language.

– We have done it and corrected a couple a further mistakes.

Answers to reviewer # 3

The authors wish to thank the reviewer for careful review of the manuscript. According to the comments pointed out by the reviewer, the revised manuscript has been improved as follows:

1. Most of the authors' responses to previous comments are satisfactory, but the response to the first comment, dealing with the definition of the gravity anomaly, is not. The relevant formula, Eq. (1), is simply not a formula for a gravity anomaly, and to say otherwise is a factual error. It is the formula for the gravitational attraction of a mass body. It may be seen approximately as the formula for the influence of a mass body on the gravity anomaly, since for small bodies the effect on gravity is the dominant part of the effect on the gravity anomaly. Or it may be seen exactly as the formula for the influence of a small mass body on the gravity disturbance, which is defined in such a way that effect of the body on gravity potential is irrelevant. Perhaps to address the two concerns cited by the authors to justify retaining the term, i.e. that the term is also misapplied elsewhere, it can simply be stated that the term "gravity anomaly" in this paper is not being used in the most correct sense, but is rather being used throughout to indicate the effect of a mass body on gravity. This indeed (for small bodies) corresponds to the largest part of the body's effect on the gravity anomaly. In this way, the issue can be addressed painlessly but without loss of consistency with the other publications referenced (on inversion, or the 2-d paper), while acknowledging that the terminology is problematic. The citation of the Vaníček et al. (2004) paper is not necessary if the above change is made–that paper was cited by me only as an example of a discussion of the complete effect of mass-density on the gravity anomaly, to clarify the issue for the authors. However, they may retain it if they wish as an example of how the effect of a mass body on the gravity anomaly may be formulated in a more theoretically consistent manner. I also note a minor issue in the wording of the additional paragraph near the bottom of p. 4, regarding gravimetry. The word "compute" in this paragraph should be changed to "measure", as that is the task of the gravimeter. Any computation done when using digital meters is ancillary to their primary task. Also, strictly speaking, the vertical direction at the gravimeter is not the normal to the geoid, unless the gravimeter is located at the geoid. Rather, the vertical is a direction perpendicular to the local horizontal, or more analogously to the wording used, normal to an equipotential surface passing through the instrument. I believe "the vertical component of the gravity field" is sufficient to indicate this direction, leaving aside any reference to the geoid or equipotential surfaces.

   Thanks again for your detailed and illuminating comment. We have included two new paragraphs after formula (1) in order to (hopefully) properly address the points raised by you.

   In the first paragraph after formula (2) we have changed "compute" to measure and deleted the expression "i.e. the component normal to the geoid".

Surveys in Geophysics manuscript No.
(will be inserted by the editor)

Gravity Anomaly of Polyhedral Bodies Having a Polynomial Density Contrast

M.G. D'Urso · S. Trotta

Received: date / Accepted: date

Abstract We analytically evaluate the gravity anomaly associated with a polyhedral body having an arbitrary geometrical shape and a polynomial density contrast in both the horizontal and vertical directions. The gravity anomaly is evaluated at an arbitrary point that does not necessarily coincide with the origin of the reference frame in which the density function is assigned. Density contrast is assumed to be a third-order polynomial as a maximum but the general approach exploited in the paper can be easily extended to higher-order polynomial functions. Invoking recent results of potential theory, the solution derived in the paper is shown to be singularity-free and is expressed as sum of algebraic quantities that only depend upon the 3D coordinates of the polyhedron vertices and upon the polynomial density function. The accuracy, robustness and effectiveness of the proposed approach is illustrated by numerical comparisons with examples derived from the existing literature.

Keywords Gravity anomaly · Polyhedral bodies · Polynomial density contrast · Singularity

1 Introduction

Gravity is an economic tool for exploring and discovering natural resources (Jacoby and Smilde, 2009). In this respect density is one of the most diagnostic physical property of a mineral deposit, and is also fundamental to oil and gas exploration. To date, density has been one of the most difficult property to measure and infer.

During the last decade, there has been significant development in gravity survey, particularly with the advent of GPS and gravity gradiometry. In conventional gravity survey, Earth's gravity acceleration is measured using gravimeter whereas in gravity gradiometer survey, the gravity gradient or how the gravitational acceleration changes over distance (or in some cases time) is measured.

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Recent reviews (LaFehr, 1980; Paterson and Reeves, 1985; Hansen, 2001) document the continuous evolution of instruments, field operations, data-processing techniques, and methods of interpretation. A steady progression in instrumentation (torsion balance, gravimeters based on land or underwater, in boreholes or on board satellites, aircraft or marine vessels, modern versions of absolute gravimeters, and gravity gradiometers) has enabled the acquisition of gravity data in nearly all environments, see, e.g., Nabighian (2005) for a quite recent historical account.

Despite being eclipsed by seismology, it is impressive to realize that about 40 different commercial gravity sensors and gravity gradiometers are available (Chapin, 2008) and about 30 different gravity sensor and gravity gradiometers designs have either been proposed or developed. In particular, gravity gradiometry is still used in exploration (Dransfield, 2007) and for regional gravity mapping (Jekeli, 2006).

Gravity data sets are effectively used to estimate locations and shapes of bodies, embedded in Earth, exhibiting anomalous mass density with respect to a constant reference value (Zhang et al., 2014). More refined Earth models can be obtained by inverting gravity data (Li and Oldenburg, 1998; Zhdanov, 2002) in conjuction with seismic and electro-magnetic induction data (Moorkamp et al., 2011; Aydemir et al., 2014; Roberts et al., 2016).

Recent improvements in gravimeter efficiency and inversion algorithms have increased the possibility of collecting and inverting huge data sets over extended areas in order to derive 3D density models (Kamm et al., 2015). In particular, gravity methods are extensively used in geoid determination (Bajracharya and Sideris, 2004) and mineral exploration (Beiki and Pedersen, 2010; Martinez et al., 2013; Abtahi et al., 2016).

In conclusion it is of paramount importance to efficiently evaluate the gravity anomaly associated with a body characterized by complex density distributions since this represents an important task in forward modelling and inversion.

Due to the mathematical complexity of the problem, the gravity anomaly of an irregular body whose density contrast is spatially variable has been first computed by approximating the body as a collection of vertical rectangular parallelepipeds (prisms) in which the density is assumed to be constant.

Numerical computations were first carried out by Talwani et al. (1959) and Bott (1960). Closed form expressions of the gravity anomaly were subsequently derived by Nagy (1966), Banerjee and Das Gupta (1977), Cady (1980), Nagy et al. (2000), Tsoulis (2000), Jiancheng and Wenbin (2010), D'Urso (2012), see also Plouff (1975, 1976), Won and Bevis (1987), Montana et al. (1992) for computer codes. The case of spheroidal shell has been addressed by Johnson and Litehiser (1972). Analytical expressions of the gravity anomaly for prisms have been derived by D'Urso (2016), for a linearly varying density, by Rao (1985, 1986, 1990), Rao et al. (1994), Gallardo-Delgado et al. (2003) for a quadratic density contrast, by García-Abdeslem (1992, 2005), for a cubic density variation with depth. A good collection of earlier references for 3D prisms can be found in Li and Chouteau (1998) who name, among others, a formula contributed in Sorokin (1951).

Non-polynomial density-contrast models for 3D bodies have been considered by Cordell (1973), Chai and Hinze (1988), Litinsky (1989), Rao et al. (1990), Chakravarthi et al. (2002), Silva et al. (2006), Chakravarthi and Sundararajan (2007), Chappell and Kusznir (2008), Zhou (2009b) and, for 2D bodies, by Gendzwill (1970), Murthy and Rao (1979), Pan (1989), Guspí (1990), Ruotoistenmäki (1992), Martín-Atienza and García-Abdeslem (1999), Zhang et al. (2001), Zhou (2008, 2009a, 2010). For more complicated forms of the density contrast, see, e.g., Cai and Wang (2005) and Mostafa (2008).

Alternative to the use of prisms, characterized by complicated functions describing density contrast, is the case of polyhedrons endowed with a a simple description of density contrast. Analytical formulas for the gravimetric analysis of polyhedra having constant density have been contributed by Paul (1974), Barnett (1976), Strakhov (1978), Okabe (1979), Waldvogel (1979), Golizdra (1981), Strakhov et al. (1986), Götze and Lahmeyer (1988), Pohanka (1988), Murthy et al. (1989), Kwok (1991b), Werner (1994), Holstein and Ketteridge (1996), Petrović (1996), Werner and Scheeres (1997), Li and Chouteau (1998), Tsoulis (2012), D'Urso (2013a, 2014a), Conway (2015), Werner (2017). Subsequent advancements have been only concerned with a linear density variation, (Pohanka, 1998; Hansen, 1999; Holstein, 2003; Hamayun et al., 2009; D'Urso, 2014b); actually, handling more complex density functions in conjunction with polyhedral models considerably increases the difficulties of the treatment, especially if analytical solutions are looked for.

For 2D bodies having density contrast depending only on depth, Zhou (2008) converted the original domain integral for gravity anomaly to a Line Integral (LI) by using Stokes theorem. In particular he derived two types of LIs for computing the gravity anomaly of bodies. In a subsequent paper (Zhou, 2009a) the author extended his method to account for density contrast functions which depended not only on depth but also on horizontal or, jointly, on horizontal and vertical directions. The gravity anomaly at observation points different from the origin has been evaluated in Zhou (2010) since, historically, gravity anomaly was computed only at the origin of the reference frame. In the same paper, Zhou dealt with the singularity of the gravity anomaly arising where the observation point is coincident with the vertices of the integration domain, an issue already discussed in Kwok (1991a), for prism-based modelling, and Tsoulis and Petrović (2001) for polyhedra.

The first approach for evaluating the gravity anomaly of bodies characterized by a complicated density contrast, even in presence of two-dimensional domains, has been either numerical or of semi-analytical nature based on the use of prisms, (Murthy and Rao, 1979; Rao et al., 1990; Chakravarthi et al., 2002; Chakravarthi and Sundararajan, 2007; Zhou, 2009b), or with 2D geometrical shapes, (Gendzwill, 1970; Murthy and Rao, 1979; Pan, 1989; Guspí, 1990; Ruotoistenmäki, 1992; Martín-Atienza and García-Abdeslem, 1999; Zhang et al., 2001; Zhou, 2008, 2009a, 2010). Actually, this last geometrical assumption, which can be used to model domains extending towards infinity in one direction, significantly simplifies the mathematical treatment of the problem.

Nevertheless, starting from the first researches on the subject (Hubbert, 1948), all authors have systematically transformed the original domain integrals into integrals of lower dimension in order to simplify the adoption of quadrature rules for the numerical evaluation of the gravity anomaly.

The derivation of analytical expressions for the gravity anomaly of polygonal bodies has been achieved only recently (D'Urso, 2015c) by exploiting the generalized Gauss theorem first presented in D'Urso (2012, 2013a), and subsequently applied to several problems ranging from geodesy (D'Urso, 2014a,b; D'Urso and Trotta, 2015b; D'Urso, 2016), to geomechanics (D'Urso and Marmo, 2009; Sessa and D'Urso, 2013; D'Urso and Marmo, 2015a), to geophysics (D'Urso and Marmo, 2013b), elasticity (Marmo and Rosati, 2016; Marmo et al., 2016a,b, 2017; Trotta et al., 2016a,b) and to heat transfer (Rosati and Marmo, 2014).

The methodology outlined in D'Urso (2015c) is here generalized in order to derive an analytical expression of the gravity anomaly for polyhedral bodies having density contrast expressed as a polynomial function of arbitrary degree in both the horizontal and vertical directions, an issue recently addressed in Ren et al. (2017). The result is obtained by first reducing the original domain integral to a 2D boundary integral by virtue of the generalized Gauss theorem. Remarkably, this also allows one to prove that the boundary integral expression of the gravity anomaly is singularity free whatever is the position of the observation point with respect to the body.

Being Ω polyhedral, the 2D expression of the gravity anomaly is written as finite sum of 2D integrals extended to the faces of Ω. By a further application of the generalized Gauss theorem each face integral is reduced to the sum of 1D integrals extended to the edges of the face. Such 1D integrals are analytically evaluated as products between the position vectors of the end vertices of each edge and scalar coefficients providing the analytical value of integrals of real variable.

Although these last integrals may exhibit a singularity when the projection of the observation point onto a face belongs to an edge, it is proved that such a singularity produces a null contribution of the i-th edge to the general expression of gravity anomaly; hence, one infers that the derived expression is singularity-free.

By exploiting a suitable change of variables, we also derive an enhanced algebraic formula which expresses the gravity anomaly at an arbitrary point P and specializes to the ordinary one when $P = O$. Remarkably, the enhanced expression of the gravity anomaly has been derived without any modification of the density contrast function since this is still defined in the original reference frame. The enhanced formula has been implemented in a MATLAB code, and its accuracy and robustness has been assessed by numerical comparisons with examples derived from the literature.

2 Gravity Anomaly of Polyhedral Bodies at the Origin O of the Reference Frame

Let us consider a Cartesian reference frame having origin at an arbitrary point O and a polyhedral body Ω. We shall assume that the density $\Delta\rho$ of the body, usually denominated density contrast, is a function of the generic point whose position with respect to O is defined by the vector $\mathbf{r}$. The symbol $\Delta\rho$ emphasizes the fact that the density of Ω is a variation with respect to that of the surrounding medium.

Denoting by G the gravitational constant, we shall first evaluate the gravity anomaly at O; it is defined by

$$\Delta\mathbf{g}(O) = G \int_{\Omega} \frac{\Delta\rho(\mathbf{r})\,\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}} dV \tag{1}$$

and the integrand function represents the magnitude of attraction on a unit mass at O arising from the infinitesimal mass $\Delta\rho dV$.

We remark that the denomination of gravity anomaly adopted to denote equation (1), though not strictly correct, is based on a common practice in the specialized literature. Actually, equation (1) is a formula for the gravitational attraction of a mass body and may be approximatively seen as the formula for the influence of a mass body on the gravity anomaly since, for small bodies, the effect on gravity is the dominant part of the effect on the gravity anomaly.

An in-depth discussion on this topic is reported in Vaníček et al. (2004) where the interested reader can find an example of how the effect of a mass body on the gravity anomaly can be formulated in a theoretically consistent manner.

The vertical component of the gravity anomaly at O is provided by

$$\Delta g_z(O) = G \int_{\Omega} \frac{\Delta\rho(\mathbf{r})\mathbf{r}\cdot\mathbf{k}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}} dV\,, \tag{2}$$

$\mathbf{k}$ being the unit vector directed along the vertical axis. The evaluation of Δg_z at an arbitrary point P will be addressed in section 3 since a considerably more elaborate expression is arrived at.

It is usually of interest to dispose of a procedure to actually compute Δg_z since most gravimeters can only measure the vertical component of the gravity field. Nevertheless the procedure detailed in the paper can be equally applied to all components of (1) and to physical problems governed by the Poisson equation (Blakely, 2010).

The computation of the integral in (2) is a hard task since the density contrast function $\Delta\rho$ does usually have a very complicated expression for the necessity of modelling 3D anomalies of Earth. For simplicity this can be modeled as an ensemble of 3D anomalies in a layered medium or a sequence of strata with horizontally undulated interfaces, e.g., sedimentary basins and underlying bedrock. In each layer mass density typically exhibits depth-dependent variations (García-Abdeslem, 1992).

However geological processes of exogenetic (fluvial, coastal, glacial,...) and endogenetic (rock diagenesis, plate tectonics, volcano eruptions, earthquakes,...) nature can induce both horizontal and vertical variations in mass density (Martín-Atienza and García-Abdeslem, 1999). Thus, a suitable expression of the density variation can allow for potentially faithful representations of the Earth subsurface with a relatively smaller amount of computations and parameters. Additionally, disposing of analytical expressions of the gravity anomaly associated with complicated expressions $\Delta\rho$ can be useful for benchmarking numerical approaches.

A quite general expression for $\Delta\rho$, able to accommodate a large variety of geological formations, is given by a triple polynomial in x, y and z, (García-Abdeslem, 2005; Zhou, 2009b; Ren et al., 2017)

$$\Delta\rho(\mathbf{r}) = \theta(x,y,z) = \sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\sum_{k=0}^{N_z} c_{ijk}\, x^i y^j z^k \tag{3}$$

where N_x, N_y and N_z represent the maximum power of the polynomial density variation along x, y and z respectively. In the sequel we shall confine the treatment to the case

$$N_x + N_y + N_z = 3 \tag{4}$$

since this will suffice to address the majority of the practical applications and, at the same time, to present our formulation at a degree of generality sufficient to be generalized to the cases $N_x + N_y + N_z > 3$.

Thus, under the assumption (4), equation (3) specializes to

$$\begin{aligned}\theta(\mathbf{r}) = c_{000} &+ c_{100}x + c_{010}y + c_{001}z + \\ &+ c_{200}x^2 + c_{020}y^2 + c_{002}z^2 + c_{110}xy + c_{011}yz + c_{101}xz + \\ &+ c_{300}x^3 + c_{030}y^3 + c_{003}z^3 + c_{210}x^2y + c_{021}y^2z + c_{102}xz^2 + \\ &+ c_{120}xy^2 + c_{012}yz^2 + c_{201}x^2z + c_{111}xyz.\end{aligned} \tag{5}$$

The scalars c_{ijk} represent the coefficients of the polynomial law; they can be estimated from the known data points by a least-square approach (Jacoby and Smilde, 2009).

Paralleling the analogous treatment developed in D'Urso (2015c), we first reformulate the general expression (3) of the density contrast by writing

$$\theta(\mathbf{r}) = \theta_{\mathbf{o}} + \mathbf{c}\cdot\mathbf{r} + \mathbf{C}\cdot\mathbf{D}_{\mathbf{rr}} + \mathbb{C}\cdot\mathbb{D}_{\mathbf{rrr}} \tag{6}$$

where θ_o is a scalar denoting the density at $\mathbf{o} = (0,0,0)$, $\mathbf{c}$ is a vector, $\mathbf{C}$ and $\mathbf{D_{rr}}$ are symmetric second-order tensors, $\mathbb{C}$ and $\mathbb{D}_{\mathbf{rrr}}$ are third-order tensors; furthermore, it has been set

$$\mathbf{D_{rr}} = \mathbf{r} \otimes \mathbf{r} \qquad \mathbb{D}_{\mathbf{rrr}} = \mathbf{r} \otimes \mathbf{r} \otimes \mathbf{r} \,. \tag{7}$$

The second-order (rank-two) tensor $\mathbf{r} \otimes \mathbf{r}$ has the following matrix representation

$$[\mathbf{r} \otimes \mathbf{r}] = \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix}, \tag{8}$$

so that, being:

$$\mathbf{C} \cdot (\mathbf{r} \otimes \mathbf{r}) = C_{11} x^2 + 2C_{12} xy + 2C_{13} xz + C_{22} y^2 + 2C_{23} yz + C_{33} z^2 \,, \tag{9}$$

a quadratic distribution of density can be assigned by suitably defining the coefficients of the symmetric tensor $\mathbf{C}$. Analogously, the third-order tensors $\mathbb{C}$ and $\mathbf{r} \otimes \mathbf{r} \otimes \mathbf{r}$, are represented in matrix form as:

$$\mathbb{C} = \begin{bmatrix} \mathbb{C}_{111} & \mathbb{C}_{112} & \mathbb{C}_{113} \\ \mathbb{C}_{121} & \mathbb{C}_{122} & \mathbb{C}_{123} \\ \mathbb{C}_{131} & \mathbb{C}_{132} & \mathbb{C}_{133} \\ \hdashline \mathbb{C}_{211} & \mathbb{C}_{212} & \mathbb{C}_{213} \\ \mathbb{C}_{221} & \mathbb{C}_{222} & \mathbb{C}_{223} \\ \mathbb{C}_{231} & \mathbb{C}_{232} & \mathbb{C}_{233} \\ \hdashline \mathbb{C}_{311} & \mathbb{C}_{312} & \mathbb{C}_{313} \\ \mathbb{C}_{321} & \mathbb{C}_{322} & \mathbb{C}_{323} \\ \mathbb{C}_{331} & \mathbb{C}_{332} & \mathbb{C}_{333} \end{bmatrix} \qquad \mathbf{r} \otimes (\mathbf{r} \otimes \mathbf{r}) = \begin{bmatrix} x \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix} \\ \hdashline y \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix} \\ \hdashline z \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix} \end{bmatrix}, \tag{10}$$

i.e. as vectors of rank-two tensors. Being

$$\begin{aligned} \mathbb{C} \cdot (\mathbf{r} \otimes \mathbf{r} \otimes \mathbf{r}) = \; & \mathbb{C}_{111} x^3 + \mathbb{C}_{222} y^3 + \mathbb{C}_{333} z^3 + \\ & + \left(\mathbb{C}_{112} + \mathbb{C}_{121} + \mathbb{C}_{211}\right) x^2 y + \left(\mathbb{C}_{113} + \mathbb{C}_{131} + \mathbb{C}_{311}\right) x^2 z + \\ & + \left(\mathbb{C}_{223} + \mathbb{C}_{232} + \mathbb{C}_{322}\right) y^2 z + \left(\mathbb{C}_{122} + \mathbb{C}_{221} + \mathbb{C}_{212}\right) x y^2 + \\ & + \left(\mathbb{C}_{133} + \mathbb{C}_{331} + \mathbb{C}_{313}\right) x z^2 + \left(\mathbb{C}_{233} + \mathbb{C}_{332} + \mathbb{C}_{323}\right) y z^2 + \\ & + \left(\mathbb{C}_{123} + \mathbb{C}_{132} + \mathbb{C}_{213} + \mathbb{C}_{231} + \mathbb{C}_{312} + \mathbb{C}_{321}\right) xyz, \end{aligned} \tag{11}$$

the representation (3) of the density contrast is recovered from (6) by setting

$$\begin{aligned} & \theta_0 = c_{000} \qquad c_1 = c_{100} \qquad c_2 = c_{010} \qquad c_3 = c_{001} \\ & C_{11} = c_{200} \qquad C_{22} = c_{020} \qquad C_{33} = c_{002} \\ & C_{12} = c_{110}/2 \quad C_{13} = c_{101}/2 \quad C_{23} = c_{011}/2 \end{aligned} \tag{12}$$

and

$$
\begin{aligned}
&\mathbb{C}_{111} = c_{300} && \mathbb{C}_{222} = c_{030} && \mathbb{C}_{333} = c_{003} \\
&\mathbb{C}_{112} = \mathbb{C}_{121} = \mathbb{C}_{211} = c_{210}/3 && \mathbb{C}_{113} = \mathbb{C}_{131} = \mathbb{C}_{311} = c_{201}/3 && \\
&\mathbb{C}_{223} = \mathbb{C}_{232} = \mathbb{C}_{322} = c_{021}/3 && \mathbb{C}_{122} = \mathbb{C}_{221} = \mathbb{C}_{212} = c_{120}/3 && \\
&\mathbb{C}_{133} = \mathbb{C}_{331} = \mathbb{C}_{313} = c_{102}/3 && \mathbb{C}_{233} = \mathbb{C}_{332} = \mathbb{C}_{323} = c_{012}/3 && \\
&\mathbb{C}_{123} = \mathbb{C}_{132} = \mathbb{C}_{213} = \mathbb{C}_{231} = \mathbb{C}_{312} = \mathbb{C}_{321} = c_{111}/6 .
\end{aligned} \tag{13}
$$

In conclusion, we derive from (2) the following expression of the gravity anomaly

$$
\Delta g_z(\mathbf{o}) = G\Big[\theta_{\mathbf{o}} d^{\Omega}_{\mathbf{r}} + \mathbf{c}\cdot\mathbf{d}^{\Omega}_{\mathbf{r}} + \mathbf{C}\cdot\mathbf{D}^{\Omega}_{\mathbf{rr}} + \mathbb{C}\cdot\mathbb{D}^{\Omega}_{\mathbf{rrr}}\Big] \tag{14}
$$

where

$$
d^{\Omega}_{\mathbf{r}} = \int_{\Omega} \frac{\mathbf{r}\cdot\mathbf{k}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}} dV \qquad \mathbf{d}^{\Omega}_{\mathbf{r}} = \int_{\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}} dV \tag{15}
$$

and

$$
\mathbf{D}^{\Omega}_{\mathbf{rr}} = \int_{\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})\mathbf{r}\otimes\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}} dV \qquad \mathbb{D}^{\Omega}_{\mathbf{rrr}} = \int_{\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})\mathbf{r}\otimes\mathbf{r}\otimes\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}} dV . \tag{16}
$$

In order to transform the previous domain integrals into boundary integrals we apply Gauss theorem in the generalized form illustrated in D'Urso (2013a, 2014a) so as to correctly take into account the singularity at $\mathbf{r} = \mathbf{o} = (0,0,0)$.

