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Abstract

The theory of logical gates in quantum computation has inspired the development of new forms of quantum logic, called quantum computational logics. The basic semantic idea is the following: the meaning of a formula is identified with a quantum information quantity, represented by a density operator, whose dimension depends on the logical complexity of the formula. At the same time, the logical connectives are interpreted as operations defined in terms of quantum gates.

In this framework, some possible relations between fuzzy representations based on continuous t-norms for quantum gates and the probabilistic behavior of quantum computational finite-valued connectives are investigated.

A fuzzy approach to quantum logical computation

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Abstract

The theory of logical gates in quantum computation has inspired the development of new forms of quantum logic, called quantum computational logics. The basic semantic idea is the following: the meaning of a formula is identified with a quantum information quantity, represented by a density operator, whose dimension depends on the logical complexity of the formula. At the same time, the logical connectives are interpreted as operations defined in terms of quantum gates.

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1. Introduction

The mathematical formalism of quantum theory has inspired the development of different forms of non-classical logics, called *quantum logics*. In many cases the semantic characterizations of these logics are based on special classes of algebraic structures defined in a Hilbert-space environment. Interesting generalizations of quantum logic introduced by Birkhoff and von Neumann are the so called *unsharp (or fuzzy) quantum logics* that can be

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semantically characterized by referring to different classes of algebraic structures whose support is the set of all *effects* of a Hilbert space [6].

A different approach to quantum logic has been developed in the framework of *quantum computational logics*, inspired by the theory of quantum computation [8, 9, 2]. While sharp and unsharp quantum logics refer to possible structures of physical events, the basic objects of quantum computational logics are *pieces of quantum information*: possible states of quantum systems that can store the information in question. The simplest piece of quantum information is a *qubit*: a unit-vector of the Hilbert space $\mathbb{C}^2$ that can be represented as a superposition $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$. The two elements of the canonical basis of $\mathbb{C}^2$, $|0\rangle = (1, 0)$ and $|1\rangle = (0, 1)$, represent the classical bits or, equivalently, the two classical truth-values. It is interesting to consider a "many-valued generalization" of qubits, represented by *qudits*: unit-vectors living in a space $\mathbb{C}^d$, where $d \geq 2$.

The aim of this paper is to study a probabilistic type representation for logical gates based on product t-norm, Łukasiewicz sum and some many valued connectives in the framework of quantum computation with density operators. Any formula of the language gives rise to a quantum circuit that transforms the density operator associated to the formula into the density operator associated the atomic subformulas in a reversible way [9]. One of the advantages of this probabilistic type representation is that we can deal with such circuits as expressions in an algebraic environment (as in the case of Boolean algebra to describe digital circuits).

The paper is organized as follows. In Sections 2-3, we introduce basic notions of quantum computational logics and recall some gates that play a special role from the logical point of view and some interesting relations between these gates and the probability function p. In Section 4, we introduce matrix basis decompositions for density matrices associated to states of d-dimensional quantum systems and describe a state tomography scheme. In Section 5, we show some interesting relations between the logical gates and continuous t-norms by probability values. Finally, in Sections 6-7, we describe the capacity for some holistic connectives of characterizing entanglement of formation both for isotropic states and for Werner states.

2. The basic notions

Let us first recall some basic definitions. As is well known, the general mathematical environment for quantum computation is the Hilbert space

$\mathcal{H}^{(n)} := \underbrace{\mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d}_{n-times}$ (n-fold tensor product where $n \geq 1$ and $d \geq 2$). The canonical orthonormal basis $\mathcal{B}^{(n)}$ of $\mathcal{H}^{(n)}$ is defined as follows:

$$\mathcal{B}^{(n)} = \left\{ |x_1, \ldots, x_n\rangle : x_1 \in \left\{0, \frac{1}{d-1}, \frac{2}{d-1}, \ldots, 1\right\}, \ldots, x_n \in \left\{0, \frac{1}{d-1}, \frac{2}{d-1}, \ldots, 1\right\}\right\},$$

where $|0\rangle = (1, 0, \ldots, 0)$, $|\frac{1}{d-1}\rangle = (0, 1, 0, \ldots, 0)$, $|\frac{2}{d-1}\rangle = (0, 0, 1, 0, \ldots, 0)$, $\ldots, |1\rangle = (0, \ldots, 0, 1)$, while $|x_1, \ldots, x_n\rangle$ is an abbreviation for the tensor product $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$.

Any piece of quantum information is represented by a density operator ρ of a space $\mathcal{H}^{(n)}$. A *quregister* (or quregister-state) is represented by a unit-vector $|\psi\rangle$ (which is a pure state) of a space $\mathcal{H}^{(n)}$ or, equivalently, by the corresponding density operator $\mathtt{P}_{|\psi\rangle}$ (the projection-operator that projects over the closed subspace determined by $|\psi\rangle$). Following a standard convention, we assume that $\mathtt{P}_{|1\rangle}$ represents the truth-value *Truth*, $\mathtt{P}_{|0\rangle}$ represents the truth-value *Falsity* and $\mathtt{P}_{|\frac{j}{d-1}\rangle}$ represent *intermediate* truth-values (where $0 < j < d-1$).

In this framework, one can define the projections that represent the *Truth*, the *Falsity* and intermediate properties in any space $\mathcal{H}^{(n)}$. A *truth-value projection* of $\mathcal{H}^{(n)}$ is a projection $P^{(n)}_{\frac{j}{d-1}}$ whose range is the closed subspace spanned by the set of all quregisters ending with $\frac{j}{d-1}$ of $\mathcal{H}^{(n)}$, where $0 \leq j \leq d-1$.

Accordingly, by applying the Born rule, one can now define the probability that ρ is true, false and an intermediate truth-value in $\mathcal{H}^{(n)}$:

$$\mathrm{p}_{\frac{j}{d-1}}(\rho) = \mathrm{tr}\Big(\rho\, P^{(n)}_{\frac{j}{d-1}}\Big),$$

where $0 \leq j \leq d-1$ and tr is the trace-functional.

From an intuitive point of view, $\mathrm{p}_{\frac{j}{d-1}}(\rho)$ represents the probability that the information stocked by the density operator ρ is the truth-value $\frac{j}{d-1}$.

One can now define the probability for any density operator ρ of $\mathcal{H}^{(n)}$ as the weighted mean of the truth-values.

Definition 1. The probability of a density operator.

$$\mathrm{p}(\rho) = \frac{1}{d-1}\sum_{j=1}^{d-1} j\, \mathrm{p}_{\frac{j}{d-1}}(\rho)$$

Clearly, we have:

$$\mathrm{p}(\rho) = \mathrm{tr}\Big(\rho\,(\mathtt{I}^{(n-1)} \otimes E)\Big),$$

where $\mathtt{I}^{(n-1)}$ is the identity operator of $\mathcal{H}^{(n-1)}$ and E is the effect of the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{d-1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{2}{d-1} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

In the particular case where ρ corresponds to the qubit

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle,$$

we obtain that $\mathrm{p}(\rho) = |c_1|^2$.

The concept of entanglement can be defined both for pure and for mixed states. Consider the product-space

$$\mathcal{H}^{(m+n+p)} = \mathcal{H}^{(m)} \otimes \mathcal{H}^{(n)} \otimes \mathcal{H}^{(p)}.$$

Any density operator ρ of $\mathcal{H}^{(m+n+p)}$ represents a possible state for a composite physical system $S = S_1 + S_2 + S_3$ (consisting of three subsystems). According to the quantum formalism, ρ determines the *reduced states* $Red^{(l)}_{[m,n,p]}(\rho)$ that represent the state of S_l (in the context ρ), with $l = 1, 2, 3$ and $Red^{(k,l)}_{[m,n,p]}(\rho)$ that represent the state of $S_k + S_l$, where $1 \leq k < l \leq 3$, respectively. In such a case, we say that ρ is a *multipartite state* with respect to the decomposition $[m, n, p]$.

It may happen that ρ is a bipartite pure state, while $Red^{(1)}_{[m,n]}(\rho)$ and $Red^{(2)}_{[m,n]}(\rho)$ are proper mixtures. In this case the information about the whole system is more precise than the pieces of information about its parts. As an example, consider the following density operator:

$$\rho = \mathtt{P}_{\frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)}$$

(the projection that projects over the closed subspace spanned by the vector $\frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$).

We have:

$$Red^{(1)}_{[1,1]}(\rho) = Red^{(2)}_{[1,1]}(\rho) = \frac{1}{2}\mathtt{I}^{(1)}.$$

Definition 2. *(Factorizability, separability and entanglement)*
Let ρ be a bipartite state of $\mathcal{H}^{(m+n)}$ (with respect to the decomposition $[m,n]$).

1) ρ is called a *(bipartite) factorized state* of $\mathcal{H}^{(m+n)}$ iff $\rho = \rho_1 \otimes \rho_2$, where ρ_1 and ρ_2 are density operators of $\mathcal{H}^{(m)}$ and $\mathcal{H}^{(n)}$, respectively;
2) ρ is called a *(bipartite) separable state* of $\mathcal{H}^{(m+n)}$ iff $\rho = \sum_i w_i \rho_i$, where each ρ_i is a bipartite factorized state of $\mathcal{H}^{(m+n)}$, $w_i \in [0,1]$ and $\sum_i w_i = 1$;
3) ρ is called a *(bipartite) entangled state* of $\mathcal{H}^{(m+n)}$ iff ρ is not separable.

Accordingly, a pure state is entangled iff it is non-factorizable. Proper mixtures, instead, may be non-factorizable, separable (and, hence, non-entangled). An example is represented by the following proper mixture:

$$\rho = \frac{1}{2}\mathtt{P}_{|0,0\rangle} + \frac{1}{2}\mathtt{P}_{|1,1\rangle}$$

A pure bipartite state ρ of $\mathcal{H}^{(m+n)}$ is called *maximally entangled* iff $Red^{(1)}_{[m,n]} = \frac{1}{2^m}\mathtt{I}^{(m)}$ or $Red^{(2)}_{[m,n]} = \frac{1}{2^n}\mathtt{I}^{(n)}$. A state ρ of $\mathcal{H}^{(m+n)}$ is called a *maximally mixed state* of $\mathcal{H}^{(m+n)}$ iff $\rho = \frac{1}{2^{(m+n)}}\mathtt{I}^{(m+n)}$.

How to measure the "entanglement-degree" of a given state? Different definitions for the concept of *entanglement-measure*, which quantify different aspects of entanglement, have been proposed in the literature [8, 11, 5]. One of the most interesting notions of entanglement-measure is the concept of *entanglement of formation*, which is defined in terms of the notion of *von Neumann-entropy*.

Let ρ be a density operator of the space $\mathcal{H}^{(n)}$. The *von Neumann-entropy* of ρ is defined as follows:

$$E_S(\rho) = -\sum_i \lambda_i \ln \lambda_i,$$

where λ_i are the eigenvalues of ρ.

