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Yukawa–Casimir wormholes

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Abstract In this work, we consider a Yukawa modification of the Casimir wormhole. With the help of an Equation of State, we impose Zero Tidal Forces. We will examine two different approaches: in a first approach, we will fix the form of the shape function of the Casimir wormholes modified by a Yukawa term in three different ways and finally a superposition of different profiles. In the second approach, we will consider the original Casimir source modified by a Yukawa term in three different ways and we will deduce the form of the shape function In both the approaches the reference energy density will be that of the Casimir source. Connection with the Absurdly Benign Traversable Wormhole are also discussed.

1 Introduction

Yukawa in 1935 [1] proposed to describe nonrelativistic strong interactions between nucleons with the help of a potential whose profile is

$$V(r) = -\frac{\alpha}{r}\exp(-\mu r)\,. \tag{1}$$

This is nothing but the screened version of the Coulomb potential with α describing the strength of the interaction and $1/\mu$ its range. This short range interaction has captured the interest of many researchers who have adapted it to the Newtonian potential to understand if it has deviations of the same kind. As a result, the Newtonian gravitational potential between two point masses m_1 and m_2 (atoms for instance) separated by a distance r, acquires a Yukawa correction which formally looks like Eq. (3) . Indeed, one gets

$$V(r) = -\frac{Gm_1m_2}{r}\left(1+\alpha\exp(-\mu r)\right), \tag{2}$$

where G is the gravitational constant. Potentials of the form (2) have been examined from the astrophysical point of view with a particular attention also on the graviton mass [2–4]. It is interesting to note that Yukawa-type forces are also predicted in the context of modified gravity theories [5–8] and also in bigravity theories [11]. To this purpose, it is important to say that the GINGER experiment offers the opportunity of constraining such theories [9]. On the other hand, in a different framework (MOG), it is possible to obtain black holes and traversable wormholes [10]. This MOG predicts also a variation of the Newton's constant G, in such a way to obtain a Yukawa term which enters the metric. Moreover, a Yukawa term seems to be directly involved in the Galaxy Rotation Curves [30]. Even in the context of Casimir effect, deviations of the Newtonian potential of the form (2) have been considered [12–14]. It is interesting to observe that a connection between the Casimir forces and the Yukawa profile has been also introduced in Ref. [23] where Van der Waals himself suggested an interaction potential of the form

$$V(r) = -\frac{A}{r}\exp(-Br)\,, \tag{3}$$

with A and B constants of appropriate dimensions. Since there exists a connection between the Casimir and the Van der Waals forces in the case of relatively large separations when the relativistic effects come into play, one can wonder if Yukawa deformations can play a fundamental role even for Traversable Wormholes. To further proceed we need to recall the Einstein's Field Equations (EFE)

$$G_{\mu\nu} = \kappa T_{\mu\nu} \qquad \kappa = \frac{8\pi G}{c^4} \tag{4}$$

in an orthonormal reference frame. In such a frame the EFE reduce to the following set of equations

$$\frac{b'}{r^2} = \kappa\rho(r)\,, \tag{5}$$

$$\frac{2}{r}\left(1-\frac{b(r)}{r}\right)\phi' - \frac{b}{r^3} = \kappa p_r(r)\,, \tag{6}$$

$$\times\left\{\left(1-\frac{b}{r}\right)\left[\phi''+\phi'\left(\phi'+\frac{1}{r}\right)\right]\right.$$

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$$-\frac{b'r-b}{2r^2}\left(\phi'+\frac{1}{r}\right)\Bigg\} = \kappa p_t(r), \tag{7}$$

in which $\rho(r)$ is the energy density,[1] $p_r(r)$ is the radial pressure, and $p_t(r)$ is the lateral pressure. The EFE (5), (6) and (7) have been obtained with the help of the line element

$$ds^2 = -e^{2\phi(r)}\,dt^2 + \frac{dr^2}{1-b(r)/r} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{8}$$

representing a spherically symmetric and static wormhole [15,16]. $b(r)$ is the shape function, while $\phi(r)$ is the redshift function. $\phi(r)$ and $b(r)$ are arbitrary functions of the radial coordinate $r \in [r_0, +\infty)$. A fundamental property of a traversable wormhole is that a flaring out condition of the throat, given by $(b - b'r)/b^2 > 0$, must be satisfied [15,16]. Furthermore, at the throat $b(r_0) = r_0$ and the condition $b'(r_0) < 1$ is imposed to have wormhole solutions. Another condition that needs to be satisfied is $1-b(r)/r > 0$. For the wormhole to be traversable, one must demand that there are no horizons present, which are identified as the surfaces with $e^{2\phi} \to 0$, so that $\phi(r)$ must be finite everywhere. The last condition is satisfied if we adopt a Zero Tidal Forces model (ZTF) represented by $\phi'(r) = 0$. Such a condition can be imposed by means of an inhomogeneous Equation of State (EoS) of the form

$$p_r(r) = \omega(r)\,\rho(r) \tag{9}$$

and

$$b(r) + \kappa p_r(r)\, r^3 = 0, \tag{10}$$

where we have used Eqs. (5) and (6). Equations (9) and (10) lead to

$$\omega(r) = -\frac{b(r)}{b'(r)\, r}. \tag{11}$$

In Ref. [17], we have found that the Casimir wormhole described by

$$\phi(r) = \ln\left(\frac{4r}{3r+r_0}\right) \quad \text{and} \quad b(r) = \frac{2r_0}{3} + \frac{r_0^2}{3r}, \tag{12}$$

does not satisfy the ZTF condition. In this paper, we will consider the Casimir wormhole shape function deformed by a Yukawa profile satisfying also the ZTF condition: in this way we have the possibility of building a new family of solutions which have a vanishing redshift function. We also assume that the Casimir relationship $\omega = 3$ holds, at least on the throat. There exists another reason to consider a Yukawa deformation to the Casimir wormhole. Indeed, in Ref. [18] we have considered a shape function of the form