This will be done in the following two subsections while in the subsequent ones the boundary integrals extended to the faces of Ω will be further reduced to 1D integrals extended to the edges of each face by means of a further application of Gauss theorem. These last integrals will be first expressed as function of the 2D coordinates of the vertices in the reference frame local to each face and then reformulated in terms of the 3D coordinates representing the basic geometric data defining the polyhedron.

2.1 Analytical Expression of the Gravity Anomaly at O in Terms of 2D Integrals

Let us now illustrate a general approach to express the 3D integrals in (14) as 2D integrals extended to the faces constituting the boundary of Ω. Generality lies in the fact that, owing to the symmetry of the integrals, application of Gauss theorem can be based upon a unique formula. Actually, we are going to prove the result

$$
\int_{\Omega} \frac{k_{\mathbf{r}}[\otimes\mathbf{r},m]}{(\mathbf{r}\cdot\mathbf{r})^{3/2}} dV = \frac{1}{m+1} \int_{\partial\Omega} \frac{k_{\mathbf{r}}[\otimes\mathbf{r},m](\mathbf{r}\cdot\mathbf{n})}{(\mathbf{r}\cdot\mathbf{r})^{3/2}} dA \quad m = 0,1,\ldots \tag{17}
$$

where $k_{\mathbf{r}} = \mathbf{r}\cdot\mathbf{k}$, $\mathbf{n}$ is the 3D outward unit normal to the boundary $\partial\Omega$ of the polyhedral body and $[\otimes\mathbf{r},m]$ denotes a rank-m tensor defined by

$$
[\otimes\mathbf{r},m] = \begin{cases} 1 & \text{if} \quad m = 0 \\ \mathbf{r} & \text{if} \quad m = 1 \\ \mathbf{r}\otimes\mathbf{r} & \text{if} \quad m = 2 \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ \underbrace{\mathbf{r}\otimes\mathbf{r}\otimes\cdots\otimes\mathbf{r}}_{m\ times} & \text{if} \quad m > 2 . \end{cases} \tag{18}
$$

To fix the ideas we shall prove the identity (17) for $m = 2$

$$\int_{\Omega} \frac{k_{\mathbf{r}} \mathbf{r} \otimes \mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}} dV = \frac{1}{3} \int_{\partial\Omega} \frac{k_{\mathbf{r}} (\mathbf{r} \otimes \mathbf{r})(\mathbf{r} \cdot \mathbf{n})}{(\mathbf{r} \cdot \mathbf{r})^{3/2}} dA \tag{19}$$

since it allows us to illustrate our approach to a degree of generality sufficient to extend the final result to all integrals in (14) and to the additional ones, not reported in (14), containing tensors of rank superior to three, i.e. tensors of the kind $[\otimes \mathbf{r}, m]$ where $m > 3$.

Recalling the identity proved in the appendix of D'Urso (2015c)

$$\begin{aligned} \operatorname{div}[\psi(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c})] \; = \; & (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) \operatorname{grad} \psi + \psi [(\operatorname{grad} \mathbf{a}) \mathbf{c}] \otimes \mathbf{b} + \\ & + \psi \mathbf{a} \otimes [(\operatorname{grad} \mathbf{b}) \mathbf{c}] + \psi (\mathbf{a} \otimes \mathbf{b}) \operatorname{div} \mathbf{c} \end{aligned} \tag{20}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ (ψ) are vector (scalar) differentiable fields, we have

$$\begin{aligned} \operatorname{div}\Big[k_{\mathbf{r}} (\mathbf{r} \otimes \mathbf{r}) \otimes \frac{\mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}}\Big] = & \Big[(\mathbf{r} \otimes \mathbf{r}) \otimes \frac{\mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}}\Big] \operatorname{grad} k_{\mathbf{r}} + k_{\mathbf{r}} \Big[(\operatorname{grad} \mathbf{r}) \frac{\mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}}\Big] \otimes \mathbf{r} + \\ & + k_{\mathbf{r}} \mathbf{r} \otimes \Big[(\operatorname{grad} \mathbf{r}) \frac{\mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}}\Big] + k_{\mathbf{r}} (\mathbf{r} \otimes \mathbf{r}) \operatorname{div} \frac{\mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}} . \end{aligned} \tag{21}$$

Applying the further identity proved in the appendix of D'Urso (2015c)

$$\operatorname{grad}(\mathbf{a} \cdot \mathbf{b}) = [\operatorname{grad} \mathbf{a}]^T \mathbf{b} + [\operatorname{grad} \mathbf{b}]^T \mathbf{a} \tag{22}$$

where $(\cdot)^T$ stands for transpose, one gets

$$\operatorname{grad} k_{\mathbf{r}} = \operatorname{grad}(\mathbf{r} \cdot \mathbf{k}) = (\operatorname{grad} \mathbf{r}) \mathbf{k} = \mathbf{k} \tag{23}$$

since $\mathbf{k}$ is a constant vector field and $\operatorname{grad} \mathbf{r} = \mathbf{I}$, being $\mathbf{I}$ the rank-two identity tensor. Substituting the previous relation in (21) one obtains

$$\begin{aligned} \operatorname{div}\Big[k_{\mathbf{r}} (\mathbf{r} \otimes \mathbf{r}) \otimes \frac{\mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}}\Big] = & \Big[(\mathbf{r} \otimes \mathbf{r}) \otimes \frac{\mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}}\Big] \mathbf{k} + k_{\mathbf{r}} \Big[\frac{\mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}} \otimes \mathbf{r} + \mathbf{r} \otimes \frac{\mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}}\Big] + \\ & + k_{\mathbf{r}} (\mathbf{r} \otimes \mathbf{r}) \operatorname{div} \frac{\mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}} = \\ = \; & 3 k_{\mathbf{r}} \frac{\mathbf{r} \otimes \mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}} + k_{\mathbf{r}} (\mathbf{r} \otimes \mathbf{r}) \operatorname{div} \frac{\mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}} . \end{aligned} \tag{24}$$

Finally, integrating the previous identity over Ω yields

$$\int_{\Omega} k_{\mathbf{r}} \frac{\mathbf{r} \otimes \mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}} dV = \frac{1}{3} \int_{\Omega} \operatorname{div}\Big[k_{\mathbf{r}} (\mathbf{r} \otimes \mathbf{r}) \otimes \frac{\mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}}\Big] dV - \frac{1}{3} \int_{\Omega} k_{\mathbf{r}} (\mathbf{r} \otimes \mathbf{r}) \operatorname{div} \frac{\mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}} dV . \tag{25}$$

The second integral on the right-hand side can be computed by means of the general result (Tang, 2006)

$$\int_{\Omega} \varphi(\mathbf{r}) \operatorname{div}\left[\frac{\mathbf{r}}{(\mathbf{r} \cdot \mathbf{r})^{3/2}}\right] dV = \begin{cases} 0 & \text{if} \quad \mathbf{o} \notin \Omega \\ \alpha_V(\mathbf{o}) \varphi(\mathbf{o}) & \text{if} \quad \mathbf{o} \in \Omega \end{cases} \tag{26}$$

where φ is a continuous scalar field and the quantity α_V represents the angular measure, expressed in steradians, of the intersection between Ω and a spherical neighbourhood of the singularity point $\mathbf{r} = \mathbf{o}$, see D'Urso (2012, 2013a, 2014a) for additional details.

The previous expression can be extended to arbitrary tensors by applying it to each scalar component of the tensor.

On account of (26) one infers that the second integral on the right-hand side of (25) is the null rank-two tensor $\mathbf{O}$ since

$$\int_{\Omega} k_{\mathbf{r}}(\mathbf{r}\otimes\mathbf{r})\,\mathrm{div}\,\frac{\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}dV = \begin{cases} \mathbf{O} & \text{if} \quad \mathbf{o}\notin\Omega \\ [k_{\mathbf{r}}\mathbf{r}\otimes\mathbf{r}]_{\mathbf{r}=\mathbf{o}}\,\alpha_V(\mathbf{o}) & \text{if} \quad \mathbf{o}\in\Omega\,. \end{cases} \tag{27}$$

However, the expression $[k_{\mathbf{r}}(\mathbf{r}\otimes\mathbf{r})]_{\mathbf{r}=\mathbf{o}}$ amounts to evaluating the quantity $k_{\mathbf{r}}(\mathbf{r}\otimes\mathbf{r})$ at the singularity point $\mathbf{r}=\mathbf{o}$, what yields trivially the null tensor $\mathbf{O}$. Hence, according to (27), the last integral in (25) is always the null tensor, independently from the position of singularity point $\mathbf{r}=\mathbf{o}$ with respect to the domain Ω of integration.

In conclusion, upon application of Gauss theorem to the second integral in (25), we finally infer the identity (19). Remarkably, the derivation of this identity has also allowed us to prove that the singularity at $\mathbf{r}=\mathbf{o}$, of the integrand function appearing on the left-hand side of (19), can be actually ignored.

Furthermore, it is not difficult to rephrase the path of reasoning detailed in formulas (21)-(27) so as to prove the more general formula (17). Hence, defining

$$d_{\mathbf{r}}^{\partial\Omega} = \int_{\partial\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})(\mathbf{r}\cdot\mathbf{n})}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}dA \qquad \mathbf{d}_{\mathbf{r}}^{\partial\Omega} = \int_{\partial\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})\,\mathbf{r}\,(\mathbf{r}\cdot\mathbf{n})}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}dA \tag{28}$$

$$\mathbf{D}_{\mathbf{rr}}^{\partial\Omega} = \int_{\partial\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})\,\mathbf{r}\otimes\mathbf{r}\,(\mathbf{r}\cdot\mathbf{n})}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}dA \qquad \mathbb{D}_{\mathbf{rrr}}^{\partial\Omega} = \int_{\partial\Omega} \frac{(\mathbf{r}\cdot\mathbf{k})\,\mathbf{r}\otimes\mathbf{r}\otimes\mathbf{r}\,(\mathbf{r}\cdot\mathbf{n})}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}dA\,, \tag{29}$$

one has, recalling definitions (15) and (16)

$$d_{\mathbf{r}}^{\Omega} = d_{\mathbf{r}}^{\partial\Omega} \qquad \mathbf{d}_{\mathbf{r}}^{\Omega} = \frac{\mathbf{d}_{\mathbf{r}}^{\partial\Omega}}{2} \qquad \mathbf{D}_{\mathbf{rr}}^{\Omega} = \frac{\mathbf{D}_{\mathbf{rr}}^{\partial\Omega}}{3} \qquad \mathbb{D}_{\mathbf{rrr}}^{\Omega} = \frac{\mathbb{D}_{\mathbf{rrr}}^{\partial\Omega}}{4}\,. \tag{30}$$

In conclusion, application of formula (17) allows us to rewrite formula (14) as follows

$$\Delta g_z(\mathbf{o}) = G\Big[\theta_{\mathbf{o}} d_{\mathbf{r}}^{\partial\Omega} + \frac{\mathbf{c}\cdot\mathbf{d}_{\mathbf{r}}^{\partial\Omega}}{2} + \frac{\mathbf{C}\cdot\mathbf{D}_{\mathbf{rr}}^{\partial\Omega}}{3} + \frac{\mathbb{C}\cdot\mathbb{D}_{\mathbf{rrr}}^{\partial\Omega}}{4}\Big], \tag{31}$$

an expression that will be further elaborated in the next subsection by transforming the 2D integrals (28), (29) in 1D integrals.

2.2 Analytical Expression of the Gravity Anomaly at *O* in terms of Face Integrals

In order to derive an expression suitable for programming, we specialize formula (31) to polyhedral domains since this is by far the most general case in the gravity inversion problems.

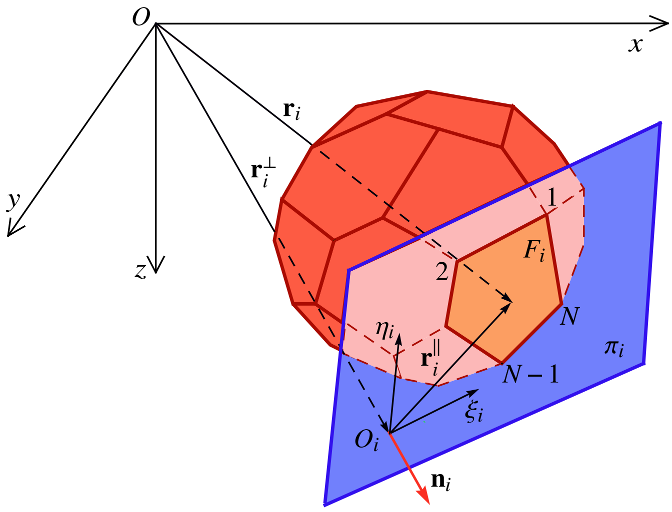


Fig. 1 Polyhedral domain Ω and decomposition of the position vector of a point on a face.

For a polyhedral body characterized by N_F faces, the integrals in (28)-(29) can be written as

$$
\begin{aligned}
d_{\mathbf{r}}^{\partial\Omega} &= \sum_{i=1}^{N_F} \int_{F_i} \frac{(\mathbf{r}_i\cdot\mathbf{k})(\mathbf{r}_i\cdot\mathbf{n}_i)}{(\mathbf{r}_i\cdot\mathbf{r}_i)^{3/2}} dA_i = \sum_{i=1}^{N_F} d_i \int_{F_i} \frac{\mathbf{r}_i\cdot\mathbf{k}}{(\mathbf{r}_i\cdot\mathbf{r}_i)^{3/2}} dA_i \\
\mathbf{d}_{\mathbf{r}}^{\partial\Omega} &= \sum_{i=1}^{N_F} \int_{F_i} \frac{(\mathbf{r}_i\cdot\mathbf{k})\mathbf{r}_i(\mathbf{r}_i\cdot\mathbf{n}_i)}{(\mathbf{r}_i\cdot\mathbf{r}_i)^{3/2}} dA_i = \sum_{i=1}^{N_F} d_i \int_{F_i} \frac{(\mathbf{r}_i\cdot\mathbf{k})\mathbf{r}_i}{(\mathbf{r}_i\cdot\mathbf{r}_i)^{3/2}} dA_i \\
\mathbf{D}_{\mathbf{rr}}^{\partial\Omega} &= \sum_{i=1}^{N_F} \int_{F_i} \frac{(\mathbf{r}_i\cdot\mathbf{k})(\mathbf{r}_i\otimes\mathbf{r}_i)(\mathbf{r}_i\cdot\mathbf{n}_i)}{(\mathbf{r}_i\cdot\mathbf{r}_i)^{3/2}} dA_i = \sum_{i=1}^{N_F} d_i \int_{F_i} \frac{(\mathbf{r}_i\cdot\mathbf{k})\mathbf{r}_i\otimes\mathbf{r}_i}{(\mathbf{r}_i\cdot\mathbf{r}_i)^{3/2}} dA_i \\
\mathbb{D}_{\mathbf{rrr}}^{\partial\Omega} &= \sum_{i=1}^{N_F} \int_{F_i} \frac{(\mathbf{r}_i\cdot\mathbf{k})(\mathbf{r}_i\otimes\mathbf{r}_i\otimes\mathbf{r}_i)(\mathbf{r}_i\cdot\mathbf{n}_i)}{(\mathbf{r}_i\cdot\mathbf{r}_i)^{3/2}} dA_i = \sum_{i=1}^{N_F} d_i \int_{F_i} \frac{(\mathbf{r}_i\cdot\mathbf{k})\mathbf{r}_i\otimes\mathbf{r}_i\otimes\mathbf{r}_i}{(\mathbf{r}_i\cdot\mathbf{r}_i)^{3/2}} dA_i
\end{aligned}
\tag{32}
$$

where the second equality in each formula above stems from the fact that the vector $\mathbf{r}_i$ spanning the i-th face, see, e.g., fig. 1, can be decomposed as follows

$$
\mathbf{r}_i = \mathbf{r}_i^{\perp} + \mathbf{r}_i^{\parallel}, \tag{33}
$$

i.e. as sum of a vector $\mathbf{r}_i^{\perp}$ orthogonal to F_i and a vector $\mathbf{r}_i^{\parallel}$ parallel to the face. Accordingly, denoting by $\mathbf{n}_i$ the unit vector pointing outwards Ω, one can set $\mathbf{r}_i\cdot\mathbf{n}_i = \mathbf{r}_i^{\perp}\cdot\mathbf{n}_i = d_i$, since d_i represents the signed distance between the origin and the i-th face F_i measured orthogonally to this last one.

The 2D integrals above can be transformed to a line integral by a further application of Gauss theorem. To this end we denote by O_i the orthogonal projection on F_i of the observation point O and assume O_i as origin of a 2D reference frame local to the face.

Furthermore, we express formula (33) in the alternative form

$$\mathbf{r}_i = \mathbf{r}_i^{\perp} + \mathbf{r}_i^{\parallel} = (\mathbf{r}_i \cdot \mathbf{n}_i)\mathbf{n}_i + \mathbf{r}_i^{\parallel} = d_i \mathbf{n}_i + \mathbf{T}_{F_i} \boldsymbol{\rho}_i \tag{34}$$

where the vector $\boldsymbol{\rho}_i = (\xi_i, \eta_i)$ represents the position vector of a generic point of the i-th face with respect to O_i and

$$\mathbf{T}_{F_i} = \begin{bmatrix} \mathbf{u}_{i1} & \mathbf{v}_{i1} \\ \mathbf{u}_{i2} & \mathbf{v}_{i2} \\ \mathbf{u}_{i3} & \mathbf{v}_{i3} \end{bmatrix} \tag{35}$$

is the linear operator mapping the 2D vector $\boldsymbol{\rho}_i$ to the 3D one $\mathbf{r}_i^{\parallel}$. In turn $\mathbf{u}_i$ and $\mathbf{v}_i$ represent two distinct, yet arbitrary, 3D unit vectors parallel to F_i.

We emphasize the use of roman and greek letters in (34) to denote, respectively, 3D and 2D vectors. The same notational distinction will be adopted throughout the paper.

Setting

$$\mathbf{r}_i \cdot \mathbf{k} = d_i \mathbf{n}_i \cdot \mathbf{k} + \mathbf{T}_{F_i} \boldsymbol{\rho}_i \cdot \mathbf{k} = d_i n_{i3} + \boldsymbol{\rho}_i \cdot \mathbf{T}_{F_i}^T \mathbf{k} = d_i n_{i3} + \boldsymbol{\rho}_i \cdot \boldsymbol{\kappa}_i \,, \tag{36}$$

the first two integrals in (32) become

$$d_{\mathbf{r}}^{\partial\Omega} = \sum_{i=1}^{N_F} d_i \left\{ d_i n_{i3} \int_{F_i} \frac{dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} + \boldsymbol{\kappa}_i \cdot \int_{F_i} \frac{\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \right\} \tag{37}$$

$$\begin{aligned} \mathbf{d}_{\mathbf{r}}^{\partial\Omega} = \sum_{i=1}^{N_F} d_i \Bigg\{ & d_i^2 n_{i3} \mathbf{n}_i \int_{F_i} \frac{dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} + d_i n_{i3} \int_{F_i} \frac{\mathbf{T}_{F_i} \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i + \\ & + d_i \mathbf{n}_i \left[\int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \cdot \boldsymbol{\kappa}_i \right] + \int_{F_i} \frac{\mathbf{T}_{F_i} \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \boldsymbol{\kappa}_i \Bigg\}. \end{aligned} \tag{38}$$

Thus, defining

$$\varphi_{F_i} = \int_{F_i} \frac{dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \quad \boldsymbol{\varphi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \quad \boldsymbol{\Phi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \,, \tag{39}$$

one finally has

$$d_{\mathbf{r}}^{\partial\Omega} = \sum_{i=1}^{N_F} d_i \left\{ d_i n_{i3} \varphi_{F_i} + \boldsymbol{\kappa}_i \cdot \boldsymbol{\varphi}_{F_i} \right\} \tag{40}$$

and

$$\mathbf{d}_{\mathbf{r}}^{\partial\Omega} = \sum_{i=1}^{N_F} d_i \left\{ d_i^2 n_{i3} \varphi_{F_i} \mathbf{n}_i + d_i n_{i3} \mathbf{T}_{F_i} \boldsymbol{\varphi}_{F_i} + d_i \mathbf{n}_i \left(\boldsymbol{\kappa}_i \cdot \boldsymbol{\varphi}_{F_i} \right) + \mathbf{T}_{F_i} \boldsymbol{\Phi}_{F_i} \boldsymbol{\kappa}_i \right\}. \tag{41}$$

To suitably shorten the expression of the last two integrals in (32) we set

$$\mathfrak{C}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \qquad \mathfrak{D}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \tag{42}$$

$$\mathfrak{C}_{F_i}\boldsymbol{\kappa}_i = \int_{F_i} \frac{(\boldsymbol{\rho}_i \cdot \boldsymbol{\kappa}_i)\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \qquad \mathfrak{D}_{F_i}\boldsymbol{\kappa}_i = \int_{F_i} \frac{(\boldsymbol{\rho}_i \cdot \boldsymbol{\kappa}_i)\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \tag{43}$$

and introduce the formal operator $\mathbb{T}_{F_i}^{b...b}$ where the symbol $b...b$ denotes an arbitrary sequence of 0 and 1. In particular

$$\mathbb{T}_{F_i}^{11}\boldsymbol{\Phi}_{F_i} = \mathbb{T}_{F_i}^{11} \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i = \mathbf{T}_{F_i}\boldsymbol{\Phi}_{F_i}\mathbf{T}_{F_i}^T , \tag{44}$$

$$\mathbb{T}_{F_i}^{111}\mathfrak{C}_{F_i} = \mathbb{T}_{F_i}^{111} \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \tag{45}$$

and

$$\mathbb{T}_{F_i}^{1010}\mathfrak{D}_{F_i} = \mathbb{T}_{F_i}^{1010} \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \tag{46}$$

since the suffix 1 (0) of $\mathbb{T}_{F_i}$ indicates that the operator $\mathbf{T}_{F_i}$ has (not) to be applied to the vector $\boldsymbol{\rho}_i$.