Definition 3. *(The entanglement of formation)*
Let ρ be a bipartite state of the space $\mathcal{H}^{(m+n)}$. The *entanglement of formation* of ρ is defined as follows:

$$E_F(\rho) = \inf\left\{\sum_i w_i E_S(Red^{(j)}_{[m,n]}(\mathtt{P}_{|\psi_i\rangle})) : \rho = \sum_i w_i \mathtt{P}_{|\psi_i\rangle}\right\},$$

where $j \in \{1, 2\}$.

Apparently, the number $E_F(\rho)$ is determined by the set of all values of the von Neumann-entropy of the two pure reduced states that correspond to all possible representations of ρ as a mixture of pure states.

3. Quantum logical gates

Pure pieces of quantum information are processed by *quantum logical gates* (briefly, *gates*): unitary operators that transform quregisters into quregisters in a reversible way.

The quantum realization of d–valued one–input/one–output gates can be done by considering single quantum systems whose Hamiltonian on $\mathbb{C}^d$ is:

$$H = \begin{bmatrix} \varepsilon_0 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon_0 + \Delta\varepsilon & 0 & \cdots & 0 \\ 0 & 0 & & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \varepsilon_0 + (d-1)\Delta\varepsilon \end{bmatrix}$$

The energy eigenvalues $\varepsilon_j = \varepsilon_0 + j\Delta\varepsilon$ of H, starting from the ground energy state ε_0 and equispaced by the quantum of energy $\Delta\varepsilon$, are the ones of the infinite dimensional quantum harmonic oscillator truncated at the $d-1$ excited level (see Fig. 1).

The unit vector $|H = \varepsilon_j\rangle = |\frac{j}{d-1}\rangle$, for $j \in \{0, 1, \ldots, d-1\}$, is the eigenvector of the state of energy $\varepsilon_0 + j\Delta\varepsilon$. The spectral resolution of the above truncated harmonic oscillator Hamiltonian is:

$$H = \sum_{j=0}^{d-1} (\varepsilon_0 + j\Delta\varepsilon) P_{\varepsilon_j}$$

where each orthogonal projection $P_{\varepsilon_j} = P^{(1)}_{\frac{j}{d-1}}$ is the quantum realization of the sharp event "a measure of the system energy yields the value $\varepsilon_0 + j\Delta\varepsilon$".

The operators $a^\dagger$ and a are non–Hermitian, adjoints of each other. The action of a on the vectors of the canonical orthonormal basis of $\mathbb{C}^d$ is the following: $a^\dagger|\frac{j}{d-1}\rangle = \sqrt{j+1}|\frac{j+1}{d-1}\rangle$ for $j \in \{0, 1, \ldots, d-2\}$, $a^\dagger|1\rangle = \mathbf{0}$; whereas the action of a is: $a|\frac{j}{d-1}\rangle = \sqrt{j}|\frac{j-1}{d-1}\rangle$ for $j \in \{1, 2, \ldots, d-1\}$, $a|0\rangle = \mathbf{0}$.

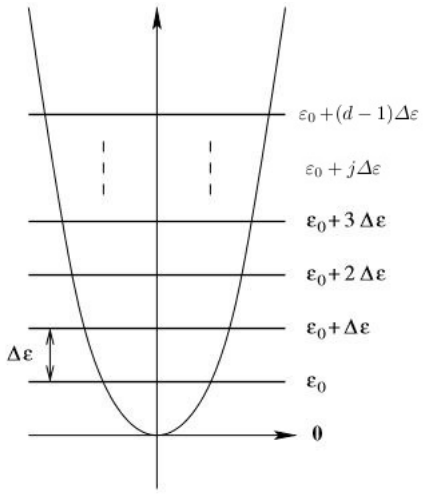

Figure 1: Energy levels of the truncated harmonic oscillator

Creation and *annihilation* operators on the Hilbert space $\mathbb{C}^d$ have the following forms:

$$a^\dagger = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sqrt{d-1} & 0 \end{bmatrix} \qquad a = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{d-1} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Using $a^\dagger$ and a, we can introduce the following operators representing the d–dimensional extension of the two–dimensional case:

$$N = a^\dagger\, a = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d-2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & d-1 \end{bmatrix} \qquad N' = a\, a^\dagger = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d-1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

The eigenvalues of the self–adjoint operator N are $0, 1, 2, \ldots, d-1$, and the eigenvector corresponding to the generic eigenvalue j is $|N = j\rangle = |\frac{j}{d-1}\rangle$.

One possible physical interpretation of N is that it describes the *number of particles* of physical systems consisting of a maximum number of $d-1$ particles. In order to add a particle to the j particles state $|N=j\rangle$ (thus making it switch to the "next" state $|N=j+1\rangle$) we apply the creation operator $a^\dagger$, while to remove a particle from this system (thus making it switch to the "previous" state $|N=j-1\rangle$) we apply the annihilation operator a. Since the maximum number of particles that can be simultaneously in the system is $d-1$, the application of the creation operator to a full $d-1$ particles system does not have any effect on the system, and returns the null vector. Analogously, the application of the annihilation operator to an empty particle system does not affect the system and returns the null vector as a result.

Another physical interpretation of operators $a^\dagger$ and a, by operator N, follows from the possibility of expressing the Hamiltonian as follows:

$$H = \varepsilon_0 \mathbb{I} + \Delta\varepsilon \, N = \varepsilon_0 \mathbb{I} + \Delta\varepsilon \, a^\dagger \, a$$

In this case $a^\dagger$ (resp., a) realizes the transition from the eigenstate of energy $\varepsilon_k = \varepsilon_0 + j\,\Delta\varepsilon$ to the "next" (resp., "previous") eigenstate of energy $\varepsilon_{j+1} = \varepsilon_0 + (j+1)\,\Delta\varepsilon$ (resp., $\varepsilon_{j-1} = \varepsilon_0 + (j-1)\,\Delta\varepsilon$) for any $0 \le j < d-1$ (resp., $0 < j \le d-1$), while it collapses the last excited (resp., ground) state of energy $\varepsilon_0 + (d-1)\,\Delta\varepsilon$ (resp., ε_0) to the null vector.

As is well known, for a fixed integer $d \ge 2$ the angular momentum based on the Hilbert space $\mathbb{C}^d$ consists of the triple of self–adjoint operators $\mathbf{J} = (J_x, J_y, J_z)$. Moreover, for $k = \frac{d-1}{2}$, the real value $k(k+1)$ is an eigenvalue of the operator $\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2$. The matrix representation of the z component of this angular momentum with respect to the orthonormal basis of its eigenvectors is:

$$J_z = \begin{bmatrix} \frac{d-1}{2} & 0 & \dots & 0 & 0 \\ 0 & \frac{d-3}{2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{3-d}{2} & 0 \\ 0 & 0 & \dots & 0 & \frac{1-d}{2} \end{bmatrix}$$

Thus, the z component of the angular momentum can assume d possible eigenvalues: $m = \frac{d-(2j+1)}{2}$ for $j \in \{0, 1, \dots, d-1\}$ with corresponding eigenvectors $|J_z = \frac{d-(2j+1)}{2}\rangle = |\frac{j}{d-1}\rangle$.

Let us consider the two operators J_+ and J_- on the Hilbert space $\mathbb{C}^d$ which are obtained from the general angular momentum operators as:

$$J_+ = J_x + iJ_y \qquad\qquad J_- = J_x - iJ_y$$

The operators J_+ and J_- are non–Hermitian, adjoints of each other, and satisfy the canonical commutation relation $[J_+, J_-] = 2J_z$. In matrix form they can be expressed as follows:

$$J_+ = \begin{bmatrix} 0 & \sqrt{d-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{2(d-2)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{2(d-2)} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{d-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$J_- = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ \sqrt{d-1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \sqrt{2(d-2)} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sqrt{2(d-2)} & 0 & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{d-1} & 0 \end{bmatrix}.$$

As is well known, the action of operators J_+ and J_- on the vectors of the orthonormal basis of $\mathbb{C}^d$ formed by the eigenvectors of J_z is the following: $J_+|J_z = m\rangle = \sqrt{k(k+1) - m(m+1)}|J_z = m+1\rangle$ for $m = -k, \ldots, k$ and $J_-|J_z = m\rangle = \sqrt{k(k+1) - m(m-1)}|J_z = m-1\rangle$ for $m = -k, \ldots, k$. Thus, we can interpret these operators as follows: the application of J_+ has the effect of changing the z component of the angular momentum to the next value. If applied to a system which has already a maximum value of J_z, J_+ leaves the system unchanged and returns the null vector. Analogously, the application of J_- has the effect of switching the system to the previous value of the z component of the angular momentum. If applied to a system which has already a minimum value of J_z, J_- does not affect the system and returns the null vector. In analogy to the creation and annihilation operators, we call J_+ the *spin–rising* operator and J_- the *spin–lowering* operator on $\mathbb{C}^d$.

The actions of J_+ and J_- on the vectors of the qudit orthonormal basis are the following: $J_+|\frac{j}{d-1}\rangle = \sqrt{j(d-j)}|\frac{j-1}{d-1}\rangle$ for $j \in \{1, 2 \ldots, d-1\}$, $J_+|0\rangle = \mathbf{0}$ and $J_-|\frac{j}{d-1}\rangle = \sqrt{(j+1)(d-(j+1))}|\frac{j+1}{d-1}\rangle$ for $j \in \{0, 1, \ldots, d-2\}$, $J_-|1\rangle = \mathbf{0}$. Thus, we have that J_+ behaves as a *spin–rising* and, simultaneously, as a *truth value annihilation* operator, whereas J_- behaves as a *spin–lowering* and as a *truth value creation* operator.

When dealing with two truth values, it holds:

$$a^\dagger = J_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad a = J_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore it holds also $N = J_-J_+$ and $N' = J_+J_-$, whereas in general, for $d > 2$, such equalities do not hold.

It is expedient to recall the definition of some gates introduced in [3] that play a special role from the logical point of view.