$$b(r) = r_0 \exp\left(-\mu\,(r - r_0)\right) \tag{13}$$

obeying Eq. (11) with

$$\omega(r) = \frac{1}{\mu r} \tag{14}$$

and therefore satisfying the ZTF property. However, if we simply assume that

$$\mu = r_0 \kappa \rho_C, \tag{15}$$

where

$$\rho_C = \frac{\hbar c \pi^2}{720 d^4}, \tag{16}$$

then the energy density on the throat becomes

$$\rho(r) = -\frac{r_0\mu}{\kappa r^2}\exp\left(-\mu\,(r - r_0)\right) \underset{r=r_0}{=} -\frac{\mu}{\kappa r_0} = -\rho_C, \tag{17}$$

namely the Casimir energy density. Moreover, with the help of the relationship (14) on the throat one gets

$$\omega(r_0) = \frac{1}{\mu r_0} = \frac{1}{r_0^2 \kappa \rho_C} \tag{18}$$

and by imposing that $\omega(r_0) = 3$, one finds

$$r_0 = \sqrt{\frac{1}{3\kappa\rho_C}} = \frac{d^2}{l_P \pi}\sqrt{\frac{30}{\pi}}, \tag{19}$$

in agreement with what found in Ref. [36] but with a factor $\sqrt{3}$ missing. This example suggests that the mixing between the Casimir wormhole and a Yukawa wormhole seems to be promising. The paper is organized as follows: in Sect. 2 we study three different combinations of the Casimir wormhole shape function with a Yukawa term, in Sect. 3 we explore the consequences of a superposition of the profiles considered in Sect. 2, in Sect. 4 we adopt the reverse procedure, namely we fix the form of the energy density and we deduce the form of the shape function, investigating three different profiles. We summarize and conclude in Sect. 5. Units in which $\hbar = c = k = 1$ are used throughout the paper and will be reintroduced whenever it is necessary.

2 Casimir–Yukawa wormholes

The Casimir wormhole obtained in Ref. [17] has as a source the original Casimir energy density with a slight but fundamental difference: the plates separation has been promoted to be a variable instead of being a fixed quantity. To satisfy the EFE a non vanishing redshift function has been computed described in Eq. (12). In this section we are interested in examining some modifications of the original Casimir wormhole shape function satisfying the ZTF condition, which can

[1] However, if $\rho(r)$ represents the mass density, then we have to replace $\rho(r)$ with $\rho(r)\,c^2$.

be obtained with the help of the EoS (9). We will take under consideration three shape function profiles. We begin with

2.1 $b(r) = r_0 \exp(-\mu(r - r_0))(2 + r_0/3r)$

The shape function of the Casimir wormhole is defined by

$$b(r) = \frac{2r_0}{3} + \frac{r_0^2}{3r}. \tag{20}$$

We wonder what are the effects of an additional Yukawa term on the original Casimir shape function whose profile becomes

$$b(r) = \left(\frac{2r_0}{3} + \frac{r_0^2}{3r}\right) \exp(-\mu(r - r_0)), \tag{21}$$

where μ is a positive mass scale to be identified. The original Casimir shape function can be reobtained when $\mu = 0$. The profile (21) satisfies the usual properties, namely the throat condition $b(r_0) = r_0$, the asymptotic flatness and the flare out condition of the throat, written into the form

$$b'(r_0) = -\frac{1}{3}(1 + 3\mu r_0) < 1. \tag{22}$$

This is always satisfied together with the property $1 - b(r)/r > 0$. Another additional property is

$$b(r) \to 0 \quad \text{when} \quad \mu \to \infty \quad \text{and} \quad r \to \infty. \tag{23}$$

The energy density can be easily computed and we obtain

$$\begin{aligned}\rho(r) &= \frac{b'}{\kappa r^2} \\ &= -\frac{r_0}{3\kappa r^4}\left(2\mu r^2 + \mu r r_0 + r_0\right) \exp(-\mu(r - r_0)) \\ &= -\frac{1}{\kappa r^2}\left(\mu b(r) + \frac{r_0^2}{3r^2} \exp(-\mu(r - r_0))\right).\end{aligned} \tag{24}$$

It is straightforward to see that, for $\mu \to 0$, one gets the original Casimir energy density with the plates separation considered as a variable if we make the following identification

$$\rho(r) = -\frac{r_0^2}{3\kappa r^4} = -\frac{r_1^2}{\kappa r^4} = -\frac{\hbar c \pi^2}{720 r^4}, \tag{25}$$

which is possible if [17]

$$r_0^2 = 3r_1^2. \tag{26}$$

However, the identification (26) is inconsistent with the assumption (11) because the relationship (11) leads to a vanishing redshift, while the identification (26) does not, as shown in Ref. [17] Therefore, we consider the following assumption

$$\rho(r_0) = -\frac{\mu}{\kappa r_0} - \frac{1}{3\kappa r_0^2} = -\frac{\hbar c \pi^2}{720 d^4}, \tag{27}$$

where d is the "*fixed plate distance*". This identification fixes the scale mass μ to the following value

$$\mu = \frac{\kappa r_0 \hbar c \pi^2}{720 d^4} - \frac{1}{3r_0} = \frac{r_0 l_P^2 \pi^3}{90 d^4} - \frac{1}{3r_0}. \tag{28}$$