Accordingly, the third integral in (32) becomes

$$\begin{aligned}\mathbf{D}_{\mathbf{rr}}^{\partial\Omega} = \sum_{i=1}^{N_F} d_i \Big\{ & d_i n_{i3}\Big[d_i^2 \varphi_{F_i}\mathbf{n}_i \otimes \mathbf{n}_i + d_i\Big(\mathbf{n}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\varphi}_{F_i} + \mathbf{T}_{F_i}\boldsymbol{\varphi}_{F_i} \otimes \mathbf{n}_i\Big) + \mathbf{T}_{F_i}\boldsymbol{\Phi}_{F_i}\mathbf{T}_{F_i}^T\Big] + \\ & + d_i^2 \mathbf{n}_i \otimes \mathbf{n}_i\Big(\boldsymbol{\kappa}_i \cdot \boldsymbol{\varphi}_{F_i}\Big) + d_i\Big[\mathbf{n}_i \otimes \mathbf{T}_{F_i}\Big(\boldsymbol{\Phi}_{F_i}\boldsymbol{\kappa}_i\Big) + \mathbf{T}_{F_i}\Big(\boldsymbol{\Phi}_{F_i}\boldsymbol{\kappa}_i\Big) \otimes \mathbf{n}_i\Big] + \mathbf{H}_i \Big\}\end{aligned} \tag{47}$$

where

$$\mathbf{H}_i = \mathbf{T}_{F_i}\Big(\mathfrak{C}_{F_i}\boldsymbol{\kappa}_i\Big)\mathbf{T}_{F_i}^T . \tag{48}$$

Furthermore, setting

$$\begin{aligned}\boldsymbol{\Phi}_{F_i} \wedge \mathbf{n}_i &= \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \qquad \boldsymbol{\Phi}_{F_i} \wedge \Big(\mathbf{n}_i \otimes \mathbf{n}_i\Big) = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \mathbf{n}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \\ \mathfrak{C}_{F_i} \wedge \mathbf{n}_i &= \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \qquad \mathfrak{C}_{F_i} \vee \mathbf{n}_i = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i ,\end{aligned} \tag{49}$$

it turns out to be

$$\begin{aligned}\mathbb{D}^{\partial\Omega}_{\mathbf{rrr}} = \sum_{i=1}^{N_F} d_i \Big\{ & d_i n_{i3}\Big[d_i^3 \varphi_{F_i} \mathbf{n}_i \otimes \mathbf{n}_i \otimes \mathbf{n}_i + d_i^2 \Big(\mathbf{n}_i \otimes \mathbf{n}_i \otimes \mathbf{T}_{F_i} \boldsymbol{\varphi}_{F_i} + \mathbf{n}_i \otimes \mathbf{T}_{F_i} \boldsymbol{\varphi}_{F_i} \otimes \mathbf{n}_i + \\ & + \mathbf{T}_{F_i} \boldsymbol{\varphi}_{F_i} \otimes \mathbf{n}_i \otimes \mathbf{n}_i \Big) + d_i \mathbf{n}_i \otimes \mathbb{T}^{11}_{F_i} \boldsymbol{\Phi}_{F_i} + d_i \mathbb{T}^{101}_{F_i} \Big(\boldsymbol{\Phi}_{F_i} \wedge \mathbf{n}_i \Big) + \\ & + d_i \mathbb{T}^{11}_{F_i} \boldsymbol{\Phi}_{F_i} \otimes \mathbf{n}_i + \mathbb{T}^{111}_{F_i} \mathfrak{C}_{F_i} \Big] + d_i^3 \mathbf{n}_i \otimes \mathbf{n}_i \otimes \mathbf{n}_i \Big(\boldsymbol{\kappa}_i \cdot \boldsymbol{\varphi}_{F_i} \Big) + \\ & + d_i^2 \Big[\mathbf{n}_i \otimes \mathbf{n}_i \otimes \mathbf{T}_{F_i} \Big(\boldsymbol{\Phi}_{F_i} \boldsymbol{\kappa}_i \Big) + \mathbf{n}_i \otimes \mathbb{T}^{101}_{F_i} \Big(\boldsymbol{\Phi}_{F_i} \wedge \mathbf{n}_i \Big) \boldsymbol{\kappa}_i + \\ & + \mathbb{T}^{1000}_{F_i} \boldsymbol{\Phi}_{F_i} \wedge \Big(\mathbf{n}_i \otimes \mathbf{n}_i \Big) \boldsymbol{\kappa}_i \Big] + d_i \Big[\mathbf{n}_i \otimes \mathbb{T}^{110}_{F_i} \mathfrak{C}_{F_i} \boldsymbol{\kappa}_i + \mathbb{T}^{1010}_{F_i} \Big(\mathfrak{C}_{F_i} \wedge \mathbf{n}_i \Big) \boldsymbol{\kappa}_i + \\ & + \mathbb{T}^{1100}_{F_i} \Big(\mathfrak{C}_{F_i} \vee \mathbf{n}_i \Big) \boldsymbol{\kappa}_i \Big] + \mathbb{T}^{1110}_{F_i} \mathfrak{D}_{F_i} \boldsymbol{\kappa}_i \Big\} \end{aligned} \tag{50}$$

being

$$\mathbb{T}^{101}_{F_i}\Big(\boldsymbol{\Phi}_{F_i} \wedge \mathbf{n}_i\Big) = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i , \tag{51}$$

$$\mathbb{T}^{1000}_{F_i}\boldsymbol{\Phi}_{F_i} \wedge \Big(\mathbf{n}_i \otimes \mathbf{n}_i\Big) = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \mathbf{n}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \boldsymbol{\kappa}_i , \tag{52}$$

$$\mathbb{T}^{110}_{F_i}\mathfrak{C}_{F_i}\boldsymbol{\kappa}_i = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \boldsymbol{\kappa}_i = \int_{F_i} \frac{(\boldsymbol{\rho}_i \cdot \boldsymbol{\kappa}_i)\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i , \tag{53}$$

$$\mathbb{T}^{1010}_{F_i}\Big(\mathfrak{C}_{F_i} \wedge \mathbf{n}_i\Big) = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} , \tag{54}$$

$$\mathbb{T}^{1100}_{F_i}\Big(\mathfrak{C}_{F_i} \vee \mathbf{n}_i\Big) = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{n}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} , \tag{55}$$

$$\mathbb{T}^{1110}_{F_i}\mathfrak{D}_{F_i}\boldsymbol{\kappa}_i = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} \boldsymbol{\kappa}_i = \int_{F_i} \frac{(\boldsymbol{\rho}_i \cdot \boldsymbol{\kappa}_i)\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i . \tag{56}$$

Notice that the symbols in (49), as well as the ones in (50), are purely formal since they involve the tensor product of 2D and 3D vectors. They have been deliberately introduced to focus the reader's attention on the main issues involved in the evaluation of the quantities $d^{\partial\Omega}_{\mathbf{r}}$, $\mathbf{d}^{\partial\Omega}_{\mathbf{r}}$, $\mathbf{D}^{\partial\Omega}_{\mathbf{rr}}$, and $\mathbb{D}^{\partial\Omega}_{\mathbf{rrr}}$. Actually, one first evaluates the integrals

$$\int_{F_i} \frac{[\otimes\boldsymbol{\rho}_i, m]}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \quad m \in [0,4] \tag{57}$$

as tensor product of 2D vectors, see, e.g., Appendix 1 and 2. Only subsequently the resulting formula is combined with the 2D vector $\boldsymbol{\kappa}_i$ and expressed in terms of 3D vectors, by means of the operator $\mathbf{T}_{F_i}$, or suitably combined with the 3D vector $\mathbf{n}_i$ to evaluate the integrals in (50).

The simultaneous presence in (57) of the quantity d_i and of the exponent 3/2 in the denominator makes the evaluation of the integrals in (57) by far more diffult than the analogous ones addressed in D'Urso (2015c) for polygonal bodies. Actually the case $d_i = 0$, meaning that the observation point O belongs to the face F_i, or equivalently that $O_i \equiv O$, needs to be properly addressed since the integrals can become singular.

For the same reason we shall not consider the fact that the integrals in (57) need to be composed with the vector $\boldsymbol{\kappa}_i$ producing

$$\left[\int_{F_i} \frac{[\otimes\boldsymbol{\rho}_i, m]}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i\right]\boldsymbol{\kappa}_i = \int_{F_i} \frac{[\otimes\boldsymbol{\rho}_i, m-1](\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \quad m \in [1,4], \tag{58}$$

since this would require to consider separately these cases in the discussion of the singularities of the algebraic expressions resulting from (57); instead, we shall perform the combination after the integration. Moreover, due to the presence of the exponent 3/2, the definite integrals that need to be computed to transform the integrals (57) into their algebraic counterparts do not exhibit anymore the useful recurrence property invoked in the appendix of D'Urso (2015c) so that it is more convenient to evaluate the integrals in (57) prior to their composition with $\boldsymbol{\kappa}_i$.

Last, but not least, most of the integrals in (57) have been already computed in D'Urso (2013a, 2014a,b) so that we include in the Appendix 1 only the explicit evaluation of the new ones.

2.3 Analytical Expression of Face Integrals in terms of 1D Integrals

It has been emphasized in the previous subsection that the main burden associated with the evaluation of the expressions (37), (38), (47) and (50) is the evaluation of the integrals (57). Similarly to the integrals (15) and (16), they can be transformed into simpler 1D integrals by a further application of the generalized Gauss theorem (Tang, 2006).

For some of them, namely the ones in (57) defined by $m = 0$, $m = 1$, and $m = 2$, this has been done in previous papers (D'Urso, 2013a, 2014a,b); for $m = 3$ and $m = 4$ this has been carried out in Appendix 1. For sake of clarity their expressions are collected hereafter for increasing values of m.

• Integral (57) for $m = 0$

$$\varphi_{F_i} = \int_{F_i} \frac{dA_i}{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{3/2}} = \frac{\alpha_i}{|d_i|} - \int_{\partial F_i} \frac{\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\nu}(s_i)}{\left[\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\rho}_i(s_i)\right]\left[\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2}} ds_i\,. \tag{59}$$

where s_i is the curvilinear abscissa along the boundary ∂F_i of the face F_i, $\boldsymbol{\nu}$ is the outward unit normal to F_i and α_i is a scalar, defined in Appendix 2, representing the measure, expressed in radians, of the intersection between F_i and a circular neighbourhood of the singularity point $\boldsymbol{\rho} = \boldsymbol{o}$ when $d_i = 0$.

• Integral (57) for $m = 1$

$$\boldsymbol{\varphi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i + d_i^2\right)^{3/2}} = -\int_{\partial F_i} \frac{\boldsymbol{\nu}(s_i)}{\left[\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2}} ds_i\,. \tag{60}$$

• Integral (57) for $m = 2$

$$\boldsymbol{\Phi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}} = -\int_{\partial F_i} \frac{\boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\nu}(s_i)}{\left[\boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2}} ds_i + \psi_{F_i} \mathbf{I}_{2D} \tag{61}$$

where $\mathbf{I}_{2D}$ is the rank-two two-dimensional identity tensor,

$$\psi_{F_i} = \int_{F_i} \frac{dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} = \int_{\partial F_i} \frac{\left[\boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2} \left[\boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\nu}(s_i)\right]}{\boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\rho}_i(s_i)} ds_i - \alpha_i |d_i| \tag{62}$$

and α_i has been introduced just before formula (60).

• Integral (57) for $m = 3$

$$\mathfrak{C}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}} = -\int_{\partial F_i} \frac{\boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\nu}(s_i)}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} ds_i + \mathbf{I}_{2D} \otimes_{23} \boldsymbol{\psi}_{F_i} + \boldsymbol{\psi}_{F_i} \otimes \mathbf{I}_{2D} \tag{63}$$

where the symbol $\otimes_{23}$ denotes the tensor product obtained by interchanging the second and third index of the rank-three tensor $\mathbf{I}_{2D} \otimes \boldsymbol{\psi}_{F_i}$ and

$$\boldsymbol{\psi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} = \int_{\partial F_i} \left[\boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2} \boldsymbol{\nu}(s_i) ds_i \, . \tag{64}$$

• Integral (57) for $m = 4$

$$\begin{aligned} \mathfrak{D}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}} &= -\int_{\partial F_i} \frac{\boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\nu}(s_i)}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} ds_i + \\ &+ \mathbf{I}_{2D} \otimes_{24} \boldsymbol{\Psi}_{F_i} + \boldsymbol{\Psi}_{F_i} \otimes_{23} \mathbf{I}_{2D} + \boldsymbol{\Psi}_{F_i} \otimes \mathbf{I}_{2D} \end{aligned} \tag{65}$$

where the symbol $\otimes_{24}$ denotes the tensor product obtained by interchanging the second and fourth index of the rank-four tensor $\mathbf{I}_{2D} \otimes \boldsymbol{\Psi}_{F_i}$ and

$$\begin{aligned} \boldsymbol{\Psi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i dA_i}{\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}} &= -\int_{\partial F_i} \left[\boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2} \boldsymbol{\rho}_i(s_i) \otimes \boldsymbol{\nu}(s_i) ds_i - \\ &- \frac{\mathbf{I}_{2D}}{3} \left\{ \int_{\partial F_i} \left[\boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\rho}_i(s_i) + d_i^2\right]^{1/2} \boldsymbol{\rho}_i(s_i) \cdot \boldsymbol{\nu}(s_i) ds_i - d_i^2 \psi_{F_i} \right\} . \end{aligned} \tag{66}$$

Since each face is polygonal the previous line integrals can be further expressed as sums extended to the N_{E_i} edges that define the boundary ∂F_i. For the j-th edge a suitable parameterization allows one to transform each 1D integral into an integral of a real variable; this is scaled by a suitable combination of the vectors $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$ that define the position vectors of the end vertices of the edge in the 2D reference frame local to F_i.

In particular we set

$$\hat{\boldsymbol{\rho}}_i(\lambda_j) = \boldsymbol{\rho}_j + \lambda_j (\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j) = \boldsymbol{\rho}_j + \lambda_j \Delta \boldsymbol{\rho}_j \tag{67}$$

where the function $\hat{\boldsymbol{\rho}}_i$ associates with each value of the adimensional abscissa

$$\lambda_j = s_j / l_j, \tag{68}$$

the position vector spanning the j-th edge. The quantity s_j, $s_j \in [0, l_j]$, is the curvilinear abscissa along the j-th edge and $l_j = |\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j|$ is the edge length. The position vector spanning the j-th edge of F_i can also be expressed as function of s_j and a new function $\boldsymbol{\rho}_i$, fulfilling the condition $\boldsymbol{\rho}_i(s_i) = \hat{\boldsymbol{\rho}}_i(\lambda_j)$. Hence

$$\boldsymbol{\rho}_i(s_j) \cdot \boldsymbol{\rho}_i(s_j) = \hat{\boldsymbol{\rho}}_i(\lambda_j) \cdot \hat{\boldsymbol{\rho}}_i(\lambda_j) = p_j \lambda_j^2 + 2 q_j \lambda_j + u_j = P_u(\lambda_j) \tag{69}$$

where, according to (67)

$$p_j = \Delta\boldsymbol{\rho}_j \cdot \Delta\boldsymbol{\rho}_j \qquad q_j = \boldsymbol{\rho}_j \cdot \Delta\boldsymbol{\rho}_j \qquad u_j = \boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j \,. \tag{70}$$

Furthermore

$$\boldsymbol{\rho}(s_j) \cdot \boldsymbol{\rho}(s_j) + d_i^2 = p_j \lambda_j^2 + 2 q_j \lambda_j + v_j \tag{71}$$

where $v_j = u_j + d_i^2$. We shall also set $P_v(\lambda_j) = P_u(\lambda_j) + d_i^2$.

2.4 Algebraic expression of face integrals in terms of 2D vectors

Refering to the Appendices 1 and 2 for further details we hereby report the algebraic counterparts of the integrals (57) for m=0,..,4.

• Integral (57) for $m = 0$

$$\varphi_{F_i} = \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) \int_0^1 \frac{d\lambda_j}{P_u(\lambda_j)\left[P_v(\lambda_j)\right]^{1/2}} = \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}} \varphi_j \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) \tag{72}$$

where φ_j is defined in (221). The symbol $(\cdot)^{\perp}$ denotes a clockwise rotation of the 2D vector $(\cdot)$ necessary to express the outward unit normal $\boldsymbol{\nu}_j$ to the j-th edge according to the formula

$$\boldsymbol{\nu}_j = \frac{\left(\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j\right)^{\perp}}{l_j} = \frac{\Delta\boldsymbol{\rho}_j^{\perp}}{l_j} \,. \tag{73}$$

The clockwise rotation indicated by the symbol $(\cdot)^{\perp}$ depends on the convention adopted to circulate along the boundary ∂F_i. In particular, we have assumed that the vertices of each face have been numbered consecutively by circulating along ∂F_i in a counter-clockwise sense with respect to the normal $\mathbf{n}_i$ to the face. Thus

$$\Delta\boldsymbol{\rho}_j = \begin{bmatrix} \Delta\xi_j \\ \Delta\eta_j \end{bmatrix} \Rightarrow \Delta\boldsymbol{\rho}_j^{\perp} = \begin{bmatrix} -\Delta\eta_j \\ \Delta\xi_j \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Delta\boldsymbol{\rho}_j \,. \tag{74}$$

• Integral (57) for $m = 1$

$$\boldsymbol{\varphi}_{F_i} = -\sum_{j=1}^{N_{E_i}} \Delta\boldsymbol{\rho}_j^{\perp} \int_0^1 \frac{d\lambda_j}{\left[P_v(\lambda_j)\right]^{1/2}} = -\sum_{j=1}^{N_{E_i}} I_{0j}\, \Delta\boldsymbol{\rho}_j^{\perp} \tag{75}$$

where the scalar I_{0j} is defined in (211).

• Integral (57) for $m = 2$

$$\begin{aligned}\mathbf{\Phi}_{F_i} &= -\sum_{j=1}^{N_{E_i}} \int_0^1 \frac{\hat{\boldsymbol{\rho}}_i(\lambda_j)}{\left[P_v(\lambda_j)\right]^{1/2}} d\lambda_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} + \psi_{F_i}\mathbf{I}_{2D} = \\ &= -\sum_{j=1}^{N_{E_i}} \left[I_{0j}\,\boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} + I_{1j}\,\Delta\boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j^{\perp}\right] + \psi_{F_i}\mathbf{I}_{2D}\end{aligned} \tag{76}$$

where I_{0j} is defined in (211), I_{1j} in (212) while ψ_{F_i} is provided by

$$\psi_{F_i} = \sum_{j=1}^{N_{E_i}} \int_0^1 \frac{\left[P_v(\lambda_j)\right]^{1/2}}{\left[P_u(\lambda_j)\right]} d\lambda_j = \sum_{j=1}^{N_{E_i}} \psi_j^i \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) - |d_i|\alpha_i \tag{77}$$

and ψ_j^i is defined in (219).

• Integral (57) for $m = 3$

$$\begin{aligned}\mathfrak{C}_{F_i} &= -\sum_{j=1}^{N_{E_i}} \int_0^1 \frac{\hat{\boldsymbol{\rho}}_i(\lambda_j) \otimes \hat{\boldsymbol{\rho}}_i(\lambda_j) d\lambda_j}{\left[P_v(\lambda_j)\right]^{1/2}} + \mathbf{I}_{2D} \otimes_{23} \boldsymbol{\psi}_{F_i} + \boldsymbol{\psi}_{F_i} \otimes \mathbf{I}_{2D} = \\ &= -\sum_{j=1}^{N_{E_i}} \left[I_{0j}\,\mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j}\,\mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{2j}\,\mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}\right] \otimes \Delta\boldsymbol{\rho}_j^{\perp} + \mathbf{I}_{2D} \otimes_{23} \boldsymbol{\psi}_{F_i} + \boldsymbol{\psi}_{F_i} \otimes \mathbf{I}_{2D}\end{aligned} \tag{78}$$

where I_{0j}, I_{1j}, I_{2j} are defined in (211), (212) and (213) respectively, $\mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j}$, $\mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}$ and $\mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}$ are defined in (180) and

$$\boldsymbol{\psi}_{F_i} = \sum_{j=1}^{N_{E_i}} l_j \boldsymbol{\nu}_j \int_0^1 \left[P_v(\lambda_j)\right]^{1/2} d\lambda_j = \sum_{j=1}^{N_{E_i}} I_{4j}\,\Delta\boldsymbol{\rho}_j^{\perp}, \tag{79}$$

the scalar I_{4j} being defined in (215).

• Integral (57) for $m = 4$

$$\begin{aligned}\mathfrak{D}_{F_i} &= -\sum_{j=1}^{N_{E_i}} \left\{ \int_0^1 \frac{\hat{\boldsymbol{\rho}}_i(\lambda_j) \otimes \hat{\boldsymbol{\rho}}_i(\lambda_j) \otimes \hat{\boldsymbol{\rho}}_i(\lambda_j) d\lambda_j}{\left[P_v(\lambda_j)\right]^{1/2}} \otimes \Delta\boldsymbol{\rho}_j^{\perp} \right\} + \mathbf{I}_{2D} \otimes_{24} \boldsymbol{\Psi}_{F_i} + \boldsymbol{\Psi}_{F_i} \otimes_{23} \mathbf{I}_{2D} + \boldsymbol{\Psi}_{F_i} \otimes \mathbf{I}_{2D} = \\ &= -\sum_{j=1}^{N_{E_i}} \left[I_{0j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + \mathbb{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{3j}\,\mathbb{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} \right] \otimes \Delta\boldsymbol{\rho}_j^{\perp} + \\ &\quad + \mathbf{I}_{2D} \otimes_{24} \boldsymbol{\Psi}_{F_i} + \boldsymbol{\Psi}_{F_i} \otimes_{23} \mathbf{I}_{2D} + \boldsymbol{\Psi}_{F_i} \otimes \mathbf{I}_{2D}\end{aligned} \tag{80}$$

where I_{0j}, I_{1j}, I_{2j}, I_{3j} are defined in (211), (212), (213) and (214) respectively, $\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\boldsymbol{\rho}_j}$, $\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}$, $\mathbb{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}$ and $\mathbb{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}$ are defined in (191), (192) and (193) and

$$\begin{aligned}\boldsymbol{\Psi}_{F_i} = & \sum_{j=1}^{N_{E_i}} \left\{ \left[\int_0^1 \left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j)+d_i^2\right]^{1/2} \left(\boldsymbol{\rho}_j+\lambda_j\Delta\boldsymbol{\rho}_j\right) d\lambda_j \right] \otimes \Delta\boldsymbol{\rho}_j^{\perp} - \right. \\ & \left. -\frac{\mathbf{I}_{2D}}{3}\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right)\int_0^1 \left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j)+d_i^2\right]^{1/2} d\lambda_j \right\} + \frac{d_i^2}{3}\left(\psi_i - |d_i|\alpha_i\right) = \\ = & \sum_{j=1}^{N_{E_i}} \left[\left(I_{4j}\boldsymbol{\rho}_j + I_{5j}\Delta\boldsymbol{\rho}_j\right)\otimes\Delta\boldsymbol{\rho}_j^{\perp} - \frac{\mathbf{I}_{2D}}{3}(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp})I_{4j}\right] + \frac{d_i^2}{3}\left(\psi_i - |d_i|\alpha_i\right),\end{aligned} \tag{81}$$

I_{4j}, I_{5j}, and ψ_i being defined in (215), (216) and (219) respectively.