For any $n \geq 1$, the diametrical negation gate is the linear operator $F_{\neg}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$:

$$F_{\neg}^{(n)}(|x_1, \ldots, x_n\rangle) = |x_1, \ldots, x_{n-1}\rangle \otimes |1 - x_n\rangle.$$

For any $n \geq 1$, the intuitionistic gate is the linear operator $F_{\sim}^{(n,1)}$ such that for every element $|x_1, \ldots, x_{n+1}\rangle$ of the computational basis $\mathcal{B}^{(n+1)}$:

$$F_{\sim}^{(n,1)}(|x_1, \ldots, x_{n+1}\rangle) = \begin{cases} |x_1, \ldots, x_n\rangle \otimes |1\rangle & \text{if } x_n = 0 \text{ and } x_{n+1} = 0 \\ |x_1, \ldots, x_n\rangle \otimes |0\rangle & \text{if } x_n = 0 \text{ and } x_{n+1} = 1 \\ |x_1, \ldots, x_{n+1}\rangle & \text{otherwise} \end{cases}$$

For any $n \geq 1$, the necessity gate $F_{\Box}^{(n,1)}$ is the linear operator such that for every element $|x_1, \ldots, x_{n+1}\rangle$ of the computational basis $\mathcal{B}^{(n+1)}$:

$$F_{\Box}^{(n,1)}(|x_1, \ldots, x_{n+1}\rangle) = \begin{cases} |x_1, \ldots, x_n\rangle \otimes |1\rangle & \text{if } x_n = 1 \text{ and } x_{n+1} = 0 \\ |x_1, \ldots, x_n\rangle \otimes |0\rangle & \text{if } x_n = 1 \text{ and } x_{n+1} = 1 \\ |x_1, \ldots, x_{n+1}\rangle & \text{otherwise} \end{cases}$$

For any $m \geq 1$ and any $n \geq 1$, the min-conjunction gate $F_{\wedge}^{(m,n,1)}$ is the linear operator such that for every element $|x_1, \ldots, x_{m+n+1}\rangle$ of the computational basis $\mathcal{B}^{(m+n+1)}$:

$$F_{\wedge}^{(m,n,1)}(|x_1, \ldots, x_{m+n+1}\rangle) = \begin{cases} |x_1, \ldots, x_m, x_{m+n+1} - x_m, x_{m+2}, \ldots, x_{m+n}, x_m\rangle \\ \qquad \text{if } x_m < x_{m+n+1} \text{ and } x_{m+1} = 0 \\ |x_1, \ldots, x_m, 0, x_{m+2}, \ldots, x_{m+n}, x_{m+1} + x_m\rangle \\ \qquad \text{if } x_m = x_{m+n+1} \text{ and } 0 < x_{m+1} \leq 1 - x_m \\ |x_1, \ldots, x_{m+n+1}\rangle \text{ otherwise} \end{cases}$$

For any $m \geq 1$ and any $n \geq 1$, the max-disjunction gate $F_{\vee}^{(m,n,1)}$ is the linear operator such that for every element $|x_1, \ldots, x_{m+n+1}\rangle$ of the computational basis $\mathcal{B}^{(m+n+1)}$:

$$F_{\vee}^{(m,n,1)}(|x_1, \ldots, x_{m+n+1}\rangle) = \begin{cases} |x_1, \ldots, x_m, 1 - x_m + x_{m+n+1}, x_{m+2}, \ldots, x_{m+n}, x_m\rangle \\ \qquad \text{if } x_{m+n+1} < x_m \text{ and } x_{m+1} = 1 \\ |x_1, \ldots, x_m, 1, x_{m+2}, \ldots, x_{m+n}, x_{m+1} + x_m - 1\rangle \\ \qquad \text{if } x_m = x_{m+n+1} \text{ and } 1 - x_m \leq x_{m+1} < 1 \\ |x_1, \ldots, x_{m+n+1}\rangle \text{ otherwise} \end{cases}$$

For any $m \geq 1$ and any $n \geq 1$, the Łukasiewicz gate is the linear operator $F_{\to_L}^{(m,n,1)}$ such that for every element $|x_1, \ldots, x_{m+n+1}\rangle$ of the computational basis $\mathcal{B}^{(m+n+1)}$:

$$F_{\to_L}^{(m,n,1)}(|x_1, \ldots, x_{m+n+1}\rangle) = \begin{cases} |x_1, \ldots, x_{m+n-1}, x_m, 1 - x_m + x_{m+n}\rangle \\ \qquad \text{if } x_{m+n} < x_m \text{ amd } x_{m+n+1} = 1 \\ |x_1, \ldots, x_{m+n-1}, x_{m+n+1} + x_m - 1, 1\rangle \\ \qquad \text{if } x_{m+n} = x_m \text{ and } 1 - x_m \leq x_{m+n+1} < 1 \\ |x_1, \ldots, x_{m+n+1}\rangle \text{ otherwise} \end{cases}$$

For any $m \geq 1$ and any $n \geq 1$, the Gödel gate is the linear operator $F_{\to_G}^{(m,n,1)}$ such that for every element $|x_1, \ldots, x_{m+n+1}\rangle$ of the computational basis $\mathcal{B}^{(m+n+1)}$:

$$F_{\to_G}^{(m,n,1)}(|x_1, \ldots, x_{m+n+1}\rangle) = \begin{cases} |x_1, \ldots, x_{m+n-1}, 1, x_{m+n}\rangle \\ \qquad \text{if } x_{m+n} < x_m \text{ and } x_{m+n+1} = 1 \\ |x_1, \ldots, x_{m+n-1}, x_{m+n+1}, 1\rangle \\ \qquad \text{if } x_{m+n+1} < x_m \text{ and } x_{m+n} = 1 \\ |x_1, \ldots, x_{m+n+1}\rangle \text{ otherwise} \end{cases}$$

For any $m \geq 1$ and any $n \geq 1$, the Monteiro gate is the linear operator $F_{\to_M}^{(m,n,1)}$ such that for every element $|x_1, \ldots, x_{m+n+1}\rangle$ of the computational basis $\mathcal{B}^{(m+n+1)}$:

$$F_{\to_M}^{(m,n,1)}(|x_1, \ldots, x_{m+n+1}\rangle) = \begin{cases} |x_1, \ldots, x_{m+n-1}, 1, x_{m+n}\rangle \\ \qquad \text{if } x_m = 1 \text{ and } x_{m+n+1} = 1 \\ |x_1, \ldots, x_{m+n-1}, x_{m+n+1}, 1\rangle \\ \qquad \text{if } x_m = 1 \text{ and } x_{m+n} = 1 \\ |x_1, \ldots, x_{m+n+1}\rangle \text{ otherwise} \end{cases}$$

For any $m \geq 1$ and any $n \geq 1$, the truncated sum gate $F_{\oplus}^{(m,n,1)}$ is the linear operator such that for every element $|x_1, \ldots, x_{m+n+1}\rangle$ of the computational basis $\mathcal{B}^{(m+n+1)}$:

$$F_{\oplus}^{(m,n,1)}(|x_1, \ldots, x_{m+n+1}\rangle) = \begin{cases} |x_1, \ldots, x_{m+n-1}, 1 - x_m, x_m + x_{m+n}\rangle \\ \qquad \text{if } x_m + x_{m+n} < 1 \text{ and } x_{m+n+1} = 1 \\ |x_1, \ldots, x_{m+n-1}, x_{m+n+1} - x_m, 1\rangle \\ \qquad \text{if } x_m \leq x_{m+n+1} < 1 \text{ and } x_{m+n} = 1 - x_m \\ |x_1, \ldots, x_{m+n+1}\rangle \text{ otherwise} \end{cases}$$

For any $m \geq 1$ and any $n \geq 1$, the Łukasiewicz conjunction gate $F_{\odot}^{(m,n,1)}$ is the linear operator such that for every element $|x_1, \ldots, x_{m+n+1}\rangle$ of the computational basis $\mathcal{B}^{(m+n+1)}$:

$$F_{\odot}^{(m,n,1)}(|x_1, \ldots, x_{m+n+1}\rangle) = \begin{cases} |x_1, \ldots, x_{m+n-1}, 1 - x_m, x_m + x_{m+n} - 1\rangle \\ \qquad \text{if } x_m + x_{m+n} > 1 \text{ and } x_{m+n+1} = 0 \\ |x_1, \ldots, x_{m+n-1}, x_{m+n} + x_{m+n+1}, 0\rangle \\ \qquad \text{if } 0 < x_{m+n+1} \leq x_m \text{ and } x_{m+n} = 1 - x_m \\ |x_1, \ldots, x_{m+n+1}\rangle \text{ otherwise} \end{cases}$$

For any $m \geq 1$ and any $n \geq 1$, the Goguen gate is the linear operator $N_{\to_\Pi}^{(m,n,1)}$ such that for every element $|x_1, \ldots, x_{m+n+1}\rangle$ of the computational basis $\mathcal{B}^{(m+n+1)}$:

$$N_{\to_\Pi}^{(m,n,1)}(|x_1, \ldots, x_{m+n+1}\rangle) = \begin{cases} |x_1, \ldots, x_{m+n-1}, 1 + x_{m+n} - \frac{x_{m+n}}{x_m}, \frac{x_{m+n}}{x_m}\rangle \\ \qquad \text{if } x_m, x_{m+n} \in GL_p,\ x_{m+n} < x_m \text{ and } x_{m+n+1} = 1 \\ |x_1, \ldots, x_{m+n-1}, x_{m+n} + x_{m+n+1} - 1, 1\rangle \\ \qquad \text{if } x_m, x_m x_{m+n+1} \in GL_p,\ x_m > 0,\ x_{m+n+1} < 1 \\ \qquad \text{and } x_{m+n} + x_{m+n+1} - 1 = x_m x_{m+n+1} \\ |x_1, \ldots, x_{m+n+1}\rangle \text{ otherwise} \end{cases}$$

where $GL_p = \{0\} \cup \{\frac{1}{2^j} | j \in \mathbb{Z} \text{ and } 0 \leq j \leq p-2\}$. The Goguen implication requires truth values which are implemented as non-equispaced rational numbers. If we let $d = 2^{p-2}+1$ then all the numbers of GL_p are also elements of $\left\{0, \frac{1}{d-1}, \frac{2}{d-1}, \ldots, 1\right\}$. This means that we can use a specially designed d-valued gate to compute the Goguen implication for a p-valued logic.

Note that the above gates are self-reversible. Besides, when $d = 2$, $F_{\to_L}^{(m,n,1)} = F_{\to_G}^{(m,n,1)} = F_{\to_M}^{(m,n,1)} = N_{\to_\Pi}^{(m,n,1)}$ and when $d = 3$, $F_{\to_G}^{(m,n,1)} = N_{\to_\Pi}^{(m,n,1)}$.

The quantum logical gates we have considered so far are, in a sense, "semiclassical". A quantum logical behaviour only emerges in the case where our gates are applied to superpositions. When restricted to basis elements, such operators turn out to behave as classical (reversible) truth-functions.