Since $\mu \geq 0$, one finds that

$$\mu = 0 \quad \text{when} \quad r_0 = \frac{d^2}{l_P \pi}\sqrt{\frac{30}{\pi}}, \tag{29}$$

which is in agreement with what found in Ref. [36] but with a factor $\sqrt{3}$ missing. Note that

$$\lim_{r \to r_0} \lim_{\mu \to \infty} \rho(r) \neq \lim_{\mu \to \infty} \lim_{r \to r_0} \rho(r), \tag{30}$$

while

$$\lim_{r \to r_0} \lim_{\mu \to 0} \rho(r) = \lim_{\mu \to 0} \lim_{r \to r_0} \rho(r). \tag{31}$$

Note also that

$$\lim_{\mu \to \infty} \rho(r) = 0. \tag{32}$$

However, due to the relationship (29), $\mu \to \infty$ is equivalent to $r_0 \to \infty$. Therefore this limiting value will be discarded. The second EFE (6) determines the value of the pressure that, differently from the Casimir wormhole, will be computed by imposing the relationship (11). A simple calculation gives

$$\omega(r) = \frac{2r + r_0}{2\mu r^2 + \mu r r_0 + r_0}. \tag{33}$$

$\omega(r)$ has the following properties

$$\omega(r_0) = \frac{3}{3\mu r_0 + 1} \tag{34}$$

$$\omega(r) \underset{r \to \infty}{\longrightarrow} 0 \tag{35}$$

$$\omega(r) \underset{\mu \to 0}{\longrightarrow} \frac{2r + r_0}{r_0} \tag{36}$$

$$\lim_{r \to r_0} \lim_{\mu \to 0} \omega(r) = \lim_{\mu \to 0} \lim_{r \to r_0} \omega(r) = 3, \tag{37}$$

which is the original relationship between the energy density and the pressure: in this case the radial pressure. With this assumption, we get

$$p_r(r) = -\frac{1}{\kappa r^3}\left(\frac{2r_0}{3} + \frac{r_0^2}{3r}\right) \exp(-\mu(r - r_0)) \tag{38}$$

and when

$$\mu \to 0, \qquad p_r(r) = -\frac{1}{\kappa r^3}\left(\frac{2r_0}{3} + \frac{r_0^2}{3r}\right) \underset{r \to r_0}{=} -\frac{1}{\kappa r_0^2}, \tag{39}$$

It remains to compute the transverse pressure

$$p_t(r) = \frac{b(r) - b'(r)r}{2\kappa r^3} = \frac{r_0}{6\kappa r^4}\left(2\mu r^2 + \mu r r_0 + 2r + 2r_0\right) \times \exp\left(-\mu\left(r - r_0\right)\right) = \frac{1}{2\kappa r^2}\left(b(r)\left(\mu + \frac{1}{r}\right) + \frac{r_0^2}{3r^2}\exp\left(-\mu\left(r - r_0\right)\right)\right), \tag{40}$$

which has the following features, for

$$\mu \to 0, \qquad p_t(r) = \frac{r_0}{3\kappa r^4}\left(r + r_0\right) \underset{r \to r_0}{=} \frac{2}{3\kappa r_0^2}, \tag{41}$$

The SET becomes

$$T_{\mu\nu} = T^a_{\mu\nu} + T^b_{\mu\nu} \tag{42}$$

where

$$T^a_{\mu\nu} = \frac{b(r)}{\kappa r^2}\left[diag\left(-\mu, -\frac{1}{r}, \frac{1}{2}\left(\mu + \frac{1}{r}\right), \times \frac{1}{2}\left(\mu + \frac{1}{r}\right)\right)\right] \tag{43}$$

and

$$T^b_{\mu\nu} = \frac{1}{\kappa r^2}\left[diag\left(-1, 0, \frac{1}{2\kappa r^2}, \frac{1}{2\kappa r^2}\right)\right] \times \frac{r_0^2}{3r^2}\exp\left(-\mu\left(r - r_0\right)\right) \tag{44}$$

On the throat the SET reduces to

$$T_{\mu\nu} = \frac{1}{\kappa r_0}\left[diag\left(-\mu - \frac{1}{3r_0}, -\frac{1}{r_0}, \frac{1}{2}\left(\mu + \frac{4}{3r_0}\right), \frac{1}{2}\left(\mu + \frac{4}{3r_0}\right)\right)\right] \tag{45}$$

and in the limit $\mu \to 0$, one gets

$$T_{\mu\nu} = \frac{1}{3\kappa r_0^2}\left[diag\left(-1, -3, 2, 2\right)\right] = \frac{\hbar c \pi^2}{720 d^4} \times \left[diag\left(-1, -3, 2, 2\right)\right] \tag{46}$$

which is verified when the relationship (29) is satisfied. Moreover the SET (46) is in agreement with the SET structure found in Ref. [17]. It is interesting to note that, for $r \to \infty$, the SET vanishes reproducing a Minkowski SET. With an abuse of language, one can say that in this limit we find a behavior that looks like a Generalized Absurdly Benign Traversable Wormhole [36]. We say "*it looks like*" because the SET vanishes for a limiting value of the radial coordinate and not for a well determined location in space time. The next profile we are going to examine is

2.2 $b(r) = r_0\left(2\exp\left(-\mu\left(r - r_0\right)\right) + r_0/r\right)/3$

For the following profile

$$b(r) = \frac{2r_0}{3}\exp\left(-\mu\left(r - r_0\right)\right) + \frac{r_0^2}{3r}, \tag{47}$$

the Yukawa modification is not distributed over the whole original shape function but only on the constant term. This little displacement has an interesting consequence, because when $\mu \to \infty$ we obtain the Ellis–Bronnikov (EB)-like wormhole [28,29]. Indeed, the EB wormhole is

$$b(r) = \frac{r_0^2}{r}. \tag{48}$$

The shape function (47) satisfies the usual properties, namely the throat condition and so on. For completeness, we write the expression of the flare-out condition, which is

$$b'(r) = -\frac{2r_0\mu}{3}\exp\left(-\mu\left(r - r_0\right)\right) - \frac{r_0^2}{3r^2} \underset{r = r_0}{=} -\frac{2r_0\mu + 1}{3} < 1. \tag{49}$$