For future reference we also include the algebraic expressions of the integrals in formula (43).

$$\mathfrak{C}_{F_i\boldsymbol{\kappa}_i} = -\sum_{j=1}^{N_{E_i}}\left(\boldsymbol{\kappa}_i\cdot\Delta\boldsymbol{\rho}_j^{\perp}\right)\left(I_{0j}\,\mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j}\,\mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{2j}\,\mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}\right) + \boldsymbol{\kappa}_i\otimes\boldsymbol{\psi}_{F_i} + \boldsymbol{\psi}_{F_i}\otimes\boldsymbol{\kappa}_i \tag{82}$$

$$\begin{aligned}\mathfrak{D}_{F_i\boldsymbol{\kappa}_i} = & -\sum_{j=1}^{N_{E_i}}\left(\boldsymbol{\kappa}_i\cdot\Delta\boldsymbol{\rho}_j^{\perp}\right)\left(I_{0j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{2j}\,\mathbb{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + \right. \\ & \left. + I_{3j}\,\mathbb{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}\right) + \boldsymbol{\Psi}_{F_i}\otimes\boldsymbol{\kappa}_i + \boldsymbol{\Psi}_{F_i}\otimes_{23}\boldsymbol{\kappa}_i + \boldsymbol{\kappa}_i\otimes\boldsymbol{\Psi}_{F_i}\,.\end{aligned} \tag{83}$$

All the previous quantities are expressed in terms of 2D vectors representing the coordinates of the end vertices of each edge in the reference frame local to each face F_i. Conversely, all tensors appearing in (37), (38), (47) and (50) have to expressed in terms of the 3D position vectors defining the vertices of the polyhedron Ω since these represent the basic geometric entities that define it. This task will be accomplished in the following subsection.

2.5 Algebraic expression of the integrals in terms of 3D vectors

The aim of this subsection is the show how the algebraic expressions derived in the previous subsection can be expressed in terms of 3D vectors in order to apply formula (31), what is fully accounted for in the next subsection. This is done by inverting (34) so as to express 2D coordinates of each vertex as function of the relevant 3D ones. In particular, premultiplying relation (34) by $\mathbf{T}_{F_i}^T$, where $(\cdot)^T$ stands for transpose, one obtains

$$\boldsymbol{\rho}_j = \mathbf{T}_{F_i}^T(\mathbf{r}_j - d_i\mathbf{n}_i) \tag{84}$$

since it is easy to check that $\mathbf{T}_{F_i}^T\mathbf{T}_{F_i} = \mathbf{I}_{2D}$.

Additional quantities that need to be expressed in terms of 3D vectors are

$$\mathbf{T}_{F_i}\Delta\boldsymbol{\rho}_j = \mathbf{r}_{j+1} - \mathbf{r}_i = \Delta\mathbf{r}_j \tag{85}$$

and

$$\mathbf{T}_{F_i}\Delta\boldsymbol{\rho}_j^{\perp} = \mathbf{T}_{F_i}\left[\mathbf{T}_{F_i}^T\Delta\mathbf{r}_j\right]^{\perp}. \tag{86}$$

We also set

$$\mathbf{f}_i = \mathbf{T}_{F_i}\boldsymbol{\varphi}_{F_i} = -\sum_{j=1}^{N_{E_i}} I_{0j}\mathbf{T}_{F_i}\Delta\boldsymbol{\rho}_j^{\perp} \tag{87}$$

according to (75) and

$$\mathbf{g}_i = \mathbf{T}_{F_i}\boldsymbol{\Phi}_{F_i}\boldsymbol{\kappa}_i = -\sum_{j=1}^{N_{E_i}} \left(\Delta\boldsymbol{\rho}_j^{\perp}\cdot\boldsymbol{\kappa}_i\right)\left[I_{0j}\mathbf{r}_j + I_{1j}\Delta\mathbf{r}_j\right] + \psi_{F_i}\mathbf{T}_{F_i}\mathbf{T}_{F_i}^T\mathbf{k} \tag{88}$$

according to (36) and (76); furthermore, we set

$$\mathbf{G}_i = \mathbf{T}_{F_i}\boldsymbol{\Phi}_{F_i}\mathbf{T}_{F_i}^T \tag{89}$$

see, e.g., formula (44).

Finally, recalling (44), (46), (48) and (49) it turns out to be

$$\mathbb{T}_{F_i}^{101}\left(\boldsymbol{\Phi}_{F_i}\wedge\mathbf{n}_i\right) = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{n}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i = \mathbf{G}_i\otimes_{23}\mathbf{n}_i\,, \tag{90}$$

$$\mathbb{T}_{F_i}^{110}\boldsymbol{\Phi}_{F_i}\otimes\mathbf{n}_i = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\otimes\mathbf{n}_i = \mathbf{G}_i\otimes\mathbf{n}_i\,, \tag{91}$$

$$\mathbb{G}_i = \mathbb{T}_{F_i}^{111}\mathfrak{C}_{F_i} = \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\,, \tag{92}$$

$$\begin{aligned}\mathbb{T}_{F_i}^{101}\left(\boldsymbol{\Phi}_{F_i}\wedge\mathbf{n}_i\right)\boldsymbol{\kappa}_i &= \mathbf{T}_{F_i}\int_{F_i} \frac{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\right)\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\otimes\mathbf{n}_i = \mathbf{T}_{F_i}\int_{F_i} \frac{\left(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\right)dA_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\boldsymbol{\kappa}_i\otimes\mathbf{n}_i = \\ &= \mathbf{T}_{F_i}\boldsymbol{\Phi}_{F_i}\boldsymbol{\kappa}_i\otimes\mathbf{n}_i = \mathbf{g}_i\otimes\mathbf{n}_i\,,\end{aligned} \tag{93}$$

$$\begin{aligned}\mathbb{T}_{F_i}^{100}\boldsymbol{\Phi}_{F_i}\wedge\left(\mathbf{n}_i\otimes\mathbf{n}_i\right)\boldsymbol{\kappa}_i &= \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{n}_i\otimes\mathbf{n}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\boldsymbol{\kappa}_i = \mathbf{T}_{F_i}\int_{F_i} \frac{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\right)\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\otimes\mathbf{n}_i\otimes\mathbf{n}_i = \\ &= \mathbf{T}_{F_i}\boldsymbol{\Phi}_{F_i}\boldsymbol{\kappa}_i\otimes\mathbf{n}_i\otimes\mathbf{n}_i = \mathbf{g}_i\otimes\mathbf{n}_i\otimes\mathbf{n}_i\,,\end{aligned} \tag{94}$$

$$\begin{aligned}\mathbb{T}_{F_i}^{110}\mathfrak{C}_{F_i}\boldsymbol{\kappa}_i &= \int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\boldsymbol{\kappa}_i = \int_{F_i} \frac{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\right)\left(\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\right)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i = \\ &= \mathbf{T}_{F_i}\int_{F_i} \frac{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\right)\left(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\right)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\mathbf{T}_{F_i}^T = \mathbf{T}_{F_i}\left[\int_{F_i} \frac{\left(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\right)dA_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\boldsymbol{\kappa}_i\right]\mathbf{T}_{F_i}^T = \\ &= \mathbf{T}_{F_i}\left(\mathfrak{C}_{F_i}\boldsymbol{\kappa}_i\right)\mathbf{T}_{F_i}^T = \mathbf{H}_i\,,\end{aligned} \tag{95}$$

$$\mathbb{T}^{1010}_{F_i}\big(\mathfrak{C}_{F_i}\wedge\mathbf{n}_i\big)\boldsymbol{\kappa}_i=\int_{F_i}\frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{n}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\boldsymbol{\kappa}_i=\int_{F_i}\frac{\big(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\big)\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{n}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i=$$

$$=\int_{F_i}\frac{\big(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\big)\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\otimes_{23}\mathbf{n}_i=\mathbf{H}_i\otimes_{23}\mathbf{n}_i\,, \tag{96}$$

$$\mathbb{T}^{1100}_{F_i}\big(\mathfrak{C}_{F_i}\vee\mathbf{n}_i\big)=\int_{F_i}\frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{n}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\boldsymbol{\kappa}_i=\int_{F_i}\frac{\big(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\big)\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{n}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i=$$

$$=\mathbf{T}_{F_i}\left[\int_{F_i}\frac{\big(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i\big)\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\right]\mathbf{T}^T_{F_i}\otimes\mathbf{n}_i=\left[\mathbf{T}_{F_i}\big(\mathfrak{C}_{F_i}\boldsymbol{\kappa}_i\big)\mathbf{T}^T_{F_i}\right]\otimes\mathbf{n}_i=\mathbf{H}_i\otimes\mathbf{n}_i\,, \tag{97}$$

$$\begin{aligned}\mathbb{H}_i=\mathbb{T}^{1110}_{F_i}\mathfrak{D}_{F_i}\boldsymbol{\kappa}_i&=\int_{F_i}\frac{(\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\boldsymbol{\kappa}_i=\\&=\int_{F_i}\frac{(\boldsymbol{\kappa}_i\cdot\boldsymbol{\rho}_i)\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i\otimes\mathbf{T}_{F_i}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i\,.\end{aligned} \tag{98}$$

The explicit evaluation of the last integral will be dealt with in the next subsection together with further considerations on actual evaluation of all third-order tensors appearing in (50).

2.6 Algebraic expression of the gravity anomaly at O

In order to make the reader fully acquainted with the operative steps required to compute the gravity anomaly at O, it is instructive to further comment on the formulas derived in the previous subsections in order to apply formula (31). As a matter of fact the evaluation of $d_{\mathbf{r}}^{\partial_i\Omega}$, $\mathbf{d}_{\mathbf{r}}^{\partial_i\Omega}$, $\mathbf{D}_{\mathbf{rr}}^{\partial_i\Omega}$, provided by formulas (37), (38) and (47), respectively, is trivial since they can be obtained by standard matrix operations.

More difficult is the evaluation of the third-order tensors appearing in (50), by taking also into account the fact that they have to first expressed in terms of 2D vectors and only subsequently, as specified in the previous subsection, reformulated in terms of 3D vectors.

To fix the ideas, let us start from the last addend in (50) that has been further detailed in (98). By means of formula (83), we actually dispose of an expression that can be written more concisely as

$$\int_{F_i}\frac{(\boldsymbol{\kappa}_i\cdot\boldsymbol{\rho}_i)\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i=\sum_{j=1}^{N_{E_i}}\Big[\alpha_j\mathbb{D}^{(j)}_{\boldsymbol{\rho}\boldsymbol{\rho}\boldsymbol{\rho}}+\boldsymbol{\Lambda}_{\boldsymbol{\rho}\boldsymbol{\rho}}\otimes\boldsymbol{\beta}+\boldsymbol{\Lambda}_{\boldsymbol{\rho}\boldsymbol{\rho}}\otimes_{23}\boldsymbol{\beta}+\boldsymbol{\beta}\otimes\boldsymbol{\Lambda}_{\boldsymbol{\rho}\boldsymbol{\rho}}\Big] \tag{99}$$

where the right-hand side is a symbolic representation of the linear combination between third-order tensors $\mathbb{D}^{(j)}_{\boldsymbol{\rho}\boldsymbol{\rho}\boldsymbol{\rho}}$, such as $\mathbb{D}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\boldsymbol{\rho}_j}$, $\mathbb{D}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}$, $\mathbb{D}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}$, $\mathbb{D}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}$, and tensor products between 2D vectors $\boldsymbol{\beta}$ and rank-two tensors $\boldsymbol{\Lambda}_{\boldsymbol{\rho}\boldsymbol{\rho}}$, this last one expressed as tensor product of 2D vectors.

Hence, to evaluate the left-hand side of (98) starting from (99) we have to transform the rank-three tensors on the right-hand side of (99) defined in terms of 2D vectors by applying the formal operator $\mathbb{T}^{111}_{F_i}$ to get,

$$\int_{F_i} \frac{\mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \mathbf{T}_{F_i}\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \boldsymbol{\kappa}_i = \mathbb{T}^{111}_{F_i} \sum_{j=1}^{N_{E_i}} \Big[\alpha_j \mathbb{D}^{(j)}_{\rho\rho\rho} + \boldsymbol{\Lambda}_{\rho\rho} \otimes \boldsymbol{\beta} + \boldsymbol{\Lambda}_{\rho\rho} \otimes_{23} \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \boldsymbol{\Lambda}_{\rho\rho} \Big]. \tag{100}$$

This is trivial for the rank-three tensor $\mathbb{D}^{(j)}_{\rho\rho\rho}$ since it is expressed as tensor product of three 2D vectors $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, $\boldsymbol{\varepsilon}$, so that

$$\mathbb{T}^{111}_{F_i} \mathbb{D}^{(j)}_{\rho\rho\rho} = \mathbb{T}^{111}_{F_i} (\boldsymbol{\gamma} \otimes \boldsymbol{\delta} \otimes \boldsymbol{\varepsilon}) = \mathbf{T}_{F_i}\boldsymbol{\gamma} \otimes \mathbf{T}_{F_i}\boldsymbol{\delta} \otimes \mathbf{T}_{F_i}\boldsymbol{\varepsilon} = \mathbf{t} \otimes \mathbf{v} \otimes \mathbf{w} \tag{101}$$

and the last tensor product between 3D vectors can be expressed in matrix form according to the rule which one adopts to define the matrix associated with a rank-three tensor, a rule that usually depends upon the adopted programming language.

For istance, extending the rule defined in (10) to three arbitrary 3D vectors one has

$$\big[\mathbf{t} \otimes (\mathbf{v} \otimes \mathbf{w})\big] = \left[t_1 \begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{pmatrix} \Bigg| \; t_2 \begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{pmatrix} \Bigg| \; t_3 \begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{pmatrix} \right]^T \tag{102}$$

where, for typographical reasons, we have represented the matrix associated with $\mathbf{t} \otimes (\mathbf{v} \otimes \mathbf{w})$ as a row rather than as a column.

Let us now apply the formal operator $\mathbb{T}^{111}_{F_i}$, already exploited in (101), to the last three addends in (100). Differently from $\mathbb{D}^{(j)}_{\rho\rho\rho}$, that is computed recursively as function of the j-th edge of F_i, the rank-two tensor $\boldsymbol{\Lambda}_{\rho\rho}$ is already available as a whole since it has been evaluated elsewhere, e.g. in a different subroutine. Hence, we already dispose of

$$\mathbb{T}^{11}_{F_i} \boldsymbol{\Lambda}_{\rho\rho} = \mathbf{T}_{F_i} \boldsymbol{\Lambda}_{\rho\rho} \mathbf{T}^T_{F_i} = \mathbf{L}_{\rho\rho} \tag{103}$$

where the roman letter $\mathbf{L}$ has been adopted to emphasize that the matrix associated with $\mathbf{L}_{\rho\rho}$ is 3×3. Accordingly

$$\mathbb{T}^{111}_{F_i} \big(\boldsymbol{\Lambda}_{\rho\rho} \otimes \boldsymbol{\beta} \big) = \mathbf{L}_{\rho\rho} \otimes \mathbf{T}_{F_i} \boldsymbol{\beta} = \mathbf{L}_{\rho\rho} \otimes \mathbf{b} \tag{104}$$

where $\mathbf{b}$ is a 3D vector.

Thus, we can exploit the general scheme in (102) by writing

$$\big[\mathbf{L} \otimes \mathbf{b}\big] = \Big[\big(\mathbf{L} \otimes \mathbf{b}\big)_1, \quad \big(\mathbf{L} \otimes \mathbf{b}\big)_2, \quad \big(\mathbf{L} \otimes \mathbf{b}\big)_3 \Big]^T . \tag{105}$$

where

$$\Big[\big(\mathbf{L} \otimes \mathbf{b}\big)_1 \Big] = \begin{bmatrix} (\mathbf{L}_{\rho\rho})_{11} b_1 & (\mathbf{L}_{\rho\rho})_{11} b_2 & (\mathbf{L}_{\rho\rho})_{11} b_3 \\ (\mathbf{L}_{\rho\rho})_{12} b_1 & (\mathbf{L}_{\rho\rho})_{12} b_2 & (\mathbf{L}_{\rho\rho})_{12} b_3 \\ (\mathbf{L}_{\rho\rho})_{13} b_1 & (\mathbf{L}_{\rho\rho})_{13} b_2 & (\mathbf{L}_{\rho\rho})_{13} b_3 \end{bmatrix}, \tag{106}$$

$$\Big[\big(\mathbf{L} \otimes \mathbf{b}\big)_2 \Big] = \begin{bmatrix} (\mathbf{L}_{\rho\rho})_{21} b_1 & (\mathbf{L}_{\rho\rho})_{21} b_2 & (\mathbf{L}_{\rho\rho})_{21} b_3 \\ (\mathbf{L}_{\rho\rho})_{22} b_1 & (\mathbf{L}_{\rho\rho})_{22} b_2 & (\mathbf{L}_{\rho\rho})_{22} b_3 \\ (\mathbf{L}_{\rho\rho})_{23} b_1 & (\mathbf{L}_{\rho\rho})_{23} b_2 & (\mathbf{L}_{\rho\rho})_{23} b_3 \end{bmatrix}, \tag{107}$$

$$\left[\left(\mathbf{L}\otimes\mathbf{b}\right)_3\right]=\begin{bmatrix}(\mathbf{L}_{\rho\rho})_{31}b_1 & (\mathbf{L}_{\rho\rho})_{31}b_2 & (\mathbf{L}_{\rho\rho})_{31}b_3\\ (\mathbf{L}_{\rho\rho})_{32}b_1 & (\mathbf{L}_{\rho\rho})_{32}b_2 & (\mathbf{L}_{\rho\rho})_{32}b_3\\ (\mathbf{L}_{\rho\rho})_{33}b_1 & (\mathbf{L}_{\rho\rho})_{33}b_2 & (\mathbf{L}_{\rho\rho})_{33}b_3\end{bmatrix}. \tag{108}$$

Analogously one has

$$\mathbb{T}_{F_i}^{111}\left(\boldsymbol{\beta}\otimes\boldsymbol{\Lambda}_{\rho\rho}\right)=\mathbf{T}_{F_i}\boldsymbol{\beta}\otimes\mathbf{L}_{\rho\rho}=\mathbf{b}\otimes\mathbf{L}_{\rho\rho} \tag{109}$$

so that the associated matrix is

$$\left[\mathbf{b}\otimes\mathbf{L}\right]=\left[\left(\mathbf{b}\otimes\mathbf{L}\right)_1,\quad \left(\mathbf{b}\otimes\mathbf{L}\right)_2,\quad \left(\mathbf{b}\otimes\mathbf{L}\right)_3\right]^T \tag{110}$$

where

$$\left[\left(\mathbf{b}\otimes\mathbf{L}\right)_1\right]=\left[b_1\begin{pmatrix}(\mathbf{L}_{\rho\rho})_{11} & (\mathbf{L}_{\rho\rho})_{12} & (\mathbf{L}_{\rho\rho})_{13}\\ (\mathbf{L}_{\rho\rho})_{21} & (\mathbf{L}_{\rho\rho})_{22} & (\mathbf{L}_{\rho\rho})_{23}\\ (\mathbf{L}_{\rho\rho})_{31} & (\mathbf{L}_{\rho\rho})_{32} & (\mathbf{L}_{\rho\rho})_{33}\end{pmatrix}\right], \tag{111}$$

$$\left[\left(\mathbf{b}\otimes\mathbf{L}\right)_2\right]=\left[b_2\begin{pmatrix}(\mathbf{L}_{\rho\rho})_{11} & (\mathbf{L}_{\rho\rho})_{12} & (\mathbf{L}_{\rho\rho})_{13}\\ (\mathbf{L}_{\rho\rho})_{21} & (\mathbf{L}_{\rho\rho})_{22} & (\mathbf{L}_{\rho\rho})_{23}\\ (\mathbf{L}_{\rho\rho})_{31} & (\mathbf{L}_{\rho\rho})_{32} & (\mathbf{L}_{\rho\rho})_{33}\end{pmatrix}\right], \tag{112}$$

$$\left[\left(\mathbf{b}\otimes\mathbf{L}\right)_3\right]=\left[b_3\begin{pmatrix}(\mathbf{L}_{\rho\rho})_{11} & (\mathbf{L}_{\rho\rho})_{12} & (\mathbf{L}_{\rho\rho})_{13}\\ (\mathbf{L}_{\rho\rho})_{21} & (\mathbf{L}_{\rho\rho})_{22} & (\mathbf{L}_{\rho\rho})_{23}\\ (\mathbf{L}_{\rho\rho})_{31} & (\mathbf{L}_{\rho\rho})_{32} & (\mathbf{L}_{\rho\rho})_{33}\end{pmatrix}\right]. \tag{113}$$

A little bit more akward is how to address the tensor product $\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}$. This case has been deliberately left at last since constructing the matrix associated with the rank-three tensor $\mathbb{T}_{F_i}^{111}\left(\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}\right)$ allows us to solve the problem concerning the tensor in (90).

Actually, if we could split the tensor $\boldsymbol{\Lambda}_{\rho\rho}$ as tensor product of two 2D vectors in the form $\boldsymbol{\Lambda}_{\rho\rho}=\boldsymbol{\gamma}\otimes\boldsymbol{\delta}$ we would trivially have

$$\mathbb{T}_{F_i}^{111}\left(\boldsymbol{\Lambda}_{\rho\rho}\otimes\boldsymbol{\beta}\right)=\mathbb{T}_{F_i}^{111}\left(\boldsymbol{\gamma}\otimes\boldsymbol{\delta}\otimes_{23}\boldsymbol{\beta}\right)=\mathbb{T}_{F_i}^{111}\left(\boldsymbol{\gamma}\otimes\boldsymbol{\beta}\otimes\boldsymbol{\delta}\right)=\mathbf{t}\otimes\mathbf{b}\otimes\mathbf{v} \tag{114}$$

and exploit the general scheme in (102) to construct the relevant matrix. Unfortunately we directly dispose of the matrix $\mathbf{L}_{\rho\rho}$ whose entries have to appear as first and third entries in the previous, purely illustrative, scheme.