The diametrical negation can be uniformly defined on the set $\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{H}^{(n)}$ for any density operator ρ of $\mathcal{H}^{(n)}$ in the expected way:

$$\mathtt{Not}(\rho) := F_{\neg}^{(n)}\,\rho\,F_{\neg}^{(n)}$$

On this basis, an intuitionistic negation `INeg`, an anti-intuitionistic negation `Con`, a possibility `Pos`, a necessity `Nec` can be defined for any density operator in $\mathcal{H}^{(n+1)}$:

$$\begin{array}{l}
\mathtt{INeg}(\rho) := F_{\sim}^{(n,1)}\,\rho\,F_{\sim}^{(n,1)} \text{ if } Red_{[n,1]}^{(2)}(\rho) = \mathsf{P}_{|0\rangle}, \\
\mathtt{Con}(\rho) := F_{\Box}^{(n,1)}\,\rho\,F_{\Box}^{(n,1)} \text{ if } Red_{[n,1]}^{(2)}(\rho) = \mathsf{P}_{|1\rangle}, \\
\mathtt{Pos}(\rho) := F_{\sim}^{(n,1)}\,\rho\,F_{\sim}^{(n,1)} \text{ if } Red_{[n,1]}^{(2)}(\rho) = \mathsf{P}_{|1\rangle}, \\
\mathtt{Nec}(\rho) := F_{\Box}^{(n,1)}\,\rho\,F_{\Box}^{(n,1)} \text{ if } Red_{[n,1]}^{(2)}(\rho) = \mathsf{P}_{|0\rangle}.
\end{array}$$

Besides, a Łukasiewicz conjunction `And`, disjunction `Or` and implication `ŁImp`, a Gödel implication `GImp`, a Monteiro implication `MImp`, a MV-conjunction `ŁOr` and MV-disjunction `ŁAnd`, a Goguen implication `NImp` can be defined for any density operator ρ in $\mathcal{H}^{(m+n+1)}$:

$$\begin{array}{l}
\mathtt{And}(\rho) := F_{\wedge}^{(m,n,1)}\,\rho\,F_{\wedge}^{(m,n,1)} \text{ if } Red_{[m,1,n]}^{(2)}(\rho) = \mathsf{P}_{|0\rangle}, \\
\mathtt{Or}(\rho) := F_{\vee}^{(m,n,1)}\,\rho\,F_{\vee}^{(m,n,1)} \text{ if } Red_{[m,1,n]}^{(2)}(\rho) = \mathsf{P}_{|1\rangle}, \\
\mathtt{ŁImp}(\rho) := F_{\to_L}^{(m,n,1)}\,\rho\,F_{\to_L}^{(m,n,1)} \text{ if } Red_{[m,n,1]}^{(3)}(\rho) = \mathsf{P}_{|1\rangle}, \\
\mathtt{GImp}(\rho) := F_{\to_G}^{(m,n,1)}\,\rho\,F_{\to_G}^{(m,n,1)} \text{ if } Red_{[m,n,1]}^{(3)}(\rho) = \mathsf{P}_{|1\rangle}, \\
\mathtt{MImp}(\rho) := F_{\to_M}^{(m,n,1)}\,\rho\,F_{\to_M}^{(m,n,1)} \text{ if } Red_{[m,n,1]}^{(3)}(\rho) = \mathsf{P}_{|1\rangle}, \\
\mathtt{ŁOr}(\rho) := F_{\oplus}^{(m,n,1)}\,\rho\,F_{\oplus}^{(m,n,1)} \text{ if } Red_{[m,n,1]}^{(3)}(\rho) = \mathsf{P}_{|1\rangle}, \\
\mathtt{ŁAnd}(\rho) := F_{\odot}^{(m,n,1)}\,\rho\,F_{\odot}^{(m,n,1)} \text{ if } Red_{[m,n,1]}^{(3)}(\rho) = \mathsf{P}_{|0\rangle}, \\
\mathtt{NImp}(\rho) := N_{\to_\Pi}^{(m,n,1)}\,\rho\,N_{\to_\Pi}^{(m,n,1)} \text{ if } Red_{[m,n,1]}^{(3)}(\rho) = \mathsf{P}_{|1\rangle}.
\end{array}$$

In [4] two universal gates f_d^1 and m_d for finite-valued reversible and conservative logics are introduced. Using the quantum implementation of the gate f_d^1, one can realize `Not`, `INeg`, `Pos`, `And`, `Or`, `ŁImp`, `GImp`, while using the quantum implementation of the gate m_d, one is able to realize `Not`, `INeg`, `Pos`, `Nec`, `ŁOr`, `ŁAnd`.

One important feature of all many–valued connectives now presented is that they are equal to the analogous Boolean connectives when only falsity and truth are involved.

The following theorem describes some interesting relations between the probability function p and some logical gates.

Theorem 1. *[7]*
Let ρ and σ be two density operators of $\mathcal{H}^{(m)}$ and $\mathcal{H}^{(n)}$ respectively. The following properties hold:

(i) $\mathrm{p}(\mathtt{Not}(\rho)) = 1 - \mathrm{p}(\rho)$;

(ii) $\mathrm{p}(\mathtt{INeg}(\rho)) = \mathrm{p}_0(\rho)$;

(iii) $\mathrm{p}(\mathtt{Con}(\rho)) = 1 - \mathrm{p}_1(\rho)$;

(iv) $\mathrm{p}(\mathtt{Pos}(\rho)) = 1 - \mathrm{p}_0(\rho)$;

(v) $\mathrm{p}(\mathtt{Nec}(\rho)) = \mathrm{p}_1(\rho)$;

(vi) $\mathrm{p}(\mathtt{And}(\rho \otimes \mathtt{P}_{|0\rangle} \otimes \sigma)) = \sum_{j=2}^{d-1} \sum_{k=1}^{j-1} \frac{k}{d-1} \mathrm{p}_{\frac{j}{d-1}}(\rho) \mathrm{p}_{\frac{k}{d-1}}(\sigma) + \sum_{j=1}^{d-1} \sum_{k=j}^{d-1} \frac{j}{d-1} \mathrm{p}_{\frac{j}{d-1}}(\rho) \mathrm{p}_{\frac{k}{d-1}}(\sigma)$;

(vii) $\mathrm{p}(\mathtt{Or}(\rho \otimes \mathtt{P}_{|1\rangle} \otimes \sigma)) = \sum_{j=1}^{d-1} \sum_{k=0}^{j-1} \frac{j}{d-1} \mathrm{p}_{\frac{j}{d-1}}(\rho) \mathrm{p}_{\frac{k}{d-1}}(\sigma) + \sum_{j=0}^{d-1} \sum_{k=j}^{d-1} \frac{k}{d-1} \mathrm{p}_{\frac{j}{d-1}}(\rho) \mathrm{p}_{\frac{k}{d-1}}(\sigma)$;

(viii) $\mathrm{p}(\text{\L}\mathtt{Imp}(\rho \otimes \sigma \otimes \mathtt{P}_{|1\rangle})) = \sum_{j=0}^{d-1} \sum_{k=0}^{j-1} (1 - \frac{j-k}{d-1}) \mathrm{p}_{\frac{j}{d-1}}(\rho) \mathrm{p}_{\frac{k}{d-1}}(\sigma) + \sum_{j=0}^{d-1} \sum_{k=j}^{d-1} \mathrm{p}_{\frac{j}{d-1}}(\rho) \mathrm{p}_{\frac{k}{d-1}}(\sigma)$;

(ix) $\mathrm{p}(\mathtt{GImp}(\rho \otimes \sigma \otimes \mathtt{P}_{|1\rangle})) = \sum_{j=2}^{d-1} \sum_{k=1}^{j-1} \frac{k}{d-1} \mathrm{p}_{\frac{j}{d-1}}(\rho) \mathrm{p}_{\frac{k}{d-1}}(\sigma) + \sum_{j=0}^{d-1} \sum_{k=j}^{d-1} \mathrm{p}_{\frac{j}{d-1}}(\rho) \mathrm{p}_{\frac{k}{d-1}}(\sigma)$;

(x) $\mathrm{p}(\mathtt{MImp}(\rho \otimes \sigma \otimes \mathtt{P}_{|1\rangle})) = 1 - \mathrm{p}_1(\rho)(1 - \mathrm{p}(\sigma))$;

(xi) $\mathrm{p}(\text{\L}\mathtt{Or}(\rho \otimes \sigma \otimes \mathtt{P}_{|1\rangle})) = \sum_{j=0}^{d-1} \sum_{k=0}^{d-2-j} \frac{j+k}{d-1} \mathrm{p}_{\frac{j}{d-1}}(\rho) \mathrm{p}_{\frac{k}{d-1}}(\sigma) + \sum_{j=0}^{d-1} \sum_{k=d-1-j}^{d-1} \mathrm{p}_{\frac{j}{d-1}}(\rho) \mathrm{p}_{\frac{k}{d-1}}(\sigma)$;

(xii) $\mathrm{p}(\text{\L}\mathtt{And}(\rho \otimes \sigma \otimes \mathtt{P}_{|0\rangle})) = \sum_{j=0}^{d-1} \sum_{k=d-1-j}^{d-1} (\frac{j+k}{d-1} - 1) \mathrm{p}_{\frac{j}{d-1}}(\rho) \mathrm{p}_{\frac{k}{d-1}}(\sigma)$;

(xiii) $\mathrm{p}(\mathtt{NImp}(\rho \otimes \sigma \otimes \mathtt{P}_{|1\rangle})) = \sum_{j=2}^{d-1} \sum_{k=1}^{j-1} \frac{k}{j} \mathrm{p}_{\frac{j}{d-1}}(\rho) \mathrm{p}_{\frac{k}{d-1}}(\sigma) + \sum_{j=0}^{d-1} \sum_{k=j}^{d-1} \mathrm{p}_{\frac{j}{d-1}}(\rho) \mathrm{p}_{\frac{k}{d-1}}(\sigma)$.

4. Generalized Pauli matrices

Matrix bases can be used to decompose density matrices associated to states of d-dimensional quantum systems. For qubits, an important basis is formed by the identity matrix and by the three Pauli matrices. A density matrix can be expressed by a 3-dimensional vector, the Bloch vector, that lies within the Poincaré-Bloch ball (sphere of radius 1). In higher dimensions, two bases play an important role: the generalized Pauli basis and the Weyl operator basis.

For any j, k, l such that $1 \leq j \leq d^2 - 1$ and $0 \leq k < l \leq d - 1$, the generalized Pauli matrices σ_j on $\mathbb{C}^d$ can be defined as follows:

$$\sigma_j = \begin{cases} |k\rangle\langle l| + |l\rangle\langle k| \\ \qquad \text{if } j \leq \frac{d(d-1)}{2} \text{ and } j = \frac{k(1-k)}{2} + (d-2)k + l; \\ -i|k\rangle\langle l| + i|l\rangle\langle k| \\ \qquad \text{if } \frac{d(d-1)}{2} < j \leq d(d-1) \text{ and } j = \frac{d(d-1)+k(1-k)}{2} + (d-2)k + l; \\ \sqrt{\frac{2}{l(l+1)}}\Big(\sum_{k=0}^{l-1} |k\rangle\langle k| - l|l\rangle\langle l| \Big) \\ \qquad \text{if } j > d(d-1) \text{ and } j = d(d-1) + l. \end{cases}$$

They are the standard $SU(d)$ generators. In particular, $\frac{d(d-1)}{2}$ matrices are symmetric, $\frac{d(d-1)}{2}$ matrices are antisymmetric, $d-1$ matrices are diagonal. Let ρ be a density operator of $\mathbb{C}^d$. The expansion of ρ with respect to the orthogonal basis $\{\mathtt{I}^{(1)}, \sigma_j : 1 \leq j \leq d^2 - 1\}$ is

$$\rho = \frac{1}{d}\Big(\mathtt{I}^{(1)} + \sqrt{\frac{d(d-1)}{2}} \sum_{j=1}^{d^2-1} b_j \sigma_j\Big),$$

where $b_j = \sqrt{\frac{d}{2(d-1)}} \mathrm{tr}(\rho\, \sigma_j) \in \mathbb{R}$.