Even in this case, we can easily compute the energy density to obtain

$$\rho(r) = \frac{b'}{\kappa r^2} = \frac{1}{\kappa r^2}\left(-\frac{2\mu r_0}{3}\exp\left(-\mu\left(r - r_0\right)\right) - \frac{r_0^2}{3r^2}\right) = -\frac{1}{\kappa r^2}\left(\mu b(r) + \frac{r_0^2}{3r^2}\left(1 - \mu r\right)\right) \tag{50}$$

which, on the throat becomes

$$\rho(r_0) = -\frac{\mu 2r_0 + 1}{3\kappa r_0^2}. \tag{51}$$

To fix the value of μ we adopt the same procedure of Sect. 2.1 and we find that even in this case the relationship (29) is satisfied. The pressure can be determined by solving the second EFE (6) and by imposing that the relationship (11) be satisfied, namely

$$\omega(r) = \frac{2r\exp\left(-\mu\left(r - r_0\right)\right) + r_0}{2\mu r^2\exp\left(-\mu\left(r - r_0\right)\right) + r_0}. \tag{52}$$

This time $\omega(r)$ has the following properties

$$\omega(r_0) = \frac{3}{2\mu r_0 + 1} \tag{53}$$

$$\omega(r) \underset{\mu \to \infty}{\longrightarrow} = 1 \tag{54}$$

$$\omega(r) \underset{r \to \infty}{\longrightarrow} = 1 \tag{55}$$

$$\omega(r) \underset{\mu \to 0}{\longrightarrow} = \frac{2r + r_0}{r_0} \tag{56}$$

$$\lim_{r \to r_0}\lim_{\mu \to 0}\omega(r) = \lim_{\mu \to 0}\lim_{r \to r_0}\omega(r) = 3, \tag{57}$$

and even in this case the original relationship between the energy density and the pressure is preserved. Thus the radial pressure is

$$p_r(r) = -\frac{1}{\kappa r^3}\left(\frac{2r_0}{3}\exp\left(-\mu\left(r-r_0\right)\right)+\frac{r_0^2}{3r}\right) \tag{58}$$

and one finds that

$$p_r(r) \underset{\mu\to 0}{=} -\frac{r_0}{3\kappa r^3}\left(2+\frac{r_0}{r}\right). \tag{59}$$

The last quantity to compute is $p_t(r)$, namely

$$p_t(r) = \frac{b(r)-b'(r)r}{2\kappa r^3} = \frac{3rb(r)\left(1+\mu r\right)+r_0^2\left(1-\mu r\right)}{6\kappa r^4}, \tag{60}$$

which has the following features, for

$$\mu \to 0, \qquad p_t(r) = \frac{r_0}{3\kappa r^4}\left(r+r_0\right) \underset{r\to r_0}{=} \frac{2}{3\kappa r_0^2}. \tag{61}$$

To summarize the SET for this particular shape function becomes

$$T_{\mu\nu} = T^a_{\mu\nu} + T^b_{\mu\nu} \tag{62}$$

where

$$T^a_{\mu\nu} = \frac{b(r)}{\kappa r^2}\left[diag\left(-\mu,-\frac{1}{r},\frac{1}{2r}\left(1+\mu r\right),\frac{1}{2r}\left(1+\mu r\right)\right)\right] \tag{63}$$

and

$$T^b_{\mu\nu} = \frac{1}{\kappa r^2}\left[diag\left(-1,0,\frac{1}{2},\frac{1}{2}\right)\right]\frac{r_0^2}{3r^2}\left(1-\mu r\right) \tag{64}$$

On the throat the SET reduces to

$$T_{\mu\nu} = \frac{1}{3\kappa r_0^2}\left[diag\left(-2\mu r_0-1,-3,2+\mu r_0,2+\mu r_0\right)\right] \tag{65}$$

and in the limit $\mu \to 0$, one gets

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{3\kappa r_0^2}\left[diag\left(-1,-3,2,2\right)\right] \\ &= \frac{\hbar c\pi^2}{720d^4}\left[diag\left(-1,-3,2,2\right)\right] \end{aligned} \tag{66}$$

which is in agreement with the SET structure found in Ref. [17] only for $\mu = 0$. Finally, we investigate the following shape function

2.3 $b(r) = r_0\left(2 + r_0\exp\left(-\mu\left(r-r_0\right)\right)/r\right)/3$

For the following profile

$$b(r) = \frac{2r_0}{3} + \frac{r_0^2}{3r}\exp\left(-\mu\left(r-r_0\right)\right), \tag{67}$$

the Yukawa modification is now put only on the variable term. Even in this modification, we have an interesting consequence, because when $\mu \to \infty$ we obtain a constant term smaller than the throat. The shape function (47) satisfies the usual properties, namely the throat condition and so on. For completeness, we verify if the flare out condition is satisfied. We find that

$$\begin{aligned} b'(r) &= -\frac{r_0^2}{3r^2}\left(1+\mu r\right)\exp\left(-\mu\left(r-r_0\right)\right) \\ &= \left(\frac{2}{3}r_0 - b(r)\right)\frac{1+\mu r}{r}, \end{aligned} \tag{68}$$

and on the throat one gets

$$b'(r_0) = -\frac{1}{3}\left(1+\mu r_0\right) < 1. \tag{69}$$

The energy density is straightforward to obtain since

$$\begin{aligned} \rho(r) = \frac{b'}{\kappa r^2} &= -\frac{r_0^2}{3\kappa r^4}\left(\mu r+1\right)\exp\left(-\mu\left(r-r_0\right)\right) \\ &= \frac{\mu r+1}{\kappa r^3}\left(\frac{2}{3}r_0 - b(r)\right) \end{aligned} \tag{70}$$

and for $\mu \to \infty$, one finds

$$\rho(r) = 0. \tag{71}$$

On the throat we obtain

$$\rho(r_0) = -\frac{\mu r_0+1}{3\kappa r_0^2} \tag{72}$$

The second Einstein's field equation (6) determines the value of the pressure that and, even in this case, we impose that the relationship (11) be satisfied. This implies that the redshift function vanishes and that

$$\omega(r) = \frac{2r\exp\left(\mu\left(r-r_0\right)\right)+r_0}{\left(\mu r+1\right)r_0}. \tag{73}$$