This does not represent a real problem since, coherently with the matrix representation (102), we can define the matrix associated with

$$\mathbb{T}_{F_i}^{111}\left(\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}\right)=\mathbf{L}_{\mathbf{rbr}} \tag{115}$$

as

$$\left[\mathbf{L}_{\mathbf{rbr}}\right]=\left[\left(\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}\right)_1,\quad \left(\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}\right)_2,\quad \left(\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}\right)_3\right]^T \tag{116}$$

where

$$\left[\left(\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}\right)_1\right]=\begin{bmatrix}b_1(\mathbf{L}_{\rho\rho})_{11} & b_1(\mathbf{L}_{\rho\rho})_{12} & b_1(\mathbf{L}_{\rho\rho})_{13}\\ b_2(\mathbf{L}_{\rho\rho})_{11} & b_2(\mathbf{L}_{\rho\rho})_{12} & b_2(\mathbf{L}_{\rho\rho})_{13}\\ b_3(\mathbf{L}_{\rho\rho})_{11} & b_3(\mathbf{L}_{\rho\rho})_{12} & b_3(\mathbf{L}_{\rho\rho})_{13}\end{bmatrix}, \tag{117}$$

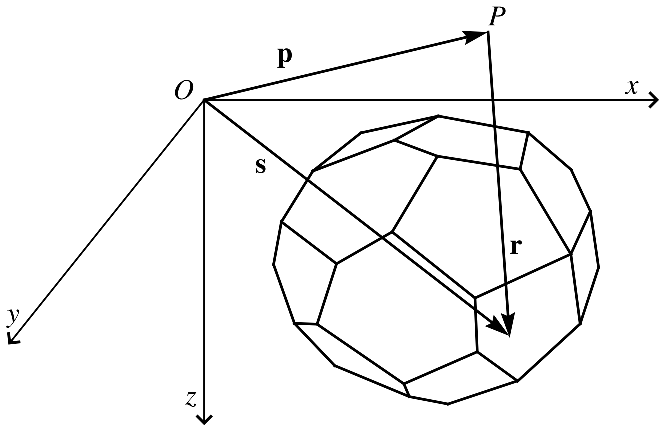


Fig. 2 Representation of geometric quantities used to assign density contrast (**s**) and define the position of Ω with respect to an arbitray point P.

$$\left[\left(\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}\right)_2\right]=\begin{bmatrix} b_1(\mathbf{L}_{\rho\rho})_{21} & b_1(\mathbf{L}_{\rho\rho})_{22} & b_1(\mathbf{L}_{\rho\rho})_{23} \\ b_2(\mathbf{L}_{\rho\rho})_{21} & b_2(\mathbf{L}_{\rho\rho})_{22} & b_2(\mathbf{L}_{\rho\rho})_{23} \\ b_3(\mathbf{L}_{\rho\rho})_{21} & b_3(\mathbf{L}_{\rho\rho})_{22} & b_3(\mathbf{L}_{\rho\rho})_{23} \end{bmatrix}, \tag{118}$$

$$\left[\left(\boldsymbol{\Lambda}_{\rho\rho}\otimes_{23}\boldsymbol{\beta}\right)_3\right]=\begin{bmatrix} b_1(\mathbf{L}_{\rho\rho})_{31} & b_1(\mathbf{L}_{\rho\rho})_{32} & b_1(\mathbf{L}_{\rho\rho})_{33} \\ b_2(\mathbf{L}_{\rho\rho})_{31} & b_2(\mathbf{L}_{\rho\rho})_{32} & b_2(\mathbf{L}_{\rho\rho})_{33} \\ b_3(\mathbf{L}_{\rho\rho})_{31} & b_3(\mathbf{L}_{\rho\rho})_{32} & b_3(\mathbf{L}_{\rho\rho})_{33} \end{bmatrix}, \tag{119}$$

and $\mathbf{L}_{\rho\rho}$ is obtained from (103) and $\mathbf{b} = \mathbf{T}_{F_i}\boldsymbol{\beta}$.

Remarkably, the same notational scheme as in the previous formula can be exploited for the tensor in (90) since $\mathbf{G}_i$ can be obtained from (44) by standard matrix operations.

Furthermore, setting $\mathbb{M} = \mathbf{G}_i \otimes_{23} \mathbf{n}_i$, the matrix $[\mathbb{M}]$ can be obtained analogously to (116). Stated equivalently, to construct the matrix associated with the rank-three tensor $\mathbb{M}$, one has to first evaluate $\boldsymbol{\Phi}_{F_i}$, transform it as in (44) to get $\mathbf{G}_i$, and exploit the notational scheme (116) by replacing $\mathbf{L}_{\rho\rho}$ with $\mathbf{G}_i$.

The notational schemes detailed in (101)-(102), (104)-(105), (109)-(110) and (115)-(116) can be suitably exploited to evaluate the tensors in (91)-(97) and, hence, the tensor $\mathbb{D}^{\partial\Omega}_{\mathbf{rrr}}$ in (50). Namely, the tensors $\mathbf{G}_i \otimes \mathbf{n}_i$ in (91) and $\mathbf{H}_i \otimes \mathbf{n}_i$ in (97) can be evaluated by applying the scheme (105), the tensor $\mathbb{G}_i$ in (92) by applying the scheme (101)-(102) and the tensor $\mathbf{H}_i \otimes_{23} \mathbf{n}_i$ in (96) by applying the scheme (115)-(116). Finally, the tensors in (93) and (95) are rank-two tensors and the tensor in (94) can be evaluated as in (102).

3 Gravity anomaly of polyhedral bodies at an arbitrary point *P*

In the previous sections it has been assumed that the observation point P would coincide with the origin of the reference frame in which the anomalous density of a body is assigned.

This has allowed us to set the stage and to define the most problematic issues to address, both from the analytical and numerical point of view.

However when gravity measures are carried out at several points and/or when multiple bodies are taken into account it is by far more convenient to fix an arbitrary reference frame in which both the coordinates of each observation point and the density of all bodies are simultaneously assigned.

To suitably extend the formulas contributed in the previous section, one can exploit a coordinate transformation (Zhou, 2010) by translating the origin of the reference frame to the observation point and modifying in accordance the expression of the density contrast by expressing the coefficients of the polynomial law in the new reference frame.

Alternatively, one can follow the approach outlined in D'Urso (2015c) and define the position vector $\mathbf{r}$ entering the definition of the gravity anomaly as follows

$$\mathbf{r} = \mathbf{s} - \mathbf{p} \tag{120}$$

where $\mathbf{p}$ is the position vector of the observation point and $\mathbf{s}$ is the position vector of an arbitrary point belonging to Ω, see e.g., fig. 2. In this way we can leave the expression (6) unchanged by writing

$$\Delta\rho(\mathbf{s}) = \theta(x,y,z) = \theta_{\mathbf{o}} + \mathbf{c}\cdot\mathbf{s} + \mathbf{C}\cdot\mathbf{D}_{\mathbf{ss}} + \mathbb{C}\cdot\mathbb{D}_{\mathbf{sss}} \tag{121}$$

where $\mathbf{D}_{\mathbf{ss}}$ and $\mathbb{D}_{\mathbf{sss}}$ are defined as in (7) and write

$$\Delta g_z(P) = G\int_{\Omega} \frac{\Delta\rho(\mathbf{s})\mathbf{r}\cdot\mathbf{k}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}dV\,. \tag{122}$$

Clearly in the case of multiple observation points P_i and/or bodies one can simply write

$$\Delta g_z(P_i) = G\sum_{j=1}^{N_B}\int_{\Omega_j} \frac{\Delta\rho(\mathbf{s}_j)\mathbf{r}_j\cdot\mathbf{k}}{(\mathbf{r}_j\cdot\mathbf{r}_j)^{3/2}}dV \tag{123}$$

where Ω_j is the domain of the j-th body, N_B is the number of bodies to analyze and $\mathbf{r}_j = \mathbf{s}_j - \mathbf{p}_i$, $\mathbf{p}_i$ being the position vector of P_i with respect to the assigned reference frame having origin at an arbitrary point O. However, being mainly interested to illustrate the rationale of our approach, we shall make reference in the sequel to the case of a single observation point and a single body.

To exploit the results illustrated in the previous section, it is convenient to express $\mathbf{s}$ as function of $\mathbf{r}$ by means of (120). For brevity this is detailed only for the rank-three tensor $\mathbb{D}_{\mathbf{sss}}$ since it is the more cumbersome to handle. In particular, we infer from (120)

$$\mathbb{D}_{\mathbf{sss}} = \mathbf{s}\otimes\mathbf{s}\otimes\mathbf{s} = (\mathbf{r}+\mathbf{p})\otimes(\mathbf{r}+\mathbf{p})\otimes(\mathbf{r}+\mathbf{p}) = \mathbb{D}_{\mathbf{rrr}} + \mathbb{D}_{\mathbf{rrp}} + \mathbb{D}_{\mathbf{ppr}} + \mathbb{D}_{\mathbf{ppp}} \tag{124}$$

where $\mathbb{D}_{\mathbf{ppp}} = \mathbf{p}\otimes\mathbf{p}\otimes\mathbf{p}$,

$$\mathbb{D}_{\mathbf{rrp}} = \mathbf{r}\otimes\mathbf{r}\otimes\mathbf{p} + \mathbf{r}\otimes\mathbf{p}\otimes\mathbf{r} + \mathbf{p}\otimes\mathbf{r}\otimes\mathbf{r} \tag{125}$$

and

$$\mathbb{D}_{\mathbf{ppr}} = \mathbf{p}\otimes\mathbf{p}\otimes\mathbf{r} + \mathbf{p}\otimes\mathbf{r}\otimes\mathbf{p} + \mathbf{r}\otimes\mathbf{p}\otimes\mathbf{p} = \mathbf{D}_{\mathbf{pp}}\otimes\mathbf{r} + \mathbf{p}\otimes\mathbf{r}\otimes\mathbf{p} + \mathbf{r}\otimes\mathbf{D}_{\mathbf{pp}}\,. \tag{126}$$

Hence, the expression (122) for the gravity anomaly becomes

$$\begin{aligned}\Delta g_z(\mathbf{p}) = G\Big\{&[\theta_\mathbf{o} + \mathbf{c}\cdot\mathbf{p} + \mathbf{C}\cdot\mathbf{D_{pp}} + \mathbb{C}\cdot\mathbb{D}_\mathbf{ppp}]d_\mathbf{r}^\Omega + \mathbf{c}\cdot\mathbf{d}_\mathbf{r}^\Omega + \\ &+\mathbf{C}\cdot[\mathbf{d}_\mathbf{r}^\Omega\otimes\mathbf{p} + \mathbf{p}\otimes\mathbf{d}_\mathbf{r}^\Omega + \mathbf{D}_\mathbf{rr}^\Omega] + \mathbb{C}\cdot[\mathbf{D_{pp}}\otimes\mathbf{d}_\mathbf{r}^\Omega + \mathbf{p}\otimes\mathbf{d}_\mathbf{r}^\Omega\otimes\mathbf{p} + \mathbf{d}_\mathbf{r}^\Omega\otimes\mathbf{D_{pp}}] + \\ &+\mathbb{C}\cdot[\mathbf{D}_\mathbf{rr}^\Omega\otimes\mathbf{p} + \mathbf{d}_\mathbf{r}^\Omega\otimes\mathbf{p}\otimes\mathbf{d}_\mathbf{r}^\Omega + \mathbf{p}\otimes\mathbf{D}_\mathbf{rr}^\Omega] + \mathbb{C}\cdot\mathbb{D}_\mathbf{rrr}^\Omega\Big\},\end{aligned} \tag{127}$$

which represents the generalization of (14) to the case $\mathbf{p}\neq\mathbf{o}$.

Special attention has to be paid to the symbol $\mathbf{d}_\mathbf{r}^\Omega\otimes\mathbf{p}\otimes\mathbf{d}_\mathbf{r}^\Omega$ which is a shorthand to denote the third-order tensor

$$\mathbf{d}_\mathbf{r}^\Omega\otimes\mathbf{p}\otimes\mathbf{d}_\mathbf{r}^\Omega = \int_\Omega \frac{(\mathbf{r}\cdot\mathbf{k})\mathbf{r}\otimes\mathbf{p}\otimes\mathbf{r}}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}dV = \mathbf{D}_\mathbf{rr}^\Omega\otimes_{23}\mathbf{p}\,. \tag{128}$$

In spite of its symbol, which has been adopted to emphasize its symmetric expression, the tensor above cannot be obtained as triple tensor product of the vectors $\mathbf{d}_\mathbf{r}^\Omega$ and $\mathbf{p}$. Rather, it is conveniently computed starting from the rank-two tensor $\mathbf{D}_\mathbf{rr}^\Omega$, after having computed its algebraic expression, as detailed in subsection 2.6.

Although $\mathbf{r}$ is now defined from (120) it can be shown that formula (17) holds as well. Thus, recalling (30) and setting

$$\theta_\mathbf{p} = \mathbf{c}\cdot\mathbf{p} + \mathbf{C}\cdot\mathbf{D_{pp}} + \mathbb{C}\cdot\mathbb{D}_\mathbf{ppp}\,, \tag{129}$$

formula (127) specializes to

$$\begin{aligned}\Delta g_z(\mathbf{p}) = G\Bigg\{&(\theta_\mathbf{o}+\theta_\mathbf{p})d_\mathbf{r}^{\partial\Omega} + \frac{\mathbf{c}\cdot\mathbf{d}_\mathbf{r}^{\partial\Omega}}{2} + \mathbf{C}\cdot\Big[\frac{\mathbf{d}_\mathbf{r}^{\partial\Omega}}{2}\otimes\mathbf{p} + \mathbf{p}\otimes\frac{\mathbf{d}_\mathbf{r}^{\partial\Omega}}{2} + \frac{\mathbf{D}_\mathbf{rr}^{\partial\Omega}}{3}\Big] + \\ &+\mathbb{C}\cdot\Big[\frac{1}{2}\Big(\mathbf{D_{pp}}\otimes\mathbf{d}_\mathbf{r}^{\partial\Omega} + \mathbf{p}\otimes\mathbf{d}_\mathbf{r}^{\partial\Omega}\otimes\mathbf{p} + \mathbf{d}_\mathbf{r}^{\partial\Omega}\otimes\mathbf{D_{pp}}\Big) + \\ &+\frac{1}{3}\Big(\mathbf{D}_\mathbf{rr}^{\partial\Omega}\otimes\mathbf{p} + \mathbf{d}_\mathbf{r}^{\partial\Omega}\otimes\mathbf{p}\otimes\mathbf{d}_\mathbf{r}^{\partial\Omega} + \mathbf{p}\otimes\mathbf{D}_\mathbf{rr}^{\partial\Omega}\Big) + \frac{\mathbb{D}_\mathbf{rrr}^{\partial\Omega}}{4}\Big]\Bigg\}.\end{aligned} \tag{130}$$

Obviously, (130) coincides with (31) when $\mathbf{p}=\mathbf{o}$.

Formula (130) can be operatively evaluated for a a polyhedral body by considering formulas (37), (38), (47) and (50) for $d_\mathbf{r}^\Omega$, $\mathbf{d}_\mathbf{r}^\Omega$, $\mathbf{D}_\mathbf{rr}^\Omega$ and $\mathbb{D}_\mathbf{rrr}^\Omega$, respectively, and the procedures detailed in subsection 2.3-2.6 to express them in terms of 3D vectors. In particular the third order tensor $\mathbf{d}_\mathbf{r}^{\partial\Omega}\otimes\mathbf{p}\otimes\mathbf{d}_\mathbf{r}^{\partial\Omega}$ is obtained by applying the notational scheme (115)-(116) and replacing $\mathbf{L}_{\rho\rho}$ with $\mathbf{D}_\mathbf{rr}^\Omega$ and $\mathbf{b}$ with $\mathbf{p}$, respectively.

4 Eliminable Singularities of the Algebraic Expressions of the Gravity Anomaly

It has already been shown that the analytical expression (31) of the gravity anomaly is singularity-free in the sense that its expression holds rigorously whatever is the position of the point O with respect to Ω. The same property holds true for the expression (130) referred to an arbitrary point P. However their algebraic counterparts, being expressed by means of the quantities detailed in subsection 2.4, do include further singularities.

They are associated with the expression of the line integrals provided in the Appendices since they become singular when the generic face F_i contains the observation point, either O or P, and this belongs to the line containing the j-th edge of the boundary ∂F_i.

However, we are going to prove analytically that the contribution of the singular line integral to the domain integral in which its computation is required is zero. Hence, from the computational point of view, the singularity of the j-th line integral does not have any practical effect and it can be simply ignored when computing the associated domain integral.

As shown in Appendix 2, some of the 2D domain integrals required in the present context, have already been computed in previous papers D'Urso (2013a, 2014a,b) so that the discussion on their singularity-free nature can be found in the quoted reference. Nevertheless we shall systematically prove this property also for these last integrals, namely the ones having $\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{1/2}$ in the denominator, since we are going to use new and simpler arguments; the same arguments will be exploited to prove the singularity-free nature of the integrals having $\left(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2\right)^{3/2}$ in the denominator.

4.1 Eliminable singularity of the integral ψ_{F_i}

We know from formulas (218) and (219) that

$$\begin{aligned}\psi_{F_i} &= \int_{F_i} \frac{dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} = \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) \int_0^1 \frac{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j) \cdot \hat{\boldsymbol{\rho}}_i(\lambda_j) + d_i^2\right]^{1/2}}{\hat{\boldsymbol{\rho}}_i(\lambda_j) \cdot \hat{\boldsymbol{\rho}}_i(\lambda_j)} d\lambda_j - \alpha_i |d_i| = \\ &= \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) \int_0^1 \frac{\left(p_j \lambda_j^2 + 2 q_j \lambda_j + v_j\right)^{1/2}}{p_j \lambda_j^2 + 2 q_j \lambda_j + u_j} d\lambda_j - \alpha_i |d_i| = \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} - \alpha_i |d_i|\end{aligned} \tag{131}$$

where, see also (70), we have set

$$p_j = \Delta\boldsymbol{\rho}_j \cdot \Delta\boldsymbol{\rho}_j = l_j^2 \quad q_j = \boldsymbol{\rho}_j \cdot \Delta\boldsymbol{\rho}_j \quad u_j = \boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j \quad v_j = u_j + d_i^2 = |\mathbf{r}_j|^2 \,. \tag{132}$$

Useful in the sequel are also the quantities (D'Urso, 2013a, 2014a,b)

$$p_j + q_j = \boldsymbol{\rho}_{j+1} \cdot \Delta\boldsymbol{\rho}_j \quad p_j + 2q_j + v_j = \boldsymbol{\rho}_{j+1} \cdot \boldsymbol{\rho}_{j+1} + d_i^2 = |\mathbf{r}_{j+1}|^2 \tag{133}$$

and the discriminant $\Delta_j = q_j^2 - p_j u_j$ of the denominator in (131). In particular, it turns out to be

$$-\Delta_j = p_j u_j - q_j^2 = \left(\boldsymbol{\rho}_{j+1} \cdot \boldsymbol{\rho}_{j+1}\right) \cdot \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j\right) - \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}\right)^2 \geq 0 \tag{134}$$

by virtue of the Cauchy-Schwartz inequality (Tang, 2006).

Clearly, our main concern is when $\Delta_j = 0$. In particular, setting $\boldsymbol{o} = (0,0)$, it is apparent from the previous expression that the denominator of the j-th integral on the right-hand side of (131) can become singular if $\boldsymbol{\rho}_j = \boldsymbol{o}$, $\boldsymbol{\rho}_{j+1} = \boldsymbol{o}$ or $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$ are parallel and point in opposite directions, i.e. if the projection of the observation point onto F_i belongs to the segment $[\boldsymbol{\rho}_j, \boldsymbol{\rho}_{j+1}]$. In turn this may happen independently from the value of d_i, i.e. whether or not the i-th face of the polyhedron Ω does contain the observation point.

In both cases, $d_i \neq 0$ or $d_i = 0$, we are going to prove by mathematical arguments that the contribution of such an edge to ψ_{F_i} is zero so that its computation can be skipped. Let us first consider the case $d_i \neq 0$.

As shown in D'Urso (2013a, 2014a) the evaluation of the line integral on the right-hand side of (131) is carried out by setting $t = \lambda_j + q_j/p_j$; this yields

$$I_{6j} = \int_0^1 \frac{\left(p_j\lambda_j^2 + 2q_j\lambda_j + v_j\right)^{1/2}}{p_j\lambda_j^2 + 2q_j\lambda_j + u_j} d\lambda_j = \frac{1}{\sqrt{p_j}} \int_{q_j/p_j}^{1+q_j/p_j} \frac{\sqrt{t^2 + B_j}}{t^2 + A_j} dt \tag{135}$$

where

$$A_j = -\frac{\Delta_j}{p_j^2} = \frac{p_j u_j - q_j^2}{p_j^2} \qquad B_j = \frac{p_j v_j - q_j^2}{p_j^2} = A_j + \frac{d_i^2}{p_j} = A_j + \frac{d_i^2}{l_j^2}\,. \tag{136}$$

Notice that the denominator in (135) is positive if $-\Delta_j = p_j^2 A_j > 0$. In this case the primitive of the integrand on the right-hand side of (135) becomes

$$I_{6j} = \frac{1}{\sqrt{p_j}} \left\{ \sqrt{\frac{B_j - A_j}{A_j}} \arctan \frac{\sqrt{B_j - A_j}}{\sqrt{A_j}\,\sqrt{B_j + t^2}} + \ln\left(t + \sqrt{B_j + t^2}\right) \right\}_{q_j/p_j}^{1+q_j/p_j} \tag{137}$$

or equivalently

$$I_{6j} = \left\{ \frac{|d_i|}{\sqrt{-\Delta_j}} \arctan \frac{|d_i|}{\sqrt{-\Delta_j}\,\sqrt{B_j + t^2}} + \frac{\ln\left(t + \sqrt{B_j + t^2}\right)}{\sqrt{p_j}} \right\}_{q_j/p_j}^{1+q_j/p_j}. \tag{138}$$

Conversely, should it be $\Delta_j = 0$, and hence $A_j = 0$, the integrand on the right-hand side of (135) becomes singular at one point belonging to the interval $\left[q_j/p_j,\ 1 + q_j/p_j\right]$. Actually, we infer from (134) and the properties of the Cauchy-Schwartz inequality that $\Delta_j = 0$ if and only if $\boldsymbol{\rho}_j = \boldsymbol{o}$, $\boldsymbol{\rho}_{j+1} = \boldsymbol{o}$ or the segment $[\boldsymbol{\rho}_j,\ \boldsymbol{\rho}_{j+1}]$ contains the null vector in its interior.

Actually if $\boldsymbol{\rho}_j = \boldsymbol{o}$ $\left(\boldsymbol{\rho}_{j+1} = \boldsymbol{o}\right)$, it turns out to be $q_j/p_j = 0$ $\left(1 + q_j/p_j = 0\right)$; hence the denominator in (135) becomes singular since $t^2 + A_j = \boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j / p_j$ $\left(\boldsymbol{\rho}_{j+1} \cdot \boldsymbol{\rho}_{j+1}/p_j\right) = 0$ at the left (right) extreme of the integration integral.

Furthermore, should the projection of the observation point fall within the segment $[\boldsymbol{\rho}_j,\ \boldsymbol{\rho}_{j+1}]$, one has $\boldsymbol{\rho}_{j+1} = \beta_j \boldsymbol{\rho}_j$ $(\beta_j < 0)$ where $q_j/p_j = (\beta_j - 1)\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j / p_j < 0$ and $1 + q_j/p_j = \beta_j(\beta_j - 1)\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_j / p_j > 0$. Accordingly, the integration interval in (135) splits in two intervals having 0 as right (left) extreme. At that point, however, $t = 0$ and $A_j = -\Delta_j / p_j^2 = 0$ by assumption so that the integrand in (135) becomes singular.

However, we are going to prove that, in the previous three cases, the singularity is eliminable and that the integral attains a finite value. Let us discuss separately the three cases, namely $\boldsymbol{\rho}_j = \boldsymbol{o}$, $\boldsymbol{\rho}_{j+1} = \boldsymbol{o}$ and $\boldsymbol{\rho}_{j+1} = \beta_j \boldsymbol{\rho}_j$ $(\beta_j < 0)$.