$b = (b_1, \ldots, b_{d^2-1})$ represents the Bloch vector associated to ρ with respect to the basis $\{\mathtt{I}^{(1)}, \sigma_j : 1 \leq j \leq d^2 - 1\}$, that lies within a Bloch ball (hypersphere of radius 1). The Bloch vector has real components that can be expressed as expectation values of measurable quantities. For example, when $d = 3$, we obtain the Gell-Mann Hermitian matrices and the Bloch vector can be expressed as expectation values of spin 1 operators.

For any k, l such that $0 \leq k \leq d - 1$ and $0 \leq l \leq d - 1$, we have:

$$|k\rangle\langle l| = \begin{cases} \frac{1}{2}(\sigma_{\frac{k(1-k)}{2}+(d-2)k+l} + i\,\sigma_{\frac{d(d-1)+k(1-k)}{2}+(d-2)k+l}) & \text{if } k < l; \\ \frac{1}{2}(\sigma_{\frac{l(1-l)}{2}+(d-2)l+k} - i\,\sigma_{\frac{d(d-1)+l(1-l)}{2}+(d-2)l+k}) & \text{if } k > l; \\ \frac{1}{d}\mathtt{I}^{(1)} - \sqrt{\frac{l}{2(l+1)}}\sigma_{d^2-d+l} + \sum_{j=0}^{d-l-2} \frac{1}{\sqrt{2(j+l+1)(j+l+2)}}\sigma_{d^2-d+j+l+1} & \text{if } k = l. \end{cases}$$

For any j, k, l such that $0 \leq j \leq d^2-1$, $0 \leq k \leq d-1$ and $0 \leq l \leq d-1$, the Weyl operators W_j on $\mathbb{C}^d$ can be defined as follows:

$$W_j = \sum_{m=0}^{d-1} \omega^{km} |m\rangle\langle m+l \mod d|,$$

where $\omega = e^{\frac{2\pi i}{d}}$, $j = kd + l$ and $\mod d$ is the modulo d.

Weyl operators are non-Hermitian but unitary and form an orthonormal basis of the Hilbert space $\mathbb{C}^d$. In particular, $\mathrm{tr}(W_j^\dagger\, W_{j'}) = d\,\delta_{j\,j'}$. Clearly, $W_0 = \mathtt{I}^{(1)}$ and for any j such that $1 \leq j \leq d^2-1$, $\mathrm{tr}(W_j) = 0$. Note that the Weyl basis $\{W_0, W_1, W_2, W_3\} = \{\mathtt{I}^{(1)}, \sigma_1, \sigma_3, i\sigma_2\}$ coincides with the Pauli basis for $d = 2$.

The shift operator W_1 (in a cyclic vector space) and the clock (with d hours) operator W_d generalize σ_1 and σ_3, respectively.

We have:

$$W_j = W_d^{\lfloor \frac{j}{d} \rfloor}\, W_1^{j \bmod d}$$

where $W_d^0 = W_1^0 = \mathtt{I}^{(1)}$.

The following Vandermonde matrix generalize the Walsh-Hadamard matrix and it is used for discrete Fourier transformations:

$$V = \frac{1}{\sqrt{d}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{(d-1)} & \omega^{2(d-1)} & \cdots & \omega^{(d-1)^2} \\ 1 & \omega^{(d-2)} & \omega^{2(d-2)} & \cdots & \omega^{(d-1)(d-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega & \omega^2 & \cdots & \omega^{(d-1)} \end{bmatrix}$$

We have: $W_d = V\, W_1\, V^\dagger$.

The expansion of ρ with respect to the basis $\{W_j : 0 \leq j \leq d^2-1\}$ is

$$\rho = \frac{1}{d}\Big(\mathtt{I}^{(d)} + \sqrt{d-1} \sum_{j=1}^{d^2-1} b_j W_j\Big),$$

where $b_j = \frac{1}{\sqrt{d-1}} \mathrm{tr}(\rho\, W_j^\dagger) \in \mathbb{C}$.

$b = (b_1, \ldots, b_{d^2-1})$ represents the Bloch vector associated to ρ with respect to the basis $\{W_j : 0 \leq j \leq d^2-1\}$ that lies within a Bloch ball.

For any k, l such that $0 \leq k \leq d-1$ and $0 \leq l \leq d-1$, we have:

$$|k\rangle\langle l| = \frac{1}{d}\sum_{m=0}^{d-1} \omega^{-km}\, W_{md+(l-k \mod d)}.$$

Weyl operators has been used in quantum teleportation [1]. They are also useful in the study of the geometry of entanglement.

The gate W_1 and the following two gates play a special role in state tomography as stated by the following theorem.

Definition 4. Hadamard gate.
For any $d \geq 2$, the Hadamard gate (for the first two eigenvectors) is the linear operator $\widetilde{\sqrt{\mathtt{I}}}$ such that for every element $|x\rangle$ of the computational basis $\mathcal{B}^{(1)}$:

$$\widetilde{\sqrt{\mathtt{I}}}(|x\rangle) = \begin{cases} \frac{1}{\sqrt{2}}(|0\rangle + |\frac{1}{d-1}\rangle) & \text{if } x = 0 \\ \frac{1}{\sqrt{2}}(|0\rangle - |\frac{1}{d-1}\rangle) & \text{if } x = \frac{1}{d-1} \\ |x\rangle \text{ otherwise} \end{cases}$$

Definition 5. Square root of $\widetilde{\mathtt{Not}}$ gate.
For any $d \geq 2$, the square root of $\widetilde{\mathtt{Not}}$ gate (for the first two eigenvectors) is the linear operator $\widetilde{\sqrt{\mathtt{Not}}}$ such that for every element $|x\rangle$ of the computational basis $\mathcal{B}^{(1)}$:

$$\widetilde{\sqrt{\mathtt{Not}}}(|x\rangle) = \begin{cases} \frac{1}{2}((1+i)|0\rangle + (1-i)|\frac{1}{d-1}\rangle) & \text{if } x = 0 \\ \frac{1}{2}((1-i)|0\rangle + (1+i)|\frac{1}{d-1}\rangle) & \text{if } x = \frac{1}{d-1} \\ |x\rangle \text{ otherwise} \end{cases}$$

Clearly, $\widetilde{\sqrt{\mathtt{I}}}, \widetilde{\sqrt{\mathtt{Not}}}, W_1$ have the following matrix forms:

$$\widetilde{\sqrt{\mathtt{I}}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad \widetilde{\sqrt{\mathtt{Not}}} = \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} & 0 & \cdots & 0 \\ \frac{1-i}{2} & \frac{1+i}{2} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad W_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & \vdots & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Theorem 2. *[7]*
For any $d \geq 2$ and any density operator ρ of $\mathbb{C}^d$, there exist a function $f : [0,1]^{d-1} \to \mathbb{C}^d$ such that

$$\rho = f(p_1, \ldots, p_h, \ldots, p_{d^2-1}),$$

where p_h is a probability of a combination of three gates $\widetilde{\sqrt{\mathtt{I}}}, \widetilde{\sqrt{\mathtt{Not}}}, W_1$ applied to ρ.

5. Fuzzy representation for quantum logical gates

The following theorem shows some interesting relations between some logical gates and continuous t-norms by probability values. The probability of the gates can be described in terms of the corresponding logical operation and $\oplus, \cdot$ [10].

Theorem 3. *Let ρ and σ be two density operators of $\mathcal{H}^{(m)}$ and $\mathcal{H}^{(n)}$ respectively. The following properties hold:*

(i) $\mathrm{p}(\mathtt{Not}(\rho)) = \oplus_{j=0}^{d-1}(\neg\frac{j}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)$;

(ii) $\mathrm{p}(\mathtt{INeg}(\rho \otimes \mathtt{P}_{|0\rangle})) = \oplus_{j=0}^{d-1}(\sim \frac{j}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)$;

(iii) $\mathrm{p}(\mathtt{Con}(\rho \otimes \mathtt{P}_{|1\rangle})) = \oplus_{j=0}^{d-1}(\flat\frac{j}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)$;

(iv) $\mathrm{p}(\mathtt{Pos}(\rho \otimes \mathtt{P}_{|1\rangle})) = \oplus_{j=0}^{d-1}(\Diamond\frac{j}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)$;

(v) $\mathrm{p}(\mathtt{Nec}(\rho \otimes \mathtt{P}_{|0\rangle})) = \oplus_{j=0}^{d-1}(\Box\frac{j}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)$;

(vi) $\mathrm{p}(\mathtt{And}(\rho \otimes \mathtt{P}_{|0\rangle} \otimes \sigma)) = \oplus_{j=0}^{d-1} \oplus_{k=0}^{j-1} (\frac{j}{d-1} \wedge \frac{k}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)\mathrm{p}_{\frac{k}{d-1}}(\sigma)$;

(vii) $\mathrm{p}(\mathtt{Or}(\rho \otimes \mathtt{P}_{|1\rangle} \otimes \sigma)) = \oplus_{j=0}^{d-1} \oplus_{k=0}^{j-1} (\frac{j}{d-1} \vee \frac{k}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)\mathrm{p}_{\frac{k}{d-1}}(\sigma)$;

(viii) $\mathrm{p}(\text{Ł}\mathtt{Imp}(\rho \otimes \sigma \otimes \mathtt{P}_{|1\rangle})) = \oplus_{j=0}^{d-1} \oplus_{k=0}^{j-1} (\frac{j}{d-1} \to_{Ł} \frac{k}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)\mathrm{p}_{\frac{k}{d-1}}(\sigma)$;

(ix) $\mathrm{p}(\mathtt{GImp}(\rho \otimes \sigma \otimes \mathtt{P}_{|1\rangle})) = \oplus_{j=0}^{d-1} \oplus_{k=0}^{j-1} (\frac{j}{d-1} \to_{G} \frac{k}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)\mathrm{p}_{\frac{k}{d-1}}(\sigma)$;

(x) $\mathrm{p}(\mathtt{MImp}(\rho\otimes\sigma\otimes\mathtt{P}_{|1\rangle}))=\oplus_{j=0}^{d-1}\oplus_{k=0}^{j-1}(\frac{j}{d-1}\rightarrow_M\frac{k}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)\mathrm{p}_{\frac{k}{d-1}}(\sigma)$;

(xi) $\mathrm{p}(\mathtt{ŁOr}(\rho\otimes\sigma\otimes\mathtt{P}_{|1\rangle}))=\oplus_{j=0}^{d-1}\oplus_{k=0}^{j-1}(\frac{j}{d-1}\oplus\frac{k}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)\mathrm{p}_{\frac{k}{d-1}}(\sigma)$;

(xii) $\mathrm{p}(\mathtt{ŁAnd}(\rho\otimes\sigma\otimes\mathtt{P}_{|0\rangle}))=\oplus_{j=0}^{d-1}\oplus_{k=0}^{j-1}(\frac{j}{d-1}\odot\frac{k}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)\mathrm{p}_{\frac{k}{d-1}}(\sigma)$;

(xiii) $\mathrm{p}(\mathtt{NImp}(\rho\otimes\sigma\otimes\mathtt{P}_{|1\rangle}))=\oplus_{j=0}^{d-1}\oplus_{k=0}^{j-1}(\frac{j}{d-1}\rightarrow_N\frac{k}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)\mathrm{p}_{\frac{k}{d-1}}(\sigma)$.