This time $\omega(r)$ has the following properties

$$\omega(r_0) = \frac{3}{\mu r_0+1} \tag{74}$$

$$\omega(r) \underset{\mu\to\infty}{\longrightarrow} = \infty \tag{75}$$

$$\omega(r) \underset{r\to\infty}{\longrightarrow} = \infty \tag{76}$$

$$\omega(r) \underset{\mu\to 0}{\longrightarrow} = \frac{2r+r_0}{r_0} \tag{77}$$

$$\lim_{r\to r_0}\lim_{\mu\to 0}\omega(r) = \lim_{\mu\to 0}\lim_{r\to r_0}\omega(r) = 3. \tag{78}$$

As we can see, from the relationship (76) , one finds that $\omega(r)$ is divergent: this is a consequence of the EoS. Indeed for $r \to \infty$, the energy density (70) vanishes because of the presence of the damping exponential overall, while into the pressure the damping exponential appears only in the constant term. For this reason, this profile will be discarded. In the next section, we explore a profile which is a superposition of the previous profiles with the aim of generalizing as much as possible the features of a Yukawa–Casimir wormhole.

3 Superposing traversable wormholes shape functions

In this section we will consider a linear combination of the previous profiles described by the following shape function

$$b(r) = r_0 \left(\alpha \exp\left(-\mu\left(r-r_0\right)\right) + (1-\alpha)\left(\frac{r_0}{r}\right)^c \times \exp\left(-\nu\left(r-r_0\right)\right)\right), \tag{79}$$

with $\mu, \nu > 0, \alpha \geq 0$ and $c \in \mathbb{R}$. Note that when $\alpha = 2/3$, $c = 1$ and $\mu = \nu = 0$, it is immediate to see that the Casimir wormhole shape function is obtained. When $\alpha = 1$, we find a pure Yukawa wormhole discussed in Ref. [18] as well as for $c = 0$ and $\mu = \nu$. For $\alpha = 0$ and $c = 1$, one finds the Yukawa modification to the EB wormhole. Finally, note that for $\mu = \nu = 0$ and $c < -1$, the wormhole is no more traversable. As a first step we examine under what conditions the flare-out property is satisfied. From

$$b'(r) = -r_0\mu\alpha \exp\left(-\mu\left(r-r_0\right)\right) - (1-\alpha)\left(\frac{r_0}{r}\right)^c \times \exp\left(-\nu\left(r-r_0\right)\right)\left(\nu r_0 + c\frac{r_0}{r}\right), \tag{80}$$

one finds

$$b'(r_0) < 1 \qquad \Longleftrightarrow \qquad \alpha\left(c + r_0\nu\right) < 1 + r_0\left(\alpha\mu + \nu\right) + c. \tag{81}$$

From the Eq. (80) we can easily compute the energy density

$$\rho(r) = \frac{b'}{\kappa r^2} = -\frac{1}{\kappa r^2}\left[r_0\mu\alpha \exp\left(-\mu\left(r-r_0\right)\right) + (1-\alpha)\left(\frac{r_0}{r}\right)^c \exp\left(-\nu\left(r-r_0\right)\right)\left(\nu r_0 + c\frac{r_0}{r}\right)\right] \tag{82}$$

and on the throat we obtain

$$\rho(r_0) = -\frac{1}{\kappa r_0^2}\left[r_0\mu\alpha - (1-\alpha)\left(\nu r_0 + c\right)\right]. \tag{83}$$

As we can see, $\rho(r_0)$ can be considered as a function of the throat. In order to fix the wormhole throat, we find the stationary point of $\rho(r_0)$, assuming

$$\rho'(r_0) = 0 \qquad \Longrightarrow \qquad \bar{r}_0 = \frac{2c(\alpha-1)}{\mu\alpha - (\alpha-1)\nu}. \tag{84}$$

Plugging $\bar{r}_0$ into Eq. (83), one finds

$$\rho(\bar{r}_0) = -\frac{\left((\mu-\nu)\alpha+\nu\right)^2}{4c(\alpha-1)\kappa} = -\frac{\hbar c\pi^2}{720 d^4}, \tag{85}$$

where we have imposed that, even in this case, the source is described by Eq. (16). A solution of the previous equation is given by

$$\bar{\mu} = \frac{\nu}{\alpha}(\alpha-1) \pm \frac{\pi l_p}{3\alpha d^2}\sqrt{c(\alpha-1)\frac{2\pi}{5}}, \quad \begin{array}{ll} c > 0, & \alpha > 1 \\ c < 0, & \alpha < 1 \end{array}. \tag{86}$$