In this first case, $\boldsymbol{\rho}_j = \boldsymbol{o}$, the integration interval is $[0,\ 1]$ and we have singularity of the integrand in (135) at the left extreme while the argument of the logarithm is positive. Thus, recalling (131) and (138), the contribution of the integral I_{6j} to ψ_{F_i} is provided by

$$\left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = \boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp} \left[\frac{|d_i|}{\sqrt{-\Delta_j}} \arctan \frac{|d_i|}{\sqrt{-\Delta_j}\,\sqrt{B_j + t^2}} + \frac{\ln\left(t + \sqrt{B_j + t^2}\right)}{\sqrt{p_j}} \right]_0^1. \tag{139}$$

Setting $\boldsymbol{\rho}_j = |\boldsymbol{\rho}_j|\mathbf{e} = \varepsilon\mathbf{e}$ and observing that, on account of (134),

$$-\Delta_j = \left(\boldsymbol{\rho}_{j+1}\cdot\boldsymbol{\rho}_{j+1}\right)|\boldsymbol{\rho}_j|^2 - \left(|\boldsymbol{\rho}_j|\mathbf{e}\cdot\boldsymbol{\rho}_{j+1}\right)^2 = \varepsilon^2\left[\boldsymbol{\rho}_{j+1}\cdot\boldsymbol{\rho}_{j+1} - \left(\mathbf{e}\cdot\boldsymbol{\rho}_{j+1}\right)^2\right], \tag{140}$$

we infer that $\sqrt{-\Delta_j}$ is infinitesimal of the same order as $\varepsilon = |\boldsymbol{\rho}_j|$ when $\varepsilon \to 0$, a property we state by writing $\sqrt{-\Delta_j} = O(\varepsilon)$. Hence (139) becomes

$$\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = \lim_{\varepsilon\to 0}\varepsilon\left\{\left[\frac{|d_i|}{\sqrt{-\Delta_j(\varepsilon)}}\arctan\frac{|d_i|}{\sqrt{-\Delta_j(\varepsilon)}\sqrt{B_j+t^2}}\right]_{\varepsilon}^{1} + \frac{1}{\sqrt{p_j}}\left[\ln\left(t+\sqrt{B_j+t^2}\right)\right]_0^1\right\} \tag{141}$$

since the $\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp} = O(\varepsilon)$ if $\varepsilon \to 0$.

Since the arctan function is finite at $t = 1$ and the same does occur for the ln function at $t = 0$ and $t = 1$, we finally have

$$\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = -|d_i|\lim_{\varepsilon\to 0}\frac{\varepsilon}{\sqrt{-\Delta_j(\varepsilon)}}\arctan\frac{|d_i|}{\sqrt{-\Delta_j(\varepsilon)}\sqrt{B_j+\varepsilon^2}} = -\frac{\pi}{2}|d_i|. \tag{142}$$

However if $\boldsymbol{\rho}_j = \boldsymbol{o}$ for the j-th edge, it will turn out to be $\boldsymbol{\rho}_{j+1} = \boldsymbol{o}$ for the $(j-1)$-th edge. Hence the arctan function in (138) will be evaluated in the interval $[-1,\,\varepsilon]$, with $\varepsilon \to 0$, and one has $\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = \pi|d_i|/2$.

To conclude the total contribution provided to φ_{F_i} by the two edges for which it simultaneously happen that $\boldsymbol{\rho}_j = \boldsymbol{o}$ for the j-th edge and $\boldsymbol{\rho}_{j+1} = \boldsymbol{o}$ for the $(j-1)$-th edge is zero.

A null contribution to φ_{F_i} is also provided by edges for which the projection of the observation point is internal to the edge. In this case $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$ are parallel so that the product $\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}$ is zero. Accordingly, both $\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}$ and $\sqrt{-\Delta_j}$ are $O(\varepsilon)$, that is both of them are infinitesimal of order ε as $\varepsilon \to 0$. In conclusion (139) yields

$$\begin{aligned}\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = |d_i|\lim\nolimits_{\varepsilon\to 0}&\left\{\frac{\varepsilon}{\sqrt{-\Delta_j(\varepsilon)}}\left[\arctan\frac{|d_i|}{\sqrt{-\Delta_j(\varepsilon)}\sqrt{B_j+t^2}}\right]_{-1}^{0} + \right.\\ &\left. + \frac{\varepsilon}{\sqrt{-\Delta_j(\varepsilon)}}\left[\arctan\frac{|d_i|}{\sqrt{-\Delta_j(\varepsilon)}\sqrt{B_j+t^2}}\right]_0^1 + \frac{\varepsilon}{\sqrt{p_j}}\left[\ln\left(t+\sqrt{B_j+t^2}\right)\right]_0^1\right\} = 0.\end{aligned} \tag{143}$$

Actually, the ln function is finite both at $t = 0$ and $t = 1$. Furthermore, by repeating the arguments exploited in (142), the arctan function attains finite and opposite values both at $t = 0$ and $t \pm 1$.

In conclusion we have proved that, when $d_i \neq 0$ and the projection of the observation point does belong to the closed interval having $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$ as extremes, the contribution of the relevant edge can be skipped since the overall contribution to φ_{F_i} associated with such a singular case is lumped within the addend $\alpha_i|d_i|$.

Let us now prove that the same result is obtained if $|d_i| = 0$, i.e. if the face F_i does contain the observation point. In this case the integral in (131) can be expressed as follows

$$\psi_{F_i} = \sum_{j=1}^{N_{E_i}}\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = \sum_{j=1}^{N_{E_i}}\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right)\int_0^1\frac{d\lambda_j}{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j)\right]^{1/2}} - \alpha_i|d_i|. \tag{144}$$

Also in this case, the j-th edge characterized by $\boldsymbol{\rho}_j = \boldsymbol{o}$ or $\boldsymbol{\rho}_{j+1} = \boldsymbol{o}$ or $\boldsymbol{\rho}_{j+1} = \beta_j \boldsymbol{\rho}_j$ $(\beta_j < 0)$ does not give any contribution to φ_{F_i}. Let us examine separately the three cases

- $\boldsymbol{\rho}_j = \boldsymbol{o}$

In this case the parameterization (67) yields $\hat{\boldsymbol{\rho}}_i(\lambda_j) = \lambda_j \boldsymbol{\rho}_{j+1}$ so that the j-th integral in (144) becomes

$$I_{6j} = \int_0^1 \frac{d\lambda_j}{\lambda_j \left(\boldsymbol{\rho}_{j+1} \cdot \boldsymbol{\rho}_{j+1}\right)^{1/2}} = \frac{1}{\sqrt{p_j}} \int_0^1 \frac{d\lambda_j}{\lambda_j} . \tag{145}$$

Setting $\varepsilon = |\boldsymbol{\rho}_j|$ and being $\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}$ infinitesimal of order ε, it turns out to be

$$\left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = \frac{1}{\sqrt{p_j}} \lim_{\varepsilon \to 0} \varepsilon \left[\ln \lambda_j\right]_{\varepsilon}^{1} = 0 \tag{146}$$

since the logarithm tends to infinite with an arbitrarily low degree.

- $\boldsymbol{\rho}_{j+1} = \boldsymbol{o}$

Setting $\hat{\boldsymbol{\rho}}_i(\lambda_j) = (1 - \lambda_j)\boldsymbol{\rho}_j$ the integral in (144) can be written

$$I_{6j} = \frac{1}{\sqrt{u_j}} \int_0^1 \frac{d\lambda_j}{1 - \lambda_j} = -\frac{1}{\sqrt{u_j}} \int_1^0 \frac{d\eta_j}{\eta_j} \tag{147}$$

where $\eta_j = 1 - \lambda_j$. Hence, setting $\varepsilon = |\boldsymbol{\rho}_{j+1}|$, one has

$$\left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = -\frac{1}{\sqrt{u_j}} \lim_{\varepsilon \to 0} \varepsilon \left[\ln \eta_j\right]_{1}^{\varepsilon} = 0 \tag{148}$$

due to the behavior of the logarithm at infinity.

- $\boldsymbol{\rho}_{j+1}$ parallel to $\boldsymbol{\rho}_j$

We are considering the case in which the observation point is projected onto the face F_i inside the j-th edge $\left[\boldsymbol{\rho}_j, \boldsymbol{\rho}_{j+1}\right]$. Hence we can set $\boldsymbol{\rho}_{j+1} = \beta_j \boldsymbol{\rho}_j, \beta_j < 0$, since $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$ point in opposite directions. Setting

$$\boldsymbol{\rho}_j(\lambda_j) = \left[1 + \lambda_j(\beta_j - 1)\right] \boldsymbol{\rho}_j = \tau_j \boldsymbol{\rho}_j , \tag{149}$$

the integral in (144) becomes

$$\begin{aligned} I_{6j} &= \frac{1}{\sqrt{u_j}} \int_0^1 \frac{d\lambda_j}{|1 + \lambda_j(\beta_j - 1)|} = \frac{1}{(\beta_j - 1)\sqrt{u_j}} \int_1^{\beta_j} \frac{d\tau_j}{|\tau_j|} = \frac{1}{(1 - \beta_j)\sqrt{u_j}} \int_{\beta_j}^1 \frac{d\tau_j}{|\tau_j|} = \\ &= \frac{1}{(1 - \beta_j)\sqrt{u_j}} \left[\int_{\beta_j}^0 \frac{d\tau_j}{|\tau_j|} + \int_0^1 \frac{d\tau_j}{|\tau_j|} \right] = \\ &= \frac{1}{(1 - \beta_j)\sqrt{u_j}} \left\{ \left[\ln \tau_j\right]_0^{|\beta_j|} + \left[\ln \tau_j\right]_0^1 \right\}. \end{aligned} \tag{150}$$

Being $\boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{j+1}$ parallel, $\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp} = 0$. Hence, setting $\varepsilon = |\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}|$

$$\left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) I_{6j} = \frac{1}{(1 - \beta_j)\sqrt{u_j}} \lim_{\varepsilon \to 0} \varepsilon \left[\ln |\beta_j| - 2 \ln \varepsilon\right] = 0 \tag{151}$$

similarly to (146).

4.2 Eliminable singularity of the integral $\boldsymbol{\psi}_{F_i}$

The expression (220) of the integral

$$\begin{aligned}\boldsymbol{\psi}_{F_i} &= \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} = \sum_{j=1}^{N_{E_i}} I_{4j} \Delta\boldsymbol{\rho}_j^{\perp} = \\ &= \sum_{j=1}^{N_{E_i}} \frac{1}{2\sqrt{p_j}} \left\{ \frac{p_j v_j - q_j^2}{p_j} LN_j + \frac{1}{\sqrt{p_j}} \left[\left(p_j + q_j\right) \sqrt{p_j + 2q_j + v_j} - q_j \sqrt{v_j} \right] \right\} \Delta\boldsymbol{\rho}_j^{\perp}\end{aligned} \tag{152}$$

is composed of two addends. The second one is well-defined, according to (132) and (133), whatever is the value of d_i and the position of j-th edge with respect to the observation point.

The first addend in (152) is well defined for $d_i \neq 0$ since

$$LN_j = \ln k_j = \ln \frac{\boldsymbol{\rho}_{j+1} \cdot \left(\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j\right) + l_j |\mathbf{r}_{j+1}|}{\boldsymbol{\rho}_j \cdot \left(\boldsymbol{\rho}_{j+1} - \boldsymbol{\rho}_j\right) + l_j |\mathbf{r}_j|} \tag{153}$$

on the basis of formula (73) in D'Urso (2014b).

Conversely, should it be $d_i = 0$ and $\boldsymbol{\rho}_i = \boldsymbol{o}$ or $\boldsymbol{\rho}_j = \boldsymbol{o}$ or $\boldsymbol{\rho}_{j+1} = \beta_j \boldsymbol{\rho}_j$ ($\beta_j < 0$), one has

$$\frac{p_j v_j - q_j^2}{p_j} LN_j = \frac{-\Delta_j}{p_j} LN_j = \lim_{\varepsilon \to 0} \frac{-\Delta_j(\varepsilon^2) LN_j(\varepsilon)}{2p_j} = 0 \tag{154}$$

since $-\Delta_j$ tends to zero quadratically and LN_j tends to infinite with an arbitrary low degree.

In conclusion edges characterized by singularities of the relevant integral I_{4j} give no contribution to $\boldsymbol{\psi}_{F_i}$.

4.3 Eliminable singularity of the integral $\boldsymbol{\Psi}_{F_i}$

The expression (208) of the integral

$$\boldsymbol{\Psi}_{F_i} = \sum_{j=1}^{N_{E_i}} \left[\left(I_{4j} \boldsymbol{\rho}_j + I_{5j} \Delta\boldsymbol{\rho}_j \right) \otimes \Delta\boldsymbol{\rho}_j^{\perp} - \frac{\mathbf{I}_{2D}}{3} (\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}) I_{4j} \right] + \frac{d_i^2}{3} \left(\psi_i - |d_i| \alpha_i \right) \tag{155}$$

depends upon the integrals ψ_i, I_{4j} and I_{5j}. The discussion on the well-posedness on ψ_i has already been detailed in subsection 4.1.

Conversely, the integrals I_{4j} and I_{5j} are composed, according to their expressions (215) and (216), of the quantities

$$\sqrt{v_j} \qquad \sqrt{p_j + 2q_j + v_j} \tag{156}$$

and of the additional integral I_{0j}. On the basis of the definition (132) and (134) the radicals in (156) are well-defined whater is value of d_i and the position of the j-th edge with respect to the observation point.

The dependence of the integrals I_{4j} and I_{5j} upon I_{0j} does not give any problem since its expression, according to (211), depends upon LN_j. Differently form (152) the quantity LN_j is not scaled by $p_j v_j - q_j^2$, so that we can not invoke the result (154). However the integral

$\boldsymbol{\Psi}_{F_i}$, and hence LN_j, is required for computing the integrals $\mathfrak{C}_{F_i}$ and $\mathfrak{D}_{F_i}$ in (42) that, in turn, are scaled by d_i in the expressions (47) and (50).

Hence, when d_i is zero, what makes LN_j undefined, we can invoke a result similar to (154) by writing

$$d_i LN_j = \lim_{\varepsilon \to 0} d_i(\varepsilon) LN_j(\varepsilon) = 0\,. \tag{157}$$

Stated equivalently, when $d_i = 0$ the contribution to the integral $\boldsymbol{\Psi}_{F_i}$ provided by the face F_i can be skipped.

4.4 Eliminable singularity of the integral φ_{F_i}

The expression provided in (221) for the integral

$$\varphi_{F_i} = \int_{F_i} \frac{dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} = \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}} \left[\frac{\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}}{|d_i| \sqrt{p_j u_j - q_j^2}} (ATN1_j - AT2N_j) \right] \tag{158}$$

is well-defined whatever is the value of d_i and the position of the j-th edge with respect to the observation point.

Also the case $d_i = 0$ does not represent a problem since φ_{F_i} is premultiplied by d_i in the formulas (37), (38) (47) and (50) for $d_{\mathbf{r}}^{\Omega}$, $\mathbf{d}_{\mathbf{r}}^{\Omega}$, $\mathbf{D}_{\mathbf{rr}}^{\Omega}$ and $\mathbb{D}_{\mathbf{rrr}}^{\Omega}$ respectively. Furhermore the discussion on the well-posedness of the quantity

$$\frac{\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}}{\sqrt{p_j v_j - q_j^2}} \Big(ATN1_j - ATN2_j \Big) \tag{159}$$

when $d_i = 0$ and the projection of the observation point lies within the segment $\left[\boldsymbol{\rho}_j, \boldsymbol{\rho}_{j+1}\right]$ is completely similar to that reported in subsection 4.1

4.5 Eliminable singularity of the integral $\boldsymbol{\varphi}_{F_i}$

We know from formula (222) that

$$\boldsymbol{\varphi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} = -\sum_{j=1}^{N_{E_i}} I_{0j}\, \Delta\boldsymbol{\rho}_j^{\perp} \tag{160}$$

where I_{0j} is provided by (211). Hence, the discussion on its well-posedness can be carried out similarly to (157) when $d_i = 0$ and the j-th edge does contain the observation point in its interior.

Actually the integral $\boldsymbol{\varphi}_{F_i}$ in the expression (37), (38) (47) and (50) for $d_{\mathbf{r}}^{\Omega}$, $\mathbf{d}_{\mathbf{r}}^{\Omega}$, $\mathbf{D}_{\mathbf{rr}}^{\Omega}$ and $\mathbb{D}_{\mathbf{rrr}}^{\Omega}$ is always scaled by d_i.

4.6 Eliminable singularity of the integral $\mathbf{\Phi}_{F_i}$

Recalling the expression (223)

$$\mathbf{\Phi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} = -\sum_{j=1}^{N_{E_i}} \left[LN_j \boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} + I_{1j} \Delta\boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} \right] + \psi_{F_i} \mathbf{I}_{2D} , \tag{161}$$

we infer that $\mathbf{\Phi}_{F_i}$ is well defined whatever is the value of d_i and the position of the observation point with respect to the j-th edge of the face F_i. This is trivial if $d_i \neq 0$ since LN_j, I_{1j} and ψ_{F_i} in the previous expression are well defined.

To discuss the well-posedness of $\mathbf{\Phi}_{F_i}$ in the case $d_i = 0$ and when the projection of the observation point onto F_i does belong to the segment $\left[\boldsymbol{\rho}_j, \boldsymbol{\rho}_{j+1}\right]$ we remind that $\mathbf{\Phi}_{F_i}$, as well as φ_{F_i} and $\boldsymbol{\varphi}_{F_i}$, is scaled by d_i in the expressions (47) and (50) for $\mathbf{D}^{\Omega}_{\mathbf{rr}}$ and $\mathbb{D}^{\Omega}_{\mathbf{rrr}}$. Hence the well-posedness of $d_i LN_j$ can be assessed as in (157), while that of ψ_{F_i} has been already proved in subsection 4.1.

Finally, according to formula (212), the well-posedness of I_{1j} depends upon that of I_{0j}; in turn this last one depends upon the product $d_i LN_j$ discussed above.

In conclusion we have proved that the gravity anomaly at an arbitrary point P can be computed effectively whatever is its position with respect to the polyhedron Ω. Actually the potential singularity of the integrals involved in the formulas (37), (38), (47) and (50) for $d^{\Omega}_{\mathbf{r}}$, $\mathbf{d}^{\Omega}_{\mathbf{r}}$, $\mathbf{D}^{\Omega}_{\mathbf{rr}}$ and $\mathbb{D}^{\Omega}_{\mathbf{rrr}}$ gives no contribution to the gravity anomaly.

5 Numerical examples

The formulas developed in the previous sections have been coded in a Matlab program in order to check their correctness and robustness. They have been applied to model tests and case studies derived from the specialized literature by assuming the density contrast to vary separately along the horizontal and the vertical directions or along both of them. In all examples the density contrast is expressed in units kilograms per cubic meter while distances are expressed in kilometers; the value of the gravitational constant G is $6,67259\, 10^{-11} m^3 kg^{-1} s^{-2}$.

Results obtained by the proposed approach have been carefully checked by comparing them whith those resulting from a numerical integration of the integrals involved in the computation of the gravity anomaly. They can be useful to allow for a comparison with computations carried out by using different methods or with more complex modellings, e.g. those reqired to evaluate the gravitational effects of an arbitrary volumetric mass layer in which a laterally varying radial density change has been assumed (Kingdon et al., 2009; Tenzer et al., 2012). To give an idea of the computational burden required in both approaches we have included the computing time (CT) obtained by running the Matlab code on a INTEL CORE2 PC with 16Gb of RAM and a i7-4700HQ CPU having clock speed of 2,40 GHz.

The first test has been taken from (García-Abdeslem, 2005) and refers to a prism extending along x and y between 10 and 20 km and delimited by the planes z=0 and z=8 km. Density contrast is expressed by the function

$$\Delta\rho(z) = -747.7 + 203.435z - 26.764z^2 + 1.4247z^3 = p + qz + rz^2 + sz^3 \tag{162}$$

where the density is expressed in kg/m^3 and z in kilometers.

In order to compare our results with those reported in (García-Abdeslem, 2005), the gravity anomaly has been computed at points P having y=15 km, z=-0.15 m and x ranging from 0 to 30 km . In particular the observer location was taken by García-Abdeslem (2005) -15 cm of the top of the prism to avoid a singularity in the analytic solution occurring when the observation and the source coordinates coincide.

Although our approach is singularity-free, as proved in section 4, we have deliberately repeated the computations made by García-Abdeslem (2005) to draw the reader's attention on the uncorrect values reported in fig. 3 of the quoted paper.

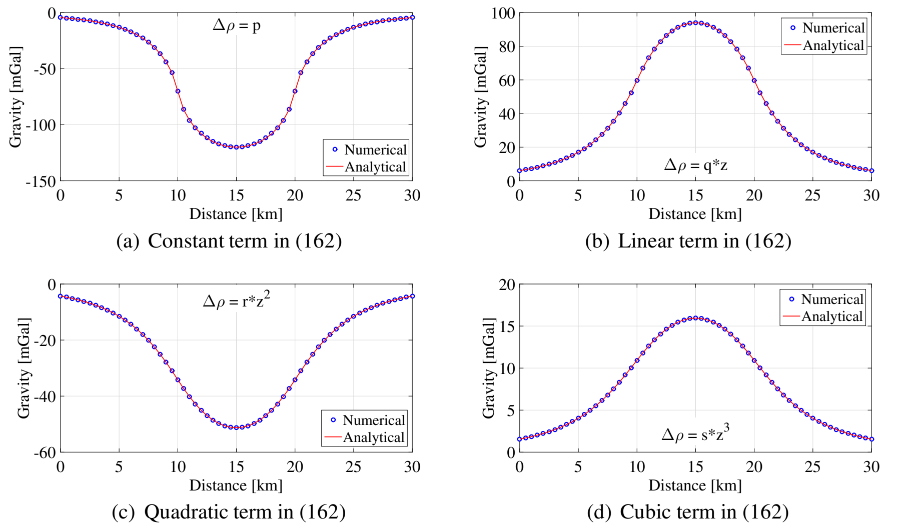
As a matter of fact all mathematical formulas in (García-Abdeslem, 2005) are correct but, for some reasons, the values of the gravity anomaly plotted in fig. 3 have been calculated by assuming wrong integration limits in formula (8) of his paper, namely x_1, y_1, z_1, x_2, y_2, z_2 (lowercase letters) instead of the correct X_1, Y_1, Z_1, X_2, Y_2, Z_2 (capital letters).

In other words formula (8) in (García-Abdeslem, 2005), reported herewith for completeness

$$I_k = \int_{X_1}^{X_2} dX \int_{Y_1}^{Y_2} dY \int_{Z_1}^{Z_2} dZ \left\{ \rho_k \frac{Z^k}{R^3} \right\} \qquad k = 1,2,3,4 \tag{163}$$

is correct but the result plotted in fig. 3 of the quoted paper have been obtained by considering x_1 instead X_1, y_1 instead Y_1 ... and so on. Please notice that, apart ρ_k, the notation in (163) is taken from the original paper so that the observation point is defined by the coordinates $P=(x_0, y_0, z_0)$ and (x,y,z) denote the source coordinates. According to García-Abdeslem (2005) the prism is bounded by the planes $x=x_1$, $y=y_1$, $z=z_1$, $x=x_2$, $y=y_2$, $z=z_2$ and it has been set $X=x-x_0$, $Y=y-y_0$, $Z=z-z_0$.