PROOF.

(i) By theorem 1, $\mathrm{p}(\mathtt{Not}(\rho))=1-\mathrm{p}(\rho)=\sum_{j=0}^{d-1}\left(1-\frac{j}{d-1}\right)\mathrm{p}_{\frac{j}{d-1}}(\rho)$. Since $\mathrm{p}_{\frac{j}{d-1}}(\rho)\in[0,1]$ and $\sum_{j=0}^{d-1}\mathrm{p}_{\frac{j}{d-1}}(\rho)=1$, $\mathrm{p}(\mathtt{Not}(\rho))=\oplus_{j=0}^{d-1}(\neg\frac{j}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)$.

(ii)-(v) Similarly.

(vi) By theorem 1, $\mathrm{p}(\mathtt{And}(\rho\otimes\mathtt{P}_{|0\rangle}\otimes\sigma))=\sum_{j=2}^{d-1}\sum_{k=1}^{j-1}\frac{k}{d-1}\mathrm{p}_{\frac{j}{d-1}}(\rho)\mathrm{p}_{\frac{k}{d-1}}(\sigma)+\sum_{j=1}^{d-1}\sum_{k=j}^{d-1}\frac{j}{d-1}\mathrm{p}_{\frac{j}{d-1}}(\rho)\mathrm{p}_{\frac{k}{d-1}}(\sigma)=\sum_{j=0}^{d-1}\sum_{k=0}^{d-1}\min\{\frac{j}{d-1},\frac{k}{d-1}\}\mathrm{p}_{\frac{j}{d-1}}(\rho)\mathrm{p}_{\frac{k}{d-1}}(\sigma)$. Since $\mathrm{p}_{\frac{j}{d-1}}(\rho),\mathrm{p}_{\frac{k}{d-1}}(\sigma)\in[0,1]$ and $\sum_{j=0}^{d-1}\mathrm{p}_{\frac{j}{d-1}}(\rho)=\sum_{k=0}^{d-1}\mathrm{p}_{\frac{k}{d-1}}(\sigma)=1$, $\mathrm{p}(\mathtt{And}(\rho\otimes\mathtt{P}_{|0\rangle}\otimes\sigma))=\oplus_{j=0}^{d-1}\oplus_{k=0}^{j-1}(\frac{j}{d-1}\wedge\frac{k}{d-1})\mathrm{p}_{\frac{j}{d-1}}(\rho)\mathrm{p}_{\frac{k}{d-1}}(\sigma)$.

(vii)-(xiii) Similarly.

$\mathrm{p}(\mathtt{And}(Red_{[m,1,n]}^{(1)}(\rho)\otimes\mathtt{P}_{|0\rangle}\otimes Red_{[m,1,n]}^{(3)}(\rho)))$ will be called the *fuzzy component* of $\mathrm{p}(\mathtt{And}(\rho))$. The fuzzy component is not related to pure states and does not characterize entanglement. Indeed, for the following states ρ_j of $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ with $j=1,\ldots,4$, $\mathrm{p}(\mathtt{And}(\rho_j))$ corresponds to the fuzzy component $\mathrm{p}(Red_{[1,1,1]}^{(1)}(\rho_j))\mathrm{p}(Red_{[1,1,1]}^{(3)}(\rho_j))$:

$$\rho_1=\mathtt{P}_{\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)}\otimes\mathtt{P}_{|0\rangle}\otimes\mathtt{P}_{\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)}$$

$$\rho_2=\frac{1}{2}\mathtt{I}^{(1)}\otimes\mathtt{P}_{|0\rangle}\otimes\frac{1}{2}\mathtt{I}^{(1)}$$

$$\rho_3=\frac{1}{4}\mathtt{P}_{|0,0,0\rangle}+\frac{1}{4}\mathtt{P}_{|1,0,1\rangle}+\frac{1}{2}\mathtt{P}_{\frac{1}{\sqrt{2}}(|0,0,1\rangle+|1,0,0\rangle)}$$

$$\rho_4=\mathtt{P}_{\frac{1}{2}(|0,0,0\rangle+|0,0,1\rangle+|1,0,0\rangle-|1,0,1\rangle)}$$

but ρ_1 is a pure state, ρ_2 is a factorized state, ρ_3 is a separable state and ρ_4 is an entangled state.

For any $d_1, d_2 \geq 2$, a density operator ρ of $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ can be written as follows [12]: $\rho = \frac{1}{d_1 d_2}\mathtt{I}_{d_1} \otimes \mathtt{I}_{d_2} + \frac{1}{2d_1}\sum_{j=1}^{d_1^2-1} \mathrm{tr}(\rho(\mathtt{I}_{d_1} \otimes \sigma_j))\mathtt{I}_{d_1} \otimes \sigma_j + \frac{1}{2d_2}\sum_{j=1}^{d_2^2-1} \mathrm{tr}(\rho(\sigma_j \otimes \mathtt{I}_{d_2}))\sigma_j \otimes \mathtt{I}_{d_2} + \frac{1}{4}\sum_{i=1}^{d_1^2-1}\sum_{j=1}^{d_2^2-1} \mathrm{tr}(\rho(\sigma_i \otimes \sigma_j))\sigma_i \otimes \sigma_j$, where $\mathtt{I}_{d_1}$ and $\mathtt{I}_{d_2}$ are the identity operators of $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$, respectively.

Let ρ be a density operator of $\mathbb{C}^2 \otimes \mathbb{C}^2$. Then, $\rho = \frac{1}{4}\left(\mathtt{I}^{(2)} + \sqrt{6}\sum_{j=1}^{15} b_j \sigma_j\right)$. Let us now consider the following operators:
$\hat{\rho} = \frac{1}{4}(\mathtt{I}^{(1)} \otimes \mathtt{P}_{|0\rangle} \otimes \mathtt{I}^{(1)} + \sum_{j=1}^{3}[\mathrm{tr}(\rho(\sigma_j \otimes \mathtt{I}^{(1)}))\sigma_j \otimes \mathtt{P}_{|0\rangle} \otimes \mathtt{I}^{(1)} + \mathrm{tr}(\rho(\mathtt{I}^{(1)} \otimes \sigma_j))\mathtt{I}^{(1)} \otimes \mathtt{P}_{|0\rangle} \otimes \sigma_j + \sum_{i=1}^{3} \mathrm{tr}(\rho(\sigma_i \otimes \sigma_j))\sigma_i \otimes \mathtt{P}_{|0\rangle} \otimes \sigma_j]);$

$$M(\rho) := \frac{1}{4}\sum_{i=1}^{3}\sum_{j=1}^{3}(\mathrm{tr}(\rho(\sigma_i \otimes \sigma_j)) - \mathrm{tr}(\rho(\sigma_i \otimes \mathtt{I}^{(1)})) - \mathrm{tr}(\rho(\mathtt{I}^{(1)} \otimes \sigma_j)))\sigma_i \otimes \mathtt{P}_{|0\rangle} \otimes \sigma_j.$$

We have:

$$\hat{\rho} = Red^{(1)}_{[1,1]}(\rho) \otimes \mathtt{P}_{|0\rangle} \otimes Red^{(2)}_{[1,1]}(\rho) + M(\rho);$$

$$\mathrm{p}(Red^{(1)}_{[1,1]}(\rho)) = \frac{1 - \sqrt{2}b_{14} - b_{15}}{2};$$

$$\mathrm{p}(Red^{(2)}_{[1,1]}(\rho)) = \frac{2 - \sqrt{6}b_{13} + \sqrt{2}b_{14} - 2b_{15}}{4};$$

$$\mathrm{p}(\mathtt{And}(\hat{\rho})) = \frac{1 - 3b_{15}}{4}.$$

Thus, $\mathrm{p}(\mathtt{And}(M(\hat{\rho}))) \neq 0$ iff $\mathrm{p}(\mathtt{And}(\hat{\rho}))$ does not correspond to the fuzzy component $(\mathrm{p}(\mathtt{And}(\hat{\rho})) \neq \mathrm{p}(Red^{(1)}_{[1,1]}(\rho))\mathrm{p}(Red^{(2)}_{[1,1]}(\rho)))$.

Note that even when $\mathrm{p}(\mathtt{And}(\hat{\rho}))$ corresponds to the fuzzy component, $\mathrm{p}(Red^{(1)}_{[1,1]}(\rho)) = \frac{1-\sqrt{2}b_{14}-b_{15}}{2}$ and $\mathrm{p}(Red^{(2)}_{[1,1]}(\rho)) = \frac{1-3b_{15}}{2(1-\sqrt{2}b_{14}-b_{15})}$ may depend on each other. Indeed, for $\rho = \mathtt{P}_{(1-k)|0,0,0\rangle + \sqrt{k(1-k)}|0,0,1\rangle + \sqrt{k(1-k)}|1,0,0\rangle - k|1,0,1\rangle}$,

$$\mathrm{p}(Red^{(1)}_{[1,1,1]}(\rho)) = k,$$

$$\mathrm{p}(Red^{(3)}_{[1,1,1]}(\rho)) = k,$$

$$\mathrm{p}(\mathtt{And}(\hat{\rho})) = k^2,$$

where $k \in (0,1)$.

6. Isotropic states and Werner states

An *isotropic state* ρ is a bipartite state of a space $\mathcal{H}^{(2)}$ that satisfies the following condition for any unitary operator U of $\mathcal{H}^{(n)}$:

$$(U \otimes U^*)\rho(U^\dagger \otimes U^{*\dagger}) = \rho.$$

Hence, any isotropic state ρ of $\mathcal{H}^{(2)}$ is invariant under the whole $U\otimes U^*$-group of transformations (where U is any unitary operator of $\mathcal{H}^{(n)}$ and U^* is the complex conjugate of U).

One can prove that the class of all isotropic states of $\mathcal{H}^{(2)}$ can be represented as a one-parameter manifold of states.

Lemma 4. *Any isotropic state of the space $\mathcal{H}^{(2)}$ can be represented as follows:*

$$\rho_\iota = \frac{1}{d^2-1}\Big[\Big(1-\frac{\iota}{d}\Big)\,\mathtt{I}^{(2)} + \Big(\iota - \frac{1}{d}\Big)\,\mathtt{P}^{(2)}\Big],$$

where $0 \leq \iota \leq d$ and $\mathtt{P}^{(2)} = \sum_{i,j=0}^{d-1} |\frac{i}{d-1}, \frac{i}{d-1}\rangle\langle\frac{j}{d-1}, \frac{j}{d-1}|$.