Plugging $\bar{\mu}$ into $\bar{r}_0$ of Eq. (84), one finds

$$\bar{r}_0 = \pm\frac{3d^2\sqrt{10\pi c(\alpha-1)}}{\pi^2 l_p}. \tag{87}$$

The value of α can be determined with the help of the relationship (11) which, in this case, becomes

$$\omega(r) = \frac{r_0\left(\alpha \exp\left(-\mu\left(r-r_0\right)\right) + (1-\alpha)\left(\frac{r_0}{r}\right)^c \exp\left(-\nu\left(r-r_0\right)\right)\right)}{r\left[r_0\mu\alpha \exp\left(-\mu\left(r-r_0\right)\right) + (1-\alpha)\left(\frac{r_0}{r}\right)^c \exp\left(-\nu\left(r-r_0\right)\right)\left(\nu r_0 + c\frac{r_0}{r}\right)\right]} \tag{88}$$

and on the throat one finds

$$\omega(r_0) = \frac{1}{\left[r_0\mu\alpha + (1-\alpha)\left(\nu r_0 + c\right)\right]}. \tag{89}$$

$\omega(r_0)$ can be further reduced to the following simple expression

$$\omega(r_0) = \frac{1}{(\alpha-1)c} = 3, \tag{90}$$

where we have used Eqs. (86) and (87) and where we have imposed the Casimir relationship between pressure and energy density on the throat. Plugging (90) into (87), one gets

$$\bar{r}_0 = \frac{\sqrt{30\pi}d^2}{\pi^2 l_p} \tag{91}$$

which is the same result of Eq. (29). With the help of the Eqs. (86), (87) and (90), it is possible to show that $\bar{r}_0$ represents the minimum of $\rho(r_0)$. This means that if we want to have a wormhole throat with a radius smaller than $\bar{r}_0$, we need to have an increasing of negative energy. To complete the analysis, we compute the transverse pressure

$$p_t(r) = -\frac{r_0\left(r_0^c(\alpha-1)\left(\nu r^{-c+1}+r^{-c}(c+1)\right)e^{-\nu(r-r_0)}-\alpha e^{-\mu(r-r_0)}(\mu r+1)\right)}{2\kappa r^3} \tag{92}$$

which, on the throat, becomes

$$p_t(r_0) = \frac{r_0((\mu-\nu)\alpha+\nu)-c\alpha+c+1}{2\kappa r_0^2}. \tag{93}$$

With the help of the relationships (91), (86) and (90), one finds

$$p_t(\bar{r}_0) = \frac{\pi^3 l_p^2}{45\kappa d^4} = 2\frac{\pi^2\hbar c}{720 d^4}. \tag{94}$$

The analytic form of the SET is quite complicated. However, it becomes very simple on the throat

$$T^a_{\mu\nu} = -\frac{1}{\kappa r_0^2} \times diag\left(r_0\mu\alpha-(1-\alpha)(\nu r_0+c),1,\right. \frac{r_0((\mu-\nu)\alpha+\nu)-c\alpha+c+1}{2}, \left.\frac{r_0((\mu-\nu)\alpha+\nu)-c\alpha+c+1}{2}\right) \tag{95}$$

and in particular in correspondence of the minimum $\bar{r}_0$, it simplifies to

$$T_{\mu\nu} = \frac{l_p^2\pi^3}{90\kappa d^4}[diag(-1,-3,2,2)] = \frac{\hbar c\pi^2}{720 d^4}[diag(-1,-3,2,2)]. \tag{96}$$

Remark It is important to observe that thanks to the EoS (9), the form of the SET is

$$\begin{aligned}T_{\mu\nu} &= \frac{r_0}{\kappa r^3}diag\left(-\frac{1}{\omega(r)},-1,\frac{1}{2\omega(r)}+\frac{1}{2},\frac{1}{2\omega(r)}+\frac{1}{2}\right)\\ &\quad\times\exp\left[-\int_{r_0}^{r}\frac{d\bar{r}}{\omega(\bar{r})\bar{r}}\right]\\ &= -\frac{b(r)}{\kappa r^3\omega(r)}\\ &\quad diag\left(1,\omega(r),-\frac{1}{2}-\frac{\omega(r)}{2},-\frac{1}{2}-\frac{\omega(r)}{2}\right)\\ &= \rho(r)\,diag\left(1,\omega(r),-\frac{1}{2}-\frac{\omega(r)}{2},-\frac{1}{2}-\frac{\omega(r)}{2}\right).\end{aligned} \tag{97}$$

By construction the SET (97) is divergenceless, but it is not traceless. However, it is always possible to rearrange the previous SET (97) in such a way to extract the traceless part. Indeed

$$T_{\mu\nu} = T^T_{\mu\nu}+\frac{T}{4}g_{\mu\nu} = \frac{\rho(r)}{2}[diag(1,2\omega(r)+1,-\omega(r),-\omega(r))-g_{\mu\nu}], \tag{98}$$

where $T^T_{\mu\nu}$ is the traceless part of the SET (97). It is interesting to observe that by imposing the following condition

$$\omega(r_0) = 1, \tag{99}$$

one finds that

$$T^T_{\mu\nu} = \frac{\rho(r_0)}{2}[diag(1,3,-1,-1)], \tag{100}$$

independently on the form of $\omega(r)$. This means that either by decomposing the SET like in Eq. (98) and fixing $\omega(r_0) = 1$ or by fixing $\omega(r_0) = 3$ without the decomposition (98), it is always possible to preserve the fundamental relationship between pressure and energy density. Note also that, from the point of view of the wormholes throat size, the choice (99) or the choice $\omega(r_0) = 3$, do not change the size of the throat size, as it should be. In the next section, we are going to examine the reverse procedure, namely we fix the form of the energy density and we will deduce the form of the shape function.