In conclusion, the correct values of the gravity anomaly at $x_0 \in [0, 30]$ km, $y_0 = 15$ km and $z_0 = -15$ cm, where we have used the notation of (García-Abdeslem, 2005), are reported in figs. 3a, 3b, 3c and 3d respectively for the separate cases of $\Delta\rho = p = \rho_1$, $\Delta\rho = qz = \rho_2$, $\Delta\rho = rz^2 = \rho_3$, $\Delta\rho = sz^3 = \rho_4$,



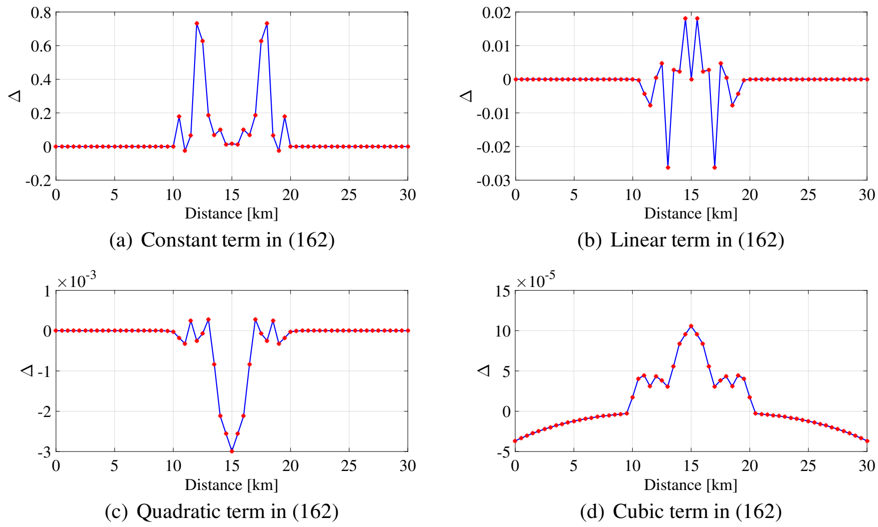
(a) Constant term in (162)

(b) Linear term in (162)

(c) Quadratic term in (162)

(d) Cubic term in (162)

Fig. 3 Gravitational attraction at P=[0,30]×15×(-0.00015) associated with the prism $\Omega \equiv [10, 20]\times[10, 20]\times[0, 8]$ (dimensions in kilometers) and density contrast given by (162).



(a) Constant term in (162)
(b) Linear term in (162)
(c) Quadratic term in (162)
(d) Cubic term in (162)

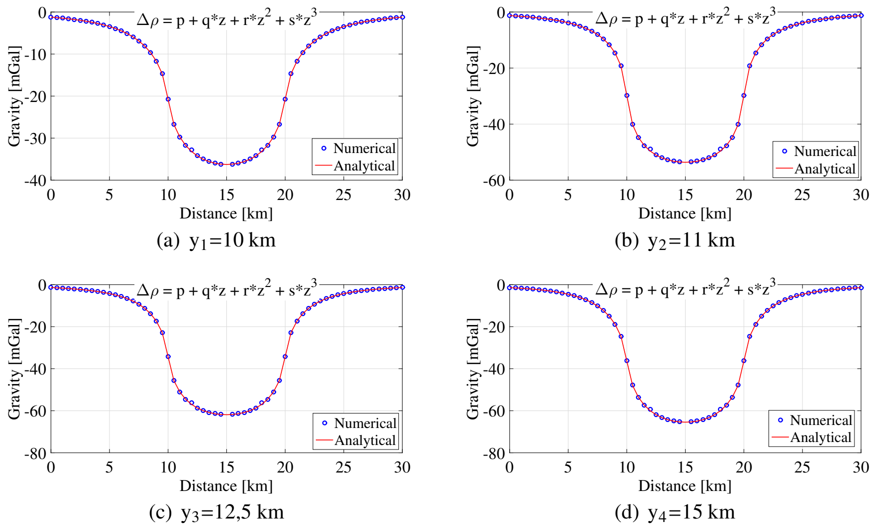
Fig. 4 Differences Δ between the analytical and numerical values plotted in fig. 3

The correctness of the values reported in fig. 3 has been checked by numerically integrating formula (162) with the aid of the adaptive quadrature procedure implemented in Matlab and by setting X_1=10-x_0, Y_1=10-y_0, Z_1=0.00015, X_2=20-x_0, Y_2=20-y_0, Z_2=8-0.00015. For completeness the differences between the analytical and numerical values reported in fig. 3 are plotted in fig. 4.

To fully test the correctness of the proposed formulation and the robustness of the relevant implementation, we have systematically carried out a comparison of the results associated with the analytical and the numerical evaluation of the integrals involved in the computation of the gravity anomaly. To emphasize the singularity-free nature of our solution, this has been done by considering the example in (García-Abdeslem, 2005) and evaluating the anomaly at z=0 and for several values of y, namely y=10, y=11 km, y=12.5 km and y=15 km.

The gravity anomaly has been evaluated for values of x ranging in the interval [0, 30] km and the relevant values are plotted in fig. 5. For completeness the analytical results are reported in table 1 together with those obtained by numerically evaluating the integrals in formula (163); for the reader's convenience the differences between the analytical and numerical values are plotted in fig. 6. The symbol NaN in table 1 for x=15 km, is due to the fact that the numerical procedure, adopted by Matlab to numerically evaluate the integrals in (163), failed to converge. Notice as well that the numerical procedure, besides being computationally more expensive, gives less precise results when the observation point belongs to Ω, i.e. y=10 km and y=15 km, and x moves towards the center of Ω; actually the numerical solution has only three significant digits at x=10 km and x=20 km.

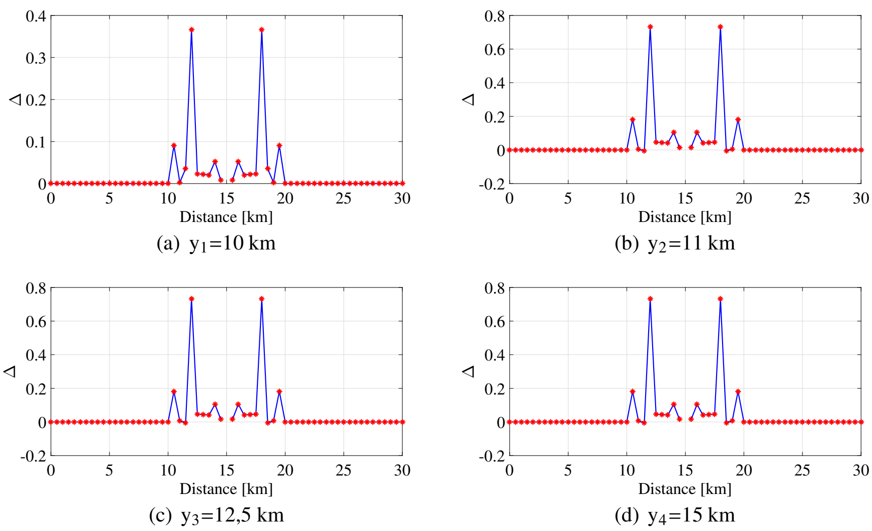
To give a quick overlook of the symmetric nature of the solution with respect to the planes x=15 km and y=15 km we have reported in fig. 7a the contour plot of the gravity anomaly at z=0. The surface distribution of the gravity anomaly becomes unsymmetric, as shown in fig. 7b, by considering a density contrast depending upon an a horizontal direction



(a) y_1=10 km (b) y_2=11 km

(c) y_3=12,5 km (d) y_4=15 km

Fig. 5 Gravitational attraction at P=[0,30]×y_k×[0] (k=1,2,3,4) associated with the prism $\Omega \equiv [10, 20] \times [10, 20] \times [0, 8]$ (dimensions in kilometers) and density contrast given by (162).



(a) y_1=10 km (b) y_2=11 km

(c) y_3=12,5 km (d) y_4=15 km

Fig. 6 Differences Δ between the analytical and numerical values plotted in fig. 5.

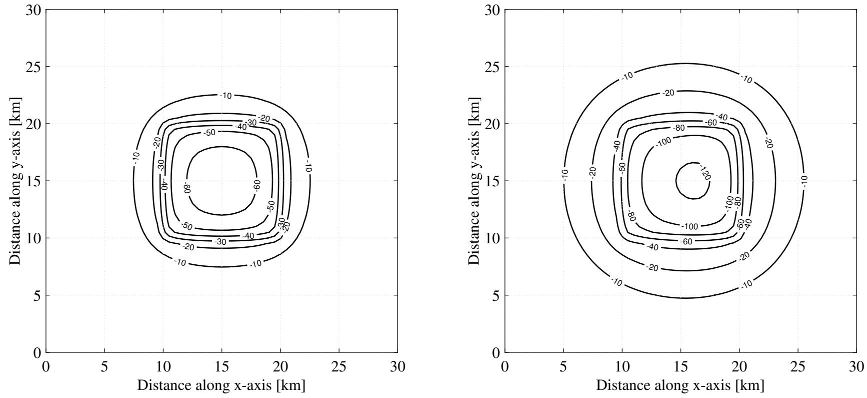
such as the expression considered in Zhou (2009b)

$$\Delta\rho(z) = -747.7 + 203.435z - 26.764z^2 + 1.4247z^3 - 23.205x. \quad (164)$$

To emphasize the dependence of the solution upon the monomials appearing in the expression of the density contrast we have plotted in fig. 8a and 8b the surface distribution of the gravity anomaly for the density contrast

$$\Delta\rho(z) = -747.7 + 203.435z - 26.764z^2 + 1.4247z^3 - 23.205y, \quad (165)$$

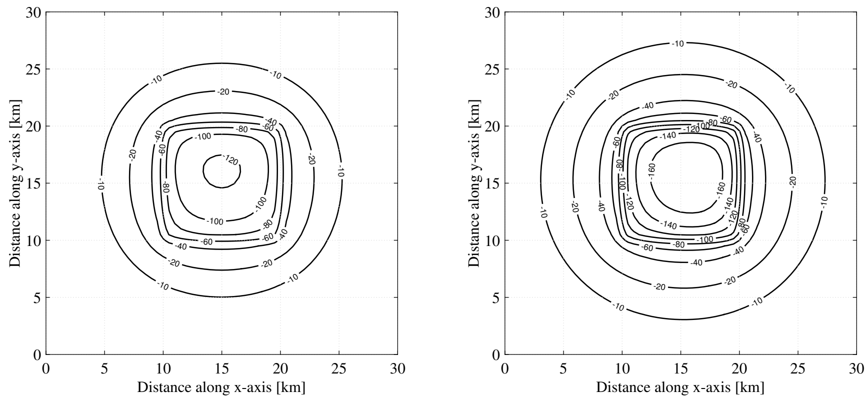
$$\Delta\rho(z) = -747.7 + 203.435z - 26.764z^2 + 1.4247z^3 - 23.205x - 23.205y. \qquad (166)$$



(a) density contrast as in (162) (b) density contrast as in (164)

Fig. 7 Gravity anomaly distribution at z=0 associated with the prism $\Omega \equiv [10,20] \times [10,20] \times [0,8]$ (dimensions in kilometers) and density contrast given by (162) (on the left) and (164) (on the right).

It is apparent from the last two plots that gravity anomaly vanishes less rapidly than in fig. 7a.



(a) density contrast as in (165) (b) density contrast as in (166)

Fig. 8 Gravity anomaly distribution at z=0 associated with the prism $\Omega \equiv [10,20] \times [10,20] \times [0,8]$ (dimensions in kilometers) and density contrast given by (165) (on the left) and (166) (on the right).

Table 1 Gravity anomaly (mGal) associated with prism $\Omega \equiv [10, 20] \times [10, 20] \times [0, 8]$ (dimensions in kilometers and density contrast (162)) at several locations; a) Analytical values; b) Numerical values. Computing Time (CT) in seconds

z=0 and	y=10 km							
x (km)	0,00	5,00	10,00	15,00	20,00	25,00	30,00	CT
a)	-1,22163576397609	-3,46372618679431	-20,7412785817980	-36,2650788733413	-20,7412785817980	-3,46372618679432	-1,22163576397614	1.9813
b)	-1,22163576397627	-3,46372618679431	-20,7413498102378	NaN	-20,7413498102377	-3,46372618679431	-1,22163576397627	143.4464
z=0 and	y=11 km							
x (km)	0,00	5,00	10,00	15,00	20,00	25,00	30,00	CT
a)	-1.28698607331256	-3.82357120782405	-29.72909079760424	-53.62521739346171	-29.72909079760428	-3.82357120782429	-1.28698607331263	1.8574
b)	-1.28698607331254	-3.82357120782415	-29.72928645482153	NaN	-29.72928645482145	-3.82357120782415	-1.28698607331254	154.6723
z=0 and	y=12,5 km							
x (km)	0,00	5,00	10,00	15,00	20,00	25,00	30,00	CT
a)	-1.36376684444623	-4.25957137389371	-34.23229607059629	-61.88280073665107	-34.23229607059632	-4.25957137389369	-1.36376684444629	1.894
b)	-1.36376684444609	-4.25957137389370	-34.23243794205016	NaN	-34.23243794205009	-4.25957137389370	-1.36376684444609	142.5479
z=0 and	y=15 km							
x (km)	0,00	5,00	10,00	15,00	20,00	25,00	30,00	CT
a)	-1,41650677516557	-4,56182411878455	-36,2650788733413	-65,4288804280923	-36,2650788733413	-4,56182411878455	-1,41650677516557	1.9127
b)	-1,41650677516342	-4,56182411878455	-36,2652685757159	NaN	-36,2652685757159	-4,56182411878455	-1,41650677516557	156.1096

6 Conclusions

The gravity anomaly at arbitrary points induced by a polyhedral body of arbitrary shape body whose shape is an arbitrary and characterized by polynomial density contrast has been obtained in closed form. It is expressed as sum of quantities that depend only upon the 3D coordinates of the vertices of the polyhedron and upon the parameters defining the density contrast. The solution procedure, based upon a generalized application of Gauss theorem, takes consistently into account the singularity intrinsic to the integrals to evaluate. In particular, by means of rigorous mathematical arguments, singularities are proved to give no contribution both to the analytical expression of the gravity anomaly and to its algebraic counterpart.

The formulation presented in the paper has been limited to polynomial density contrast varying with a cubic law as a maximum but it can be easily extended to polynomials of higher degree. The effectiveness of the proposed approach has been intensively tested by numerical comparisons, carried out by means of a Matlab code, with several example derived from the specialized literature. Future contributions will concern the cases of density contrast variable with exponential law for 2D and 3D domains.

7 Appendix 1 - Algebraic expression of integrals

We are going to show that the 2D integrals

$$\int_{F_i} \frac{[\otimes \boldsymbol{\rho}_i, m]}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \quad m \in [0,4] \tag{167}$$

can be evaluated analytically. As a matter of fact we only need to evaluate the integrals for $m = 3$ and $m = 4$

$$\mathfrak{C}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i \qquad \mathfrak{D}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i , \tag{168}$$

since the additional ones in (167) have been already computed in D'Urso (2013a, 2014a,b). For completeness these last ones are reported in Appendix 2.

A further integral, namely

$$\boldsymbol{\Psi}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} dA_i , \tag{169}$$

required for the computation of the integrals (168), will be dealt with at the end of this Appendix.

The rationale for evaluating the integrals (168) is to first apply the generalized Gauss theorem D'Urso (2013a, 2014a) to transform them into 1D integrals and, subsequently, to compute such integrals by means of algebraic expressions depending upon the 2D coordinates of the vertices that define the face F_i.

In order to apply the Gauss theorem to the integrals in (168) let us first prove the identity

$$\operatorname{grad}[\varphi(\mathbf{a} \otimes \mathbf{b})] = (\mathbf{a} \otimes \mathbf{b}) \otimes \operatorname{grad}\varphi + \varphi \operatorname{grad}\mathbf{a} \otimes \mathbf{b} + \varphi \mathbf{a} \otimes \operatorname{grad}\mathbf{b} , \tag{170}$$

holding for scalar φ and vector $(\mathbf{a}, \mathbf{b})$ differentiable fields.

It can be easily verified by applying the chain rule to the ijk component of the third-order tensor on the left-hand side

$$\left\{\mathrm{grad}\,[\varphi(\mathbf{a}\otimes\mathbf{b})]\right\}_{jkq} = \left(\varphi a_j b_k\right)_{/q} = \varphi_{/q}\, a_j\, b_k + \varphi a_{j/q}\, b_k + \varphi a_j\, b_{k/q}\,. \tag{171}$$

In a similar fashion one can prove the further differential identity involving fourt-order tensors

$$\mathrm{grad}\,[\varphi(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c})] = (\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c})\mathrm{grad}\varphi + \varphi\,\mathrm{grad}\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c} + \varphi\mathbf{a}\otimes\mathrm{grad}\mathbf{b}\otimes\mathbf{c} + \varphi\mathbf{a}\otimes\mathbf{b}\otimes\mathrm{grad}\mathbf{c}\,. \tag{172}$$

Let us now apply the identity (171) as follows

$$\left[\mathrm{grad}\left(\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}\right)\right]_{jkq} = -\left[\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\right]_{jkq} + \frac{(\boldsymbol{\rho}_i)_{j/q}(\boldsymbol{\rho}_i)_k}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}} + \frac{(\boldsymbol{\rho}_i)_j(\boldsymbol{\rho}_i)_{k/q}}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}} \tag{173}$$

since

$$\mathrm{grad}\left[\frac{1}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}\right] = -\frac{\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\,. \tag{174}$$

Thus, being $(\boldsymbol{\rho}_i)_{j/q} = \delta_{jq}$ we infer from (173)

$$\mathrm{grad}\left(\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}\right) = -\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}} + \frac{\mathbf{I}_{2D}\otimes_{23}\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}} + \frac{\boldsymbol{\rho}_i\otimes\mathbf{I}_{2D}}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}} \tag{175}$$

where $\mathbf{I}_{2D}$ is the 2D identity tensor and $\otimes_{23}$ denotes the tensor product obtained by interchanging the second and third index of the rank-three tensor $\mathbf{I}_{2D}\otimes\boldsymbol{\rho}_i$.

The integral over F_i of the first addend in the formula above can be transformed into a boundary integral by exploiting the differential identity (Bowen and Wang, 2006)

$$\int_{\Omega}\mathrm{grad}\,\mathbf{S}\,dV = \int_{\partial\Omega}\mathbf{S}\otimes\mathbf{n}\,dA \tag{176}$$

where $\mathbf{S}$ is a continuous tensor field.

Thus, integrating over F_i the previous relation and recalling the definition (64) one has

$$\int_{F_i}\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i = -\int_{\partial F_i}\frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}ds_i + \mathbf{I}_{2D}\otimes_{23}\boldsymbol{\psi}_{F_i} + \boldsymbol{\psi}_{F_i}\otimes\mathbf{I}_{2D} \tag{177}$$

where $\boldsymbol{\nu}$ is the unit normal pointing outwards the boundary ∂F_i of the i-th face F_i of the polyhedron.

Hence the first integral on the right-hand side of (177) becomes

$$\int_{\partial F_i}\frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}ds_i = \sum_{j=1}^{N_{E_i}}\int_0^{l_j}\frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)ds_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}\otimes\boldsymbol{\nu}_j \tag{178}$$

since $\boldsymbol{\nu}$ is constant on each of the N_{E_i} edges belonging to ∂F_i.

Recalling (68) and (73), formula (178) becomes

$$\int_{\partial F_i}\frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}ds_i = \sum_{j=1}^{N_{E_i}}\int_0^1\frac{\hat{\boldsymbol{\rho}}_i(\lambda_j)\otimes\hat{\boldsymbol{\rho}}_i(\lambda_j)d\lambda_j}{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j)+d_i^2\right]^{1/2}}\otimes\Delta\boldsymbol{\rho}_j^{\perp} \tag{179}$$

and the integral on the right-hand side can be further transformed by defining

$$\mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} = \boldsymbol{\rho}_j \otimes \boldsymbol{\rho}_j \quad \mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} = \boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j + \Delta\boldsymbol{\rho}_j \otimes \boldsymbol{\rho}_j \quad \mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} = \Delta\boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j\,. \tag{180}$$

Actually, recalling the parametrization (67) one has

$$\hat{\boldsymbol{\rho}}_i(\lambda_j)\otimes\hat{\boldsymbol{\rho}}_i(\lambda_j) = \mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + \lambda_j \mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + \lambda_j^2 \mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}\,, \tag{181}$$

$$\int_0^1 \frac{\hat{\boldsymbol{\rho}}_i(\lambda_j)\otimes\hat{\boldsymbol{\rho}}_i(\lambda_j)d\lambda_j}{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j)+d_i^2\right]^{1/2}} = I_{0j}\,\mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j}\,\mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{2j}\,\mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} \tag{182}$$

where the explicit expression of the integrals

$$\begin{aligned} I_{0j} &= \int_0^1 \frac{d\lambda_j}{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j)+d_i^2\right]^{1/2}} \qquad I_{1j} = \int_0^1 \frac{\lambda_j d\lambda_j}{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j)+d_i^2\right]^{1/2}} \\ I_{2j} &= \int_0^1 \frac{\lambda_j^2 d\lambda_j}{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j)+d_i^2\right]^{1/2}} \end{aligned} \tag{183}$$

is provided in Appendix 2.

In conclusion it turns out be

$$\int_{\partial F} \frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}ds_i = \sum_{j=1}^{N_{E_i}}\left[I_{0j}\,\mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j}\,\mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{2j}\,\mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}\right]\otimes\Delta\boldsymbol{\rho}_j^{\perp}\,, \tag{184}$$

so that the integral of interest can be computed as fallows on account of (177)

$$\begin{aligned} \mathfrak{C}_{F_i} = \int_{F_i} \frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i &= -\sum_{j=1}^{N_{E_i}}\left[I_{0j}\,\mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j}\,\mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{2j}\,\mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}\right]\otimes\Delta\boldsymbol{\rho}_j^{\perp} + \\ &+ \mathbf{I}_{2D}\otimes_{23}\boldsymbol{\psi}_{F_i} + \boldsymbol{\psi}_{F_i}\otimes\mathbf{I}_{2D} \end{aligned} \tag{185}$$

where the expression of $\boldsymbol{\psi}_{F_i}$ as explicit function of the position vectors defining the boundary of F_i is provided at the end of this Appendix.

Of interest is also the composition of the third-order tensor above with the vector $\boldsymbol{\kappa}_i$ since it appears in the expressions (47), (50) and (49). For this end let us first notice that

$$\begin{aligned} \left[\left(\mathbf{I}_{2D}\otimes_{23}\boldsymbol{\psi}_{F_i}\right)\boldsymbol{\kappa}_i\right]_{jk} &= \left(\mathbf{I}_{2D}\otimes_{23}\boldsymbol{\psi}_{F_i}\right)_{jkp}\left(\boldsymbol{\kappa}_i\right)_p = I_{jp}\left(\boldsymbol{\psi}_{F_i}\right)_k\left(\boldsymbol{\kappa}_i\right)_p = \\ &= \delta_{jp}\left(\boldsymbol{\kappa}_i\right)_p\left(\boldsymbol{\psi}_{F_i}\right)_k = \left(\boldsymbol{\kappa}_i\right)_j\left(\boldsymbol{\psi}_{F_i}\right)_k = \left(\boldsymbol{\kappa}_i\otimes\boldsymbol{\psi}_{F_i}\right)_{jk}\,. \end{aligned} \tag{186}$$

Hence

$$\begin{aligned} \mathfrak{C}_{F_i}\boldsymbol{\kappa}_i = \int_{F_i} \frac{(\boldsymbol{\rho}_i\cdot\boldsymbol{\kappa}_i)(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i &= -\sum_{j=1}^{N_{E_i}}\left(\boldsymbol{\kappa}_i\cdot\Delta\boldsymbol{\rho}_j^{\perp}\right)\left(I_{0j}\,\mathbf{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j} + I_{1j}\,\mathbf{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j} + I_{2j}\,\mathbf{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}\right) + \\ &+ \boldsymbol{\kappa}_i\otimes\boldsymbol{\psi}_{F_i} + \boldsymbol{\psi}_{F_i}\otimes\boldsymbol{\kappa}_i \end{aligned} \tag{187}$$

so that the right-hand side fulfills the symmetry of the tensor on the left-hand side of the previous expression.