Notice that the number ι represents the expectation value of $\mathtt{P}^{(2)}$ for the state ρ_ι (i.e. $\iota = \mathrm{tr}(P^{(2)}\rho_\iota)$) and ρ_ι can be viewed as a mixture of a maximally chaotic state and a singlet state:

$$\rho_\iota = \frac{1-\alpha}{d^2}\mathtt{I}^{(2)} + \frac{\alpha}{d}\mathtt{P}^{(2)},$$

where $\alpha = \frac{d\,\iota-1}{d^2-1}$.

The expansion of ρ with respect to the basis $\{\mathtt{I}^{(d)}, \sigma_j : 1 \leq j \leq d^2-1\}$ is

$$\rho_\iota = \frac{1}{d^2}\mathtt{I}^{(1)} \otimes \mathtt{I}^{(1)} + \frac{\alpha}{2d}\sum_{j=1}^{d^2-1} c_j \sigma_j \otimes \sigma_j,$$

where $c_j = \begin{cases} -1 & \text{if } \frac{d(d-1)}{2} < j \leq d(d-1); \\ 1 & \text{otherwise.} \end{cases}$

The expansion of ρ_ι with respect to the basis $\{W_j : 0 \leq j \leq d^2-1\}$ is

$$\rho_\iota = \frac{1}{d^2}\Big(\mathtt{I}^{(1)} \otimes \mathtt{I}^{(1)} + \alpha \sum_{j=1}^{d^2-1} W_j \otimes W_{j'}\Big),$$

where $j' = (-\lfloor \frac{j}{d} \rfloor \mod d)\, d + j \mod d$.

Similar results can be found for another interesting class of states that contains all *Werner states*, introduced in [15] in order to show that entangled bipartite states do not necessarily exhibit non-local correlations.

A *Werner state* is a bipartite state ρ of a space $\mathcal{H}^{(2)}$ that satisfies the following condition for any unitary operator U of $\mathcal{H}^{(1)}$:

$$(U \otimes U)\rho(U^\dagger \otimes U^\dagger) = \rho.$$

Hence, any Werner state is invariant under local unitary transformations.

As happens in the case of Werner states, one can also prove that the class of all Werner states of $\mathcal{H}^{(2)}$ can be represented as a one-parameter manifold of states.

Lemma 5. *Any Werner state of the space $\mathcal{H}^{(2)}$ can be represented as follows:*

$$\rho_w = \frac{1}{d^2-1}\Big[\left(1-\frac{w}{d}\right)\mathtt{I}^{(2)} + \left(w-\frac{1}{d}\right)\mathtt{SW}^{(1,1)}\Big],$$

where $-1 \le w \le 1$ (while $\mathtt{I}^{(2)}$ and $\mathtt{SW}^{(1,1)}$ are the identity operator and the swap-gate of the space $\mathcal{H}^{(2)}$, i.e., the linear operator such that, for every element $|x\rangle \otimes |y\rangle$ of the canonical basis, $\mathtt{SW}^{(1,1)}|x,y\rangle = |y,x\rangle$).

Notice that the number w represents the expectation value of $\mathtt{SW}^{(1,1)}$ for the state ρ_w (i.e. $w = \mathrm{tr}(\mathtt{SW}^{(1,1)}\rho_w)$).

7. Entanglement for isotropic states and Werner states

Unlike the general case, the probabilistic behavior of the holistic conjunction allows us to characterize entanglement both for isotropic states and for Werner states.

Lemma 6. *[13]*
Let ρ_ι be an isotropic state of $\mathcal{H}^{(2)}$.

$$E_F(\rho_\iota) = \begin{cases} h(s(\gamma(\iota)) + (1-\gamma(\iota))\log_2(d-1)) \ \mathit{if}\ \iota \in (1,d] \\ 0 \ \mathit{otherwise} \end{cases} \qquad ,$$

where $\gamma(\iota) = \frac{1}{d^2}(\sqrt{\iota} + \sqrt{(d-1)(d-\iota)})^2$, s is the binary Shannon entropy (i.e. $s(x) = -x\log_2 x - (1-x)\log_2(1-x)$) and h is the convex-hull of the inner expression (i.e. the largest convex curve nowhere larger than the given one).

On this basis one can prove that the entanglement of formation of isotropic states also can be represented in terms of the probabilistic behavior of the holistic conjunction.

Lemma 7. *Let ρ be a state of the Hilbert space $\mathcal{H}^{(3)}$ such that $\rho_\iota = Red^{(1,3)}_{[1,1,1]}(\rho)$ is an isotropic bipartite state of $\mathcal{H}^{(2)}$ and $Red^{(2)}_{[1,1,1]}(\rho) = \mathtt{P}_{|0\rangle}$. Then,*

1) $\mathrm{p}(\rho) = \mathrm{p}(\rho_\iota) = \mathrm{p}(Red^{(1)}_{[1,1]}(\rho_\iota)) = \mathrm{p}(Red^{(2)}_{[1,1]}(\rho_\iota)) = \frac{1}{2}$*;*

2) $\mathrm{p}(\mathtt{And}(\rho)) = \frac{2d-3+\iota}{6(d-1)}$.

Proof. 1) $\mathrm{p}(\rho) = \mathrm{tr}(\rho(\mathtt{I}^{(2)} \otimes E)) = \mathrm{tr}(\rho_\iota(\mathtt{I}^{(1)} \otimes E)) = \mathrm{tr}(Red^{(2)}_{[1,1]}(\rho_\iota)E) = \mathrm{tr}(Red^{(1)}_{[1,1]}(\rho_\iota)E) = \sum_{k=0}^{d-1} \frac{k}{d-1}\frac{1}{d^2-1}(d(1-\frac{\iota}{d}) + \iota - \frac{1}{d}) = \frac{1}{2}$.
2) $\mathrm{p}(\mathtt{And}(\rho)) = \mathrm{tr}\Big(F_\wedge^{(m,n,1)}\,\rho\,F_\wedge^{(m,n,1)}\,(\mathtt{I}^{(n-1)} \otimes E)\Big) = \sum_{k=0}^{d-1} \frac{k}{d-1}\frac{1}{d^2-1}(2(\sum_{j=k}^{d-1} 1 - \frac{\iota}{d}) - (1-\frac{\iota}{d}) + \iota - \frac{1}{d}) = \frac{2d-3+\iota}{6(d-1)} = \frac{1}{3} - \frac{1}{6d} - \frac{(d+1)\alpha}{6d}$, where $\alpha = \frac{d\,\iota - 1}{d^2-1}$.

In particular, for any entangled state ρ_ι (i.e. $\iota \in (1,d]$), we have: $\frac{1}{3} < \mathrm{p}(\mathtt{And}(\rho)) \leq \frac{1}{2}$. For any separable state ρ_ι (i.e. $\iota \in [0,1]$), we have: $\frac{1}{3} - \frac{1}{6(d-1)} \leq \mathrm{p}(\mathtt{And}(\rho)) \leq \frac{1}{3}$. For the factorized state $\rho_\iota = \frac{1}{d}\mathtt{I}^{(1)} \otimes \frac{1}{d}\mathtt{I}^{(1)}$ (i.e. $\iota = \frac{1}{d}$), we have: $\mathrm{p}(\mathtt{And}(\rho)) = \frac{1}{3} - \frac{1}{6d}$. Note that the fuzzy component $\mathrm{p}(\mathtt{And}(Red^{(1)}_{[1,1,1]}(\rho) \otimes \mathtt{P}_{|0\rangle} \otimes Red^{(3)}_{[1,1,1]}(\rho))) = \mathrm{p}(\frac{1}{d}\mathtt{I}^{(1)} \otimes \mathtt{P}_{|0\rangle} \otimes \frac{1}{d}\mathtt{I}^{(1)}) = \frac{1}{3} - \frac{1}{6d}$ does not depend on ι.

Theorem 8. *Let ρ be a state of the Hilbert space $\mathcal{H}^{(3)}$ such that $\rho_\iota = Red^{(1,3)}_{[1,1,1]}(\rho)$ is an isotropic bipartite state of $\mathcal{H}^{(2)}$ and $Red^{(2)}_{[1,1,1]}(\rho) = \mathtt{P}_{|0\rangle}$. Then,*

$$E_F(\rho_\iota) = \begin{cases} h\Big[s(\gamma(6(d-1)\mathrm{p}(\mathtt{And}(\rho)) - 2d + 3)) \\ +(1 - \gamma(6(d-1)\mathrm{p}(\mathtt{And}(\rho)) - 2d + 3))\log_2(d-1)\Big], \\ \quad \text{if } \frac{1}{3} < \mathrm{p}(\mathtt{And}(\rho)) \leq \frac{1}{2}; \\ 0, \ \text{otherwise.} \end{cases}$$

Proof. By Lemma 6 and Lemma 7.

A simple correlation connects the entanglement of formation for a Werner state ρ_w with the parameter w [14].

Lemma 9. *The entanglement of formation of any Werner space ρ_w of $\mathcal{H}^{(2)}$ is*

$$E_F(\rho_w) = \begin{cases} s\left(\frac{1-\sqrt{1-w^2}}{2}\right) & \text{if } w \in [-1,0]; \\ 0, & \text{otherwise} \end{cases}$$

where $s(x) = -x\log_2 x - (1-x)\log_2(1-x)$.

On this basis one can prove that the entanglement of formation of Werner states can be represented in terms of the probabilistic behavior of the holistic conjunction.

Lemma 10. *Let ρ be a state of the Hilbert space $\mathcal{H}^{(3)}$ such that $\rho_w = Red^{(1,3)}_{[1,1,1]}(\rho)$ is a Werner state of $\mathcal{H}^{(2)}$ and $Red^{(2)}_{[1,1,1]}(\rho) = \mathtt{P}_{|0\rangle}$. Then,*

1) $\mathrm{p}(\rho) = \mathrm{p}(\rho_w) = \mathrm{p}(Red^{(1)}_{[1,1]}(\rho_w)) = \mathrm{p}(Red^{(2)}_{[1,1]}(\rho_w)) = \frac{1}{2}$*;*

2) $\mathrm{p}(\mathtt{And}(\rho_w)) = \frac{2d-3+w}{6(d-1)}$*.*

Proof. Similar to Lemma 7.

Theorem 11. *Let ρ_w be a Werner state of $\mathcal{H}^{(2)}$.*

$$E_F(\rho_w) = \begin{cases} s\left(\frac{1-\sqrt{1-\left(6(d-1)\mathrm{p}(\mathtt{And}(\rho))-2d+3\right)^2}}{2}\right), \\ \quad \text{if } \frac{1}{3} - \frac{1}{3(d-1)} \leq \mathrm{p}(\mathtt{And}(\rho)) < \frac{1}{3} - \frac{1}{6(d-1)}; \\ 0, \quad \text{otherwise.} \end{cases}$$

Proof. By Lemma 9 and Lemma 10.

One might wonder whether the capacity of characterizing entanglement may depend on the specific features of the holistic conjunction. The answer to this question is negative. In fact, similar results can be obtained by using the gate `Or` instead of `And`.

Theorem 12. *Let ρ be a state of the Hilbert space $\mathcal{H}^{(3)}$ such that $\rho_\iota = Red^{(1,3)}_{[1,1,1]}(\rho)$ is an isotropic bipartite state of $\mathcal{H}^{(2)}$ and $Red^{(2)}_{[1,1,1]}(\rho) = \mathtt{P}_{|1\rangle}$. Then,*

1) $\mathrm{p}(\mathtt{Or}(\rho)) = \frac{4d-3-\iota}{6(d-1)}$*;*

2) $$E_F(\rho_\iota) = \begin{cases} h\Big[s(\gamma(6(1-d)\mathrm{p}(\mathtt{Or}(\rho)) + 4d-3)) \\ +(1-\gamma(6(1-d)\mathrm{p}(\mathtt{Or}(\rho)) + 4d-3))\log_2(d-1)\Big], \\ \quad \text{if } \frac{1}{2} < \mathrm{p}(\mathtt{Or}(\rho)) \leq \frac{2}{3}; \\ 0, \ \text{otherwise.} \end{cases}$$

PROOF. 1) $\mathrm{p}(\mathtt{Or}(\rho)) = \mathrm{tr}\Big(F_\vee^{(m,n,1)}\,\rho\,F_\vee^{(m,n,1)}\,(\mathtt{I}^{(n-1)}\otimes E)\Big) = \sum_{k=0}^{d-1}\frac{k}{d-1}\frac{1}{d^2-1}(2(\sum_{j=0}^{k}1-\frac{\iota}{d}) - (1-\frac{\iota}{d}) + \iota - 1 - \frac{1}{d}) = \frac{4d-3-\iota}{6(d-1)} = \frac{2}{3} + \frac{1}{6d} - \frac{(d+1)\alpha}{6d}$, where $\alpha = \frac{d\iota-1}{d^2-1}$.

2) By Lemma 6 and 1).

Similar results can also be obtained by using other gates that represent a binary function (in the Hilbert-space environment).

Theorem 13. *Let ρ_ι be an isotropic bipartite state (or a Werner state) of the Hilbert space $\mathcal{H}^{(2)}$. Then,*

(i) $\mathrm{p}(\text{Ł}\mathtt{Imp}(\rho_\iota \otimes \mathtt{P}_{|1\rangle})) = \frac{5d-6+\iota}{6(d-1)}$;

(ii) $\mathrm{p}(\mathtt{GImp}(\rho_\iota \otimes \mathtt{P}_{|1\rangle})) = \frac{4d^2+d-6+(2d-1)\iota}{6(d^2-1)}$;

(iii) $\mathrm{p}(\mathtt{MImp}(\rho_\iota \otimes \mathtt{P}_{|1\rangle})) = \frac{2d^2-d-2+\iota}{2(d^2-1)}$;

(iv) $\mathrm{p}(\text{Ł}\mathtt{Or}(\rho_\iota \otimes \mathtt{P}_{|1\rangle})) = \frac{d(d^2-1)(5d-6+\iota)+6(d\iota-1)\lfloor\frac{d+1}{2}\rfloor(\lfloor\frac{d+1}{2}\rfloor-d)}{6d(d^2-1)(d-1)}$;

(v) $\mathrm{p}(\text{Ł}\mathtt{And}(\rho_\iota \otimes \mathtt{P}_{|0\rangle})) = \frac{d(d^2-1)(d-\iota)+6(d\iota-1)((d-1 \mod 2)\lfloor\frac{d-1}{2}\rfloor)(1+\lfloor\frac{d-1}{2}\rfloor)}{6d(d^2-1)(d-1)}$.

PROOF.

(i) $\mathrm{p}(\text{Ł}\mathtt{Imp}(\rho_\iota \otimes \mathtt{P}_{|1\rangle})) = \frac{1}{d^2-1}(\sum_{k=0}^{d-2}\frac{k}{d-1}(k+1)(1-\frac{\iota}{d}) + \frac{d(d+1)}{2}(1-\frac{\iota}{d}) + d(\iota - \frac{1}{d})) = \frac{5d-6+\iota}{6(d-1)}$;

(ii) $\mathrm{p}(\mathtt{GImp}(\rho_\iota \otimes \mathtt{P}_{|1\rangle})) = \frac{1}{d^2-1}(\sum_{k=0}^{d-2}\frac{k}{d-1}(d-1-k)(1-\frac{\iota}{d}) + \frac{d(d+1)}{2}(1-\frac{\iota}{d}) + d(\iota-\frac{1}{d})) = \frac{4d^2+d-6+(2d-1)\iota}{6(d^2-1)}$;

(iii) $\mathrm{p}(\mathtt{MImp}(\rho_\iota \otimes \mathtt{P}_{|1\rangle})) = \frac{1}{d^2-1}(\sum_{k=0}^{d-2}\frac{k}{d-1}(1-\frac{\iota}{d}) + (d^2-d+1)(1-\frac{\iota}{d}) + d(\iota-\frac{1}{d})) = \frac{2d^2-d-2+\iota}{2(d^2-1)}$;

(iv) $\mathrm{p}(\texttt{ŁOr}(\rho_\iota \otimes \texttt{P}_{|1\rangle})) = \frac{1}{d^2-1}(\sum_{k=0}^{d-2} \frac{k}{d-1}(k+1)(1-\frac{\iota}{d}) + \frac{d(d+1)}{2}(1-\frac{\iota}{d}) + \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{2k}{d-1}(\iota-\frac{1}{d})+(d-\lfloor \frac{d+1}{2} \rfloor)(\iota-\frac{1}{d})) = \frac{d(d^2-1)(5d-6+\iota)+6(d\iota-1)\lfloor \frac{d+1}{2} \rfloor(\lfloor \frac{d+1}{2} \rfloor-d)}{6d(d^2-1)(d-1)}$;

(v) $\mathrm{p}(\texttt{ŁAnd}(\rho_\iota \otimes \texttt{P}_{|0\rangle})) = \frac{1}{d^2-1}(\sum_{k=0}^{d-1} \frac{k}{d-1}(d-k)(1-\frac{\iota}{d}) + \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor}(2k+(d-1 \mod 2))(\iota-\frac{1}{d})) = \frac{d(d^2-1)(d-\iota)+6(d\iota-1)((d-1 \mod 2)\lfloor \frac{d-1}{2} \rfloor)(1+\lfloor \frac{d-1}{2} \rfloor)}{6d(d^2-1)(d-1)}$.

8. Conclusion

We showed some interesting relations between the logical gates and continuous t-norms by probability values. On this basis, one can deal with quantum circuits as expressions in an algebraic environment (such as product many valued algebra for combinational circuits made up from Łukasiewicz gates). Some holistic connectives are useful in order to characterize the entanglement of formation both for isotropic states and for Werner states. In a future work, we will study possible applications to game theory and to the theory of communication with feedback. In particular, we will analyze holistic situations in Rényi-Ulam's games and Pelc's game.

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Figure

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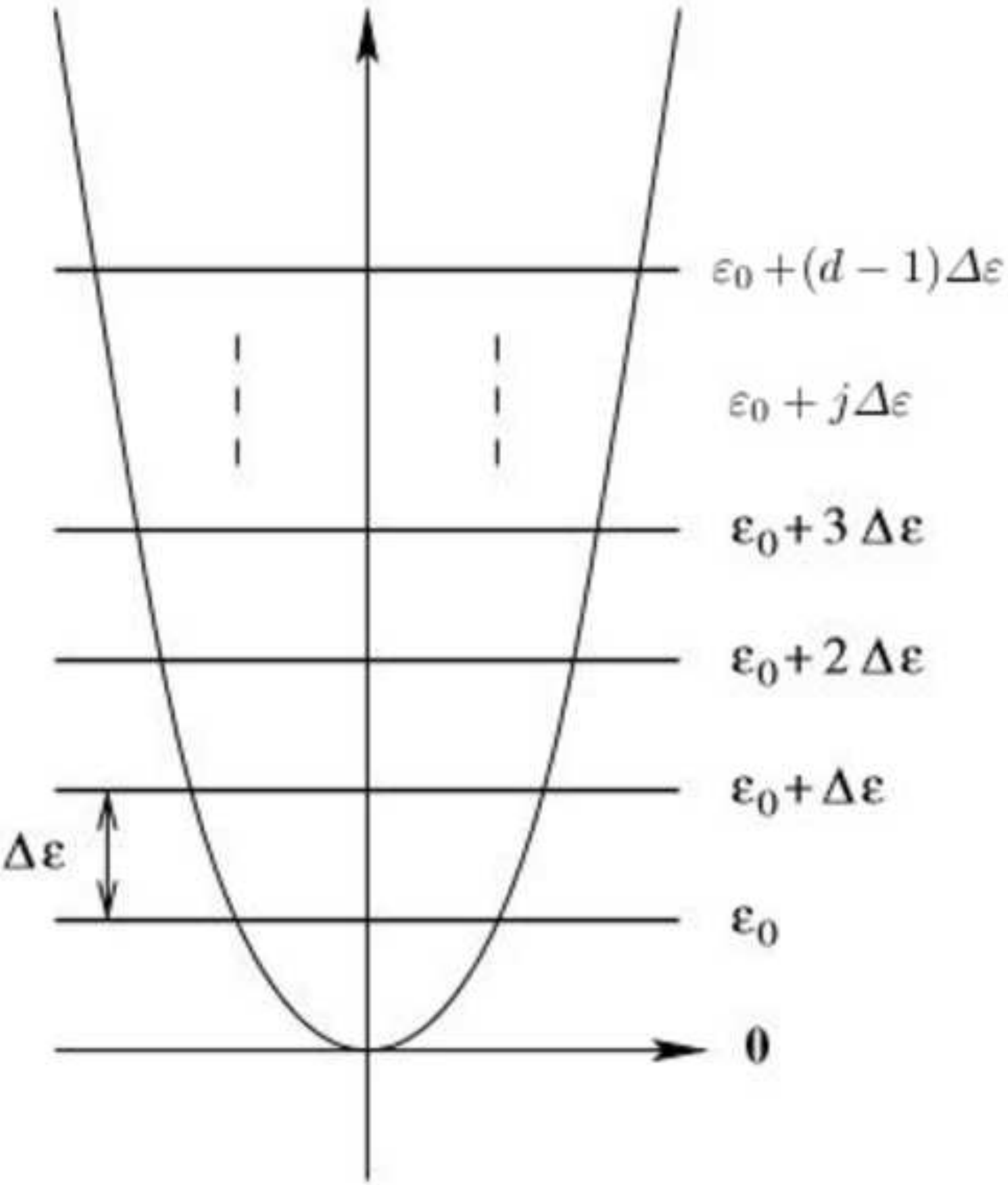

***Revision Note**

We added the required reference and shortened the paper (keeping it self-contained).
We made the minor corrections suggested by the second referee.