4 Traversable wormholes with a Yukawa energy density profile

In this section we change the strategy and we fix our attention on some energy density profiles modified by a Yukawa term and with the help of Eq. (5), we will deduce the form of the shape function. Three different forms will be examined. We begin with the following profile

4.1 $\rho(r) = -r_0\rho_C\frac{e^{-\mu(r-r_0)}}{r}$

$$\rho(r) = -r_0\rho_C\frac{e^{-\mu(r-r_0)}}{r}; \qquad \rho_0 > 0, \tag{101}$$

where ρ_C has dimensions of an energy density and μ is a positive mass scale parameter. We can identify ρ_C with the value expressed by (16). Equation (101) can be easily integrated to obtain

$$\begin{aligned}b(r) &= b(r_0)+\kappa\int_{r_0}^{r}\rho(r')r'^2dr'\\ &= r_0\left(1-\frac{\rho_C\kappa}{\mu^2}-\frac{r_0\rho_C\kappa}{\mu}+\frac{e^{-\mu(r-r_0)}\rho_C\kappa(\mu r+1)}{\mu^2}\right)\end{aligned} \tag{102}$$

where we have used the condition $b(r_0) = r_0$. It is immediate to verify that the shape function (102) satisfies the asymptotic flatness and the flare-out condition. To have ZTF, Eq. (11) must be imposed, that it means

$$\omega(r)=\frac{\left(\mu^2-\rho_C\kappa\left(1+r_0\mu\right)\right)e^{\mu(r-r_0)}+\rho_C\kappa\left(\mu r+1\right)}{\mu^2r^2\rho_C\kappa}. \tag{103}$$

As one can see, for $r\to\infty$, $\omega(r)\to\infty$. However, one can adopt another strategy to have a finite $\omega(r)$. Indeed from the shape function (102), we can find that there exists $\bar{r}>r_0$, such that $b(\bar{r})=0$, where

$$\bar{r}=-\frac{1}{\mu}\left(\mathrm{W}\left(\frac{\mu^2-\kappa\rho_C\left(\mu r_0+1\right)}{\kappa\rho_C\exp\left(\mu r_0+1\right)}\right)+1\right), \tag{104}$$

where $\mathrm{W}(x)$ is the Lambert function defined mathematically as the multivalued inverse of the function $x\exp(x)$,

$$W(x)\exp W(x)=x. \tag{105}$$

If $-1/e<x<0$, there are two real solutions, and thus two real branches of W [33–35]. Inspired by the Absurdly Benign Traversable Wormhole (ABTW) and its generalization, the Generalized Absurdly Benign Traversable Wormhole (GABTW) [36], we define the shape function (102) in such a way that

$$\begin{aligned}&b(r)=r_0\left(1-\frac{\rho_C\kappa}{\mu^2}-\frac{r_0\rho_C\kappa}{\mu}+\frac{e^{-\mu(r-r_0)}\rho_C\kappa\left(\mu r+1\right)}{\mu^2}\right)\\&\quad\text{for}\quad r<\bar{r}\\&\quad b(r)=0\quad\text{for}\quad r\geq\bar{r},\end{aligned} \tag{106}$$

where $\bar{r}$ has been defined in (104). As a consequence also $\omega(r)$ behaves in the same way and therefore also the radial pressure. Nevertheless the energy density does not vanish because $\bar{r}$ does not set to zero its value. However $\rho(\bar{r})$ can be very small and therefore even the transverse pressure. Therefore outside the region defined by $r\geq\bar{r}$, one obtains a quasi-Minkowski spacetime. To complete the analysis, we compute the value of $\omega(r)$ on the throat. We find

$$\omega(r_0)=\frac{1}{r_0^2\rho_C\kappa}, \tag{107}$$

in agreement with what found in Ref. [36] and with Eq. (19).

4.2 $\rho(r)=-\frac{\rho_C}{2}\left(\alpha+\beta r_0\frac{e^{-\mu(r-r_0)}}{r}\right)$

The second energy density profile we are going to consider is obtained with a small modification of the profile (101)

$$\rho(r)=-\frac{\rho_C}{2}\left(\alpha+\beta r_0\frac{e^{-\mu(r-r_0)}}{r}\right)\qquad\alpha,\beta\in\mathbb{R}. \tag{108}$$

As we can see, this is a linear combination between the original Casimir profile and the Yukawa profile (101). Note that for $\mu=0$, $\alpha=\beta=1$ and $r=r_0$, we obtain the pure Casimir energy density. Note also that this profile is a generalization of the potential (2). Differently from the profile (101), here we can choose α and β in such a way to have

$$\rho(\bar{r})=0\qquad\Longrightarrow\qquad\alpha=-\beta r_0\frac{e^{-\mu(\bar{r}-r_0)}}{\bar{r}}\qquad\alpha,\beta\in\mathbb{R}. \tag{109}$$

The motivation for this choice will be clarified below. The shape function can be easily computed and we find

$$\begin{aligned}b(r)=r_0&+\frac{\kappa\rho_C\left(-\alpha\mu^2r^3+\alpha r_0^3\mu^2-3\beta r_0^2\mu-3\beta r_0\right)}{6\mu^2}\\&+\frac{\kappa\rho_C\left(3\beta\mu r r_0+3\beta r_0\right)e^{-\mu(r-r_0)}}{6\mu^2}.\end{aligned} \tag{110}$$

It is easy to see that for $r\gg r_0$

$$\begin{aligned}b(r)&\simeq r_0-\frac{\kappa\rho_C\left(\alpha\mu^2r^3-\alpha r_0^3\mu^2+3\beta r_0^2\mu+3\beta r_0\right)}{6\mu^2}\\&\simeq-\frac{\kappa\rho_C\alpha}{6}r^3.\end{aligned} \tag{111}$$

This behavior is due to the constant term in (108) which is dominant and produces a divergent shape function. However, since β is not fixed, we can impose that $b(\bar{r})=0$, where $\bar{r}$ is the same of Eq. (109). Plugging the value of α found in Eq. (109) into Eq. (110), and by imposing that $b(\bar{r})=0$, one finds

$$\beta=\frac{6\bar{r}\mu^2}{\rho\kappa\left(e^{\mu(r_0-\bar{r})}\left(\mu^2r_0^3-\mu^2\bar{r}^3-3\mu\bar{r}^2-3\bar{r}\right)+\left(3\mu r_0+3\right)\bar{r}\right)} \tag{112}$$

and

$$\begin{aligned}&b(r)\\&=\frac{\left(\left(3\mu r+3\right)\bar{r}\,e^{-\mu(r-r_0)}+e^{\mu(r_0-\bar{r})}\left(\mu^2r^3-\mu^2\bar{r}^3-3\mu\bar{r}^2-3\bar{r}\right)\right)r_0}{e^{\mu(r_0-\bar{r})}\left(\mu r_0^3-\mu^2\bar{r}^3-3\mu\bar{r}^2-3\bar{r}\right)+\left(3\mu r_0+3\right)\bar{r}}\end{aligned} \tag{113}$$

Thus if we assume that for $r>\bar{r}$, $b(\bar{r})=0$, we get a feature similar to the ABTW. Moreover, to have ZTF, $\omega(r)$ must be

$$\omega(r) = -\frac{b(r)}{b'(r)\,r}$$
$$= \frac{r_0 6\mu^2 + \kappa\rho_C\left(-\alpha\,\mu^2 r^3 + \alpha r_0^3\mu^2 - 3\beta r_0^2\mu - 3\beta r_0\right) + \kappa\rho_C\left(3\beta\mu r r_0 + 3\beta r_0\right)e^{-\mu(r-r_0)}}{3r\mu^2\rho_C\left(\alpha + \beta r_0\frac{e^{-\mu(r-r_0)}}{r}\right)} \tag{114}$$

and, with the help of Eqs. (109) and (112), we get

$$\omega(r) = \frac{(-3\mu r - 3)\,\bar{r}e^{-\mu(r-r_0)} - e^{\mu(r_0-\bar{r})}\left(\mu^2 r^3 - \mu^2\bar{r}^3 - 3\mu\bar{r}^2 - 3\bar{r}\right)}{3\left(re^{\mu(r_0-\bar{r})} - e^{-\mu(r-r_0)}\bar{r}\right)\mu^2 r^2}. \tag{115}$$

For $r \to \bar{r}$, $\omega(r)$ is an indeterminated form of the kind $0/0$. However close to the point $r = \bar{r}$, the shape function can be approximated by

$$b(r) \underset{r\to\bar{r}}{\simeq} O\left((r-\bar{r})^2\right), \tag{116}$$

while $b'(r)$ can be approximated by

$$b'(r) \underset{r\to\bar{r}}{\simeq} \frac{3\bar{r}\mu^2 e^{\mu(r_0-\bar{r})}(\mu\bar{r}+1)\,r_0}{e^{\mu(r_0-\bar{r})}\left(\mu^2 r_0^3 - \mu^2\bar{r}^3 - 3\mu\bar{r}^2 - 3\bar{r}\right) + (3\mu r_0 + 3)\,\bar{r}}(r-\bar{r}) + O(r-\bar{r})^2. \tag{117}$$

Thus, even in this case, we can assume that

$$\omega(r) = 0 \qquad r \geq \bar{r}. \tag{118}$$

On the throat the analytic form of $\omega(r)$ is far to be simple. Indeed, we find

$$\omega(r_0) = \frac{(-3\mu r_0 - 3)\,\bar{r} - e^{\mu(r_0-\bar{r})}\left(\mu^2 r_0^3 - \mu^2\bar{r}^3 - 3\mu\bar{r}^2 - 3\bar{r}\right)}{3\left(r_0 e^{\mu(r_0-\bar{r})} - \bar{r}\right)\mu^2 r_0^2}. \tag{119}$$

However, one finds that

$$\omega(r_0) \underset{\mu\to 0}{\simeq} \frac{-2r_0^2 + r_0\bar{r} + \bar{r}^2}{6r_0^2} \geq 0 \quad \text{for} \quad \bar{r} \geq r_0, \tag{120}$$

while

$$\omega(r_0) \underset{\mu\to\infty}{\simeq} \frac{1}{r_0\mu} + \frac{1}{\mu^2 r_0^2} \to 0. \tag{121}$$

However when $\bar{r} \gg r_0$, we get

$$\omega(r_0) \underset{\bar{r}\gg r_0}{\simeq} \frac{1}{r_0\mu} + \frac{1}{\mu^2 r_0^2}, \tag{122}$$

which is finite and positive. Therefore we can conclude that from an energy density of the form (108), it is possible to extract another shape function which generalizes an ABTW. It is important to observe that such a generalization is realized because of the presence of a repulsive Yukawa–Casimir profile, otherwise for a choice of the form

$$\rho(\bar{r}) = -\frac{\rho_C}{2}\left(1 + r_0\frac{e^{-\mu(\bar{r}-r_0)}}{r}\right) \simeq -\frac{\rho_C}{2} \neq 0, \tag{123}$$

the energy density outside the region $r > \bar{r}$ is not Minkowski. Although interesting, the profile (108) has the defect of having a way to compare the throat radius with the original Casimir source, like in Eq. (107).

4.3 $\rho(r) = \frac{r_0\rho_C}{r}\left(\alpha e^{-\mu(r-r_0)} - (1-\alpha)\,e^{-\nu(r-r_0)}\right)$

To this purpose, we fix our attention on an energy density profile which can reproduce both a Yukawa behavior and an ABTW. This is represented by

$$\rho(r) = \frac{r_0\rho_C}{r}\left(\alpha e^{-\mu(r-r_0)} - (1-\alpha)\,e^{-\nu(r-r_0)}\right) \qquad \mu, \nu > 0. \tag{124}$$

When $\alpha = 0$, we obtain the profile (101), while when $\alpha = 1$, we obtain its repulsive version. $\forall\alpha \neq 0, 1$, we have a linear superposition