To evaluate analytically the second integral in (168) we exploit the identity (172) to get

$$\left[\operatorname{grad}\left(\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}\right)\right]_{jkpq} = -\left[\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\right]_{jkpq} + \frac{\delta_{jq}(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)_{kp}}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}} + \\ + \frac{\delta_{kq}(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)_{jp}}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}} + \frac{\delta_{pq}(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)_{jk}}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}, \tag{188}$$

or equivalently

$$\operatorname{grad}\left(\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}\right) = -\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}} + \frac{\mathbf{I}_{2D}\otimes_{24}(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}} + \\ + \frac{(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)\otimes_{23}\mathbf{I}_{2D}}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}} + \frac{(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)\otimes\mathbf{I}_{2D}}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}} \tag{189}$$

where $\otimes_{24}$ denotes the tensor product obtained by interchanging the second and fourth index of the rank-four tensor $\mathbf{I}_{2D}\otimes(\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i)$.

Integrating the previous relation over F_i and applying Gauss theorem yields

$$\mathfrak{D}_{F_i} = \int_{F_i}\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}dA_i = -\int_{\partial F_i}\frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}ds_i + \\ + \mathbf{I}_{2D}\otimes_{24}\boldsymbol{\Psi}_{F_i} + \boldsymbol{\Psi}_{F_i}\otimes_{23}\mathbf{I}_{2D} + \boldsymbol{\Psi}_{F_i}\otimes\mathbf{I}_{2D} \tag{190}$$

where $\boldsymbol{\Psi}_{F_i}$ is analytically evaluated in formula (208) of Appendix 2.

In view of the ensuing developments we further set

$$\mathbb{E}_{\rho_j\rho_j\rho_j} = \boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j \qquad \mathbb{E}_{\rho_j\rho_j\Delta\rho_j} = \boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j + \boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j + \Delta\boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j \tag{191}$$

$$\mathbb{E}_{\rho_j\Delta\rho_j\Delta\rho_j} = \boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j + \Delta\boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j + \Delta\boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j\otimes\boldsymbol{\rho}_j \tag{192}$$

$$\mathbb{E}_{\Delta\rho_j\Delta\rho_j\Delta\rho_j} = \Delta\boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j\otimes\Delta\boldsymbol{\rho}_j \tag{193}$$

yielding

$$\hat{\boldsymbol{\rho}}_i(\lambda_j)\otimes\hat{\boldsymbol{\rho}}_i(\lambda_j)\otimes\hat{\boldsymbol{\rho}}_i(\lambda_j) = \mathbb{E}_{\rho_j\rho_j\rho_j} + \lambda_j\mathbb{E}_{\rho_j\rho_j\Delta\rho_j} + \lambda_j^2\mathbb{E}_{\rho_j\Delta\rho_j\Delta\rho_j} + \lambda_j^3\mathbb{E}_{\Delta\rho_j\Delta\rho_j\Delta\rho_j}. \tag{194}$$

Accordingly, the integral on the right-hand side in (190) becomes

$$\int_{\partial F_i}\frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}ds_i = \sum_{j=1}^{N_{E_i}}\int_0^1\left\{\frac{\hat{\boldsymbol{\rho}}_i(\lambda_j)\otimes\hat{\boldsymbol{\rho}}_i(\lambda_j)\otimes\hat{\boldsymbol{\rho}}_i(\lambda_j)d\lambda_j}{\left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j)+d_i^2\right]^{1/2}}\otimes\Delta\boldsymbol{\rho}_j^{\perp}\right\} = \\ = -\sum_{j=1}^{N_{E_i}}\left[I_{0j}\,\mathbb{E}_{\rho_j\rho_j\rho_j} + I_{1j}\,\mathbb{E}_{\rho_j\rho_j\Delta\rho_j} + \right. \\ \left. + I_{2j}\,\mathbb{E}_{\rho_j\Delta\rho_j\Delta\rho_j} + I_{3j}\,\mathbb{E}_{\Delta\rho_j\Delta\rho_j\Delta\rho_j}\right]\otimes\Delta\boldsymbol{\rho}_j^{\perp} \tag{195}$$

where the integrals I_{0j}, I_{1j}, I_{2j} and I_{3j} are explicitly evaluated in the Appendix 2.

In conclusion one has

$$\int_{\partial F_i} \frac{\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}ds_i = \sum_{j=1}^{N_{E_i}}\Big[I_{0j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\boldsymbol{\rho}_j}+I_{1j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}+ + I_{2j}\,\mathbb{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}+I_{3j}\,\mathbb{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}\Big]\otimes\Delta\boldsymbol{\rho}_j^{\perp}+ + \mathbf{I}_{2D}\otimes_{24}\boldsymbol{\Psi}_{F_i}+\boldsymbol{\Psi}_{F_i}\otimes_{23}\mathbf{I}_{2D}+\boldsymbol{\Psi}_{F_i}\otimes\mathbf{I}_{2D}\,. \tag{196}$$

The composition of the previous integral with $\boldsymbol{\kappa}_i$, a quantity that is needed in (175) and (to be displayed), yields a third-order tensor. The contribution to the jkp component of this tensor provided by the tensor product $\boldsymbol{\Psi}_{F_i}\otimes_{23}\mathbf{I}_{2D}$ is given by

$$\begin{aligned}\Big[\big(\boldsymbol{\Psi}_{F_i}\otimes_{23}\mathbf{I}_{2D}\big)\,\boldsymbol{\kappa}_i\Big]_{jkp} &= \big(\boldsymbol{\Psi}_{F_i}\otimes_{23}\mathbf{I}_{2D}\big)_{jkpq}\big(\boldsymbol{\kappa}_i\big)_q = \big(\boldsymbol{\Psi}_{F_i}\big)_{jp}\big(\delta_{kq}\big)\big(\boldsymbol{\kappa}_i\big)_q = \\ &= \big(\boldsymbol{\Psi}_{F_i}\big)_{jp}\big(\boldsymbol{\kappa}_i\big)_k = \big(\boldsymbol{\Psi}_{F_i}\otimes_{23}\boldsymbol{\kappa}_i\big)_{jkp}\,.\end{aligned} \tag{197}$$

Analogously

$$\begin{aligned}\Big[\big(\mathbf{I}_{2D}\otimes_{24}\boldsymbol{\Psi}_{F_i}\big)\,\boldsymbol{\kappa}_i\Big]_{jkp} &= \big(\mathbf{I}_{2D}\otimes_{24}\boldsymbol{\Psi}_{F_i}\big)_{jkpq}\big(\boldsymbol{\kappa}_i\big)_q = \big(\delta_{jq}\big)\big(\boldsymbol{\Psi}_{F_i}\big)_{pk}\big(\boldsymbol{\kappa}_i\big)_q = \\ &= \big(\boldsymbol{\kappa}_i\big)_j\big(\boldsymbol{\Psi}_{F_i}\big)_{pk} = \big(\boldsymbol{\kappa}_i\big)_j\big(\boldsymbol{\Psi}_{F_i}\big)_{kp} = \big(\boldsymbol{\kappa}_i\otimes\boldsymbol{\Psi}_{F_i}\big)_{jkp}\end{aligned} \tag{198}$$

where the identity $\big(\boldsymbol{\Psi}_{F_i}\big)_{pk} = \big(\boldsymbol{\Psi}_{F_i}\big)_{kp}$ stems from the symmetry of $\boldsymbol{\Psi}_{F_i}$. Accordingly, we infer from (190) and (196)

$$\mathfrak{D}_{F_i}\boldsymbol{\kappa}_i = \int_{F_i}\frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{3/2}}\boldsymbol{\kappa}_i = -\sum_{j=1}^{N_{E_i}}\big(\boldsymbol{\kappa}_i\cdot\Delta\boldsymbol{\rho}_j^{\perp}\big)\big(I_{0j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\boldsymbol{\rho}_j}+I_{1j}\,\mathbb{E}_{\boldsymbol{\rho}_j\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}+ + I_{2j}\,\mathbb{E}_{\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}+I_{3j}\,\mathbb{E}_{\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j\Delta\boldsymbol{\rho}_j}\big)+ + \boldsymbol{\Psi}_{F_i}\otimes\boldsymbol{\kappa}_i+\boldsymbol{\Psi}_{F_i}\otimes_{23}\boldsymbol{\kappa}_i+\boldsymbol{\kappa}_i\otimes\boldsymbol{\Psi}_{F_i}\,. \tag{199}$$

The expression (185) for $\mathfrak{C}_{F_i}$ and (190) for $\mathfrak{D}_{F_i}$ require the computation of the integral $\boldsymbol{\Psi}_{F_i}$ defined in formula (169); it is evaluated analytically by invoking the differential identity

$$\mathrm{grad}\big[\varphi\mathbf{a}\big] = \mathbf{a}\otimes\mathrm{grad}\varphi+\varphi\,\mathrm{grad}\mathbf{a} \tag{200}$$

holding for differentiable scalar (φ) and vector ($\mathbf{a}$) fields. Actually, applying the previous identity as follows

$$\mathrm{grad}\big[(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}\boldsymbol{\rho}_i\big] = \frac{\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}}+(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2)^{1/2}\mathbf{I}_{2D}\,, \tag{201}$$

integrating over F_i and setting

$$\iota_{F_i} = \int_{F_i}\big(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2\big)^{1/2}dA_i \tag{202}$$

one has

$$\boldsymbol{\Psi}_{F_i} = \int_{\partial F_i} \left[\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\rho}_i(s_i)+d_i^2\right]^{1/2}\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}_i(s_i)ds_i - \iota_{F_i}\mathbf{I}_{2D}\,. \tag{203}$$

To compute the domain integral (202), we apply the differential identity

$$\mathrm{div}\left[\varphi\mathbf{a}\right] = \mathrm{grad}\varphi\cdot\mathbf{a}+\varphi\,\mathrm{div}\,\mathbf{a} \tag{204}$$

to the vector field $\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2\right)^{1/2}\boldsymbol{\rho}_i$ to get

$$\mathrm{div}\left[\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2\right)^{1/2}\boldsymbol{\rho}_i\right] = \frac{\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i}{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2\right)^{1/2}} + 2\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2\right)^{1/2}. \tag{205}$$

Adding and subtracting d_i^2 to the numerator yields

$$\mathrm{div}\left[\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2\right)^{1/2}\boldsymbol{\rho}_i\right] = 3\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2\right)^{1/2} - \frac{d_i^2}{\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2\right)^{1/2}}, \tag{206}$$

so that, upon integrating over F_i and applying Gauss theorem, one has

$$\iota_{F_i} = \frac{1}{3}\int_{\partial F_i}\left[\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\rho}_i(s_i)+d_i^2\right]^{1/2}\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\nu}(s_i)ds_i - \frac{d_i^2}{3}\psi_{F_i}, \tag{207}$$

by recalling definition (62). In conclusion, we infer from (203) and the previous expression

$$\begin{aligned}
\boldsymbol{\Psi}_{F_i} &= \int_{\partial F_i}\left[\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\rho}_i(s_i)+d_i^2\right]^{1/2}\boldsymbol{\rho}_i(s_i)\otimes\boldsymbol{\nu}(s_i)ds_i - \\
&\quad - \frac{\mathbf{I}_{2D}}{3}\left\{\int_{\partial F_i}\left[\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\rho}_i(s_i)+d_i^2\right]^{1/2}\boldsymbol{\rho}_i(s_i)\cdot\boldsymbol{\nu}(s_i)ds_i - d_i^2\psi_{F_i}\right\} \\
&= \sum_{j=1}^{N_{E_i}}\left\{\left[\int_0^{l_j}\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2\right)^{1/2}\boldsymbol{\rho}_i ds_j\right]\otimes\boldsymbol{\nu}_j - \right. \\
&\qquad \left. - \frac{\mathbf{I}_{2D}}{3}\left[\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\nu}_j\right)\int_0^{l_j}\left(\boldsymbol{\rho}_i\cdot\boldsymbol{\rho}_i+d_i^2\right)^{1/2}ds_j\right]\right\} + \frac{d_i^2}{3}\psi_{F_i} = \\
&= \sum_{j=1}^{N_{E_i}}\left\{\left[\int_0^1\left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j)+d_i^2\right]^{1/2}\left(\boldsymbol{\rho}_j+\lambda_j\Delta\boldsymbol{\rho}_j\right)d\lambda_j\right]\otimes\Delta\boldsymbol{\rho}_j^{\perp} - \right. \\
&\qquad \left. - \frac{\mathbf{I}_{2D}}{3}\left(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp}\right)\int_0^1\left[\hat{\boldsymbol{\rho}}_i(\lambda_j)\cdot\hat{\boldsymbol{\rho}}_i(\lambda_j)+d_i^2\right]^{1/2}d\lambda_j\right\} + \frac{d_i^2}{3}\left(\psi_i-|d_i|\alpha_i\right) = \\
&= \sum_{j=1}^{N_{E_i}}\left[\left(I_{4j}\boldsymbol{\rho}_j+I_{5j}\Delta\boldsymbol{\rho}_j\right)\otimes\Delta\boldsymbol{\rho}_j^{\perp} - \frac{\mathbf{I}_{2D}}{3}(\boldsymbol{\rho}_j\cdot\boldsymbol{\rho}_{j+1}^{\perp})I_{4j}\right] + \frac{d_i^2}{3}\left(\psi_i-|d_i|\alpha_i\right)
\end{aligned} \tag{208}$$

where ψ_i is defined in (219).

We have numerically verified that the sum over the N_{E_i} edges of the first addend on the right-hand side returns a symmetric rank-two tensor as the one the left-hand side.

8 Appendix 2 - Available expressions of integrals

We hereby collect some known formulas in order to allow the reader to implement the expression of the gravity anomaly contributed in the main body of the paper.

We first report the algebraic expression of some definite integrals that will be repeatedly referred to in the sequel; they have been computed elsewhere D'Urso (2013a, 2014a,b) though with a different denomination. Making reference to the quantities p_j, q_j, u_j, v_j introduced in formula (71), we set

$$ATN1_j = \arctan \frac{|d_i|(p_j+q_j)}{\sqrt{p_j u_j - q_j^2}\,\sqrt{p_j+2q_j+v_j}}, \tag{209}$$

$$ATN2_j = \arctan \frac{|d_i| q_j}{\sqrt{p_j u_j - q_j^2}\,\sqrt{v_j}} \tag{210}$$

where the suffix $(\cdot)_j$ has been added to remind that they all refer to the j-th edge of the generic face F_i.

Of interest are also the following integrals

$$I_{0j} = \int_0^1 \frac{d\lambda_j}{\left[p_j\lambda^2+2q_j\lambda_j+v_j\right]^{1/2}} = \ln k_j = \ln \frac{p_j+q_j+\sqrt{p_j}\sqrt{p_j+2q_j+v_j}}{q_j+\sqrt{p_j v_j}} = LN_j\,, \tag{211}$$

$$I_{1j} = \int_0^1 \frac{\lambda_j d\lambda_j}{\left[p_j\lambda^2+2q_j\lambda_j+v_j\right]^{1/2}} = \frac{1}{p_j}\left\{\sqrt{p_j+2q_j+v_j} - \sqrt{v_j} - \frac{q_j}{\sqrt{p_j}} I_{0j}\right\}, \tag{212}$$

$$\begin{aligned} I_{2j} = \int_0^1 \frac{\lambda_j^2 d\lambda_j}{\left[p_j\lambda^2+2q_j\lambda_j+v_j\right]^{1/2}} &= \frac{1}{2p_j^2}\left[(p_j-3q_j)\sqrt{p_j+2q_j+v_j}+3q_j\sqrt{v_j}\right]+ \\ &+ \frac{3q_j^2-p_jv_j}{2p_j^{5/2}} I_{0j}\,, \end{aligned} \tag{213}$$

$$\begin{aligned} I_{3j} = \int_0^1 \frac{\lambda_j^3 d\lambda_j}{\left[p_j\lambda^2+2q_j\lambda_j+v_j\right]^{1/2}} &= \frac{1}{6p_j^3}\left[(2p_j^2-5p_jq_j-4p_jv_j+15q_j^2)\sqrt{p_j+2q_j+v_j}+\right. \\ &\left. +(4p_jv_j-15q_j^2)\sqrt{v_j}\right]+\frac{3p_jq_jv_j-5q_j^3}{2p_j^{7/2}} I_{0j}\,, \end{aligned} \tag{214}$$

$$I_{4j} = \int_0^1 \left[p_j\lambda^2+2q_j\lambda_j+v_j\right]^{1/2} d\lambda_j = \frac{(p_j+q_j)\sqrt{p_j+2q_j+v_j}-q_j\sqrt{v_j}}{2p_j} + \frac{p_jv_j-q_j^2}{2p_j^{3/2}} I_{0j}\,, \tag{215}$$

$$\begin{aligned} I_{5j} = \int_0^1 \lambda_j\left[p_j\lambda^2+2q_j\lambda_j+v_j\right]^{1/2} d\lambda_j &= \frac{1}{6p_j^2}\left[(2p_j^2+p_jq_j+2p_jv_j-3q_j^2)\sqrt{p_j+2q_j+v_j}-\right. \\ &\left. -(2p_jv_j-3q_j^2)\sqrt{v_j}\right]+\frac{q_j^3-p_jq_jv_j}{2p_j^{5/2}} I_{0j}\,, \end{aligned} \tag{216}$$

$$I_{6j} = \int_0^1 \frac{\left[p_j\lambda^2 + 2q_j\lambda_j + v_j\right]^{1/2}}{p_j\lambda^2 + 2q_j\lambda_j + u_j} d\lambda_j = \frac{|d_i|}{\sqrt{p_j u_j - q_j^2}}\left[ATN1_j - ATN2_j\right] + \frac{1}{\sqrt{p_j}} LN_j . \quad (217)$$

Let us now consider the evaluation of 2D integrals having either $(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}$ or $(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}$ in the denominator. The first domain integral to consider is

$$\psi_{F_i} = \int_{F_i} \frac{dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} = \psi_i - |d_i|\alpha_i \quad (218)$$

where

$$\begin{aligned} \psi_i &= \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\nu}_j\right) \int_0^{l_j} \frac{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}}{\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i} ds_j = \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) \int_0^1 \frac{(p_j\lambda_j^2 + 2q_j\lambda_j + v_j)^{1/2}}{p_j\lambda_j^2 + 2q_j\lambda_j + u_j} d\lambda_j = \\ &= \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) \left\{ \frac{|d_i|}{\sqrt{p_j u_j - q_j^2}} \left[ATN1_j - ATN2_j\right] + \frac{1}{\sqrt{p_j}} LN_j \right\} = \sum_{j=1}^{N_{E_i}} \psi_j^i \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right). \end{aligned} \quad (219)$$

The derivation of the previous expression can be found, e.g., in formula (19) of D'Urso (2013a) and (23) of D'Urso (2014a).

The scalar α_i in (218) is the two-dimensional counterpart of the quantity α_V in (26) and accounts for the singularity of ψ_{F_i} when $d_i = 0$ and $\boldsymbol{\rho} = \boldsymbol{o}$ where $\boldsymbol{o} = (0, 0)$. Thus α_i represents the angular measure, expressed in radians, of the intersection between F_i and a circular neighbourhood of the singularity point $\boldsymbol{\rho} = \boldsymbol{o}$, see D'Urso (2013a, 2014a,b) for additional details. Although its computation is not required in the ensuing developments, we specify for completeness that α_i can be computed by means of the general algorithm detailed in D'Urso and Russo (2002).

Analogously formulas (19), (77) and (79) of D'Urso (2014b) yield

$$\begin{aligned} \boldsymbol{\psi}_{F_i} &= \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} = \sum_{j=1}^{N_{E_i}} \boldsymbol{\nu}_j \int_0^{l_j} (\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2} ds_i = \\ &= \sum_{j=1}^{N_{E_i}} l_j \boldsymbol{\nu}_j \int_0^1 \left[p_j\lambda_j^2 + 2q_j\lambda_j + v_j\right]^{1/2} d\lambda_j = \sum_{j=1}^{N_{E_i}} I_{4j} \Delta\boldsymbol{\rho}_j^{\perp} \end{aligned} \quad (220)$$

while formulas (37) and (81) of D'Urso (2014b)

$$\begin{aligned} \varphi_{F_i} &= \int_{F_i} \frac{dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} = \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}} \left[\left(\boldsymbol{\rho}_j \cdot \boldsymbol{\nu}_j\right) \int_0^{l_j} \frac{ds_j}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i)(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} \right] = \\ &= \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right) \int_0^1 \frac{\lambda_j}{(p_j\lambda_j^2 + 2q_j\lambda_j + u_j)(p_j\lambda_j^2 + 2q_j\lambda_j + v_j)^{1/2}} = \\ &= \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}} \left[\frac{\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}}{|d_i| \sqrt{p_j u_j - q_j^2}} (ATN1_j - ATN2_j) \right] = \frac{\alpha_i}{|d_i|} - \sum_{j=1}^{N_{E_i}} \varphi_j \left(\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_{j+1}^{\perp}\right). \end{aligned} \quad (221)$$

Furthermore, on account of formulas (38) and (82) of D'Urso (2014b) it turns out to be

$$\begin{aligned}\boldsymbol{\varphi}_{F_i} &= \int_{F_i} \frac{\boldsymbol{\rho}_i dA_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} = -\sum_{j=1}^{N_{E_i}} \left(\boldsymbol{\nu}_j \int_0^{l_j} \frac{ds_j}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} \right) = \\ &= -\sum_{j=1}^{N_{E_i}} \Delta\boldsymbol{\rho}_j^{\perp} \int_0^1 \frac{d\lambda_j}{(p_j \lambda_j^2 + 2q_j \lambda_j + v_j)^{1/2}} = -\sum_{j=1}^{N_{E_i}} I_{0j}\, \Delta\boldsymbol{\rho}_j^{\perp}\end{aligned} \tag{222}$$

while one infers from formulas (40) and (83) of D'Urso (2014b)

$$\begin{aligned}\boldsymbol{\Phi}_{F_i} &= \int_{F_i} \frac{\boldsymbol{\rho}_i \otimes \boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{3/2}} dA_i = \\ &= -\sum_{j=1}^{N_{E_i}} \int_0^{l_j} \frac{\boldsymbol{\rho}_i}{(\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i + d_i^2)^{1/2}} ds_i \otimes \boldsymbol{\nu}_j + \psi_{F_i} \mathbf{I}_{2D} = \\ &= -\sum_{j=1}^{N_{E_i}} \int_0^1 \frac{\boldsymbol{\rho}_j + \lambda_j \Delta\boldsymbol{\rho}_j}{(p_j \lambda_j^2 + 2q_j \lambda_j + v_j)^{1/2}} d\lambda_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} + \psi_{F_i} \mathbf{I}_{2D} \\ &= -\sum_{j=1}^{N_{E_i}} \left[LN_j\, \boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} + I_{1j}\, \Delta\boldsymbol{\rho}_j \otimes \Delta\boldsymbol{\rho}_j^{\perp} \right] + \psi_{F_i} \mathbf{I}_{2D}\end{aligned} \tag{223}$$

where $\mathbf{I}_{2D}$ is the rank-two two-dimensional identity tensor.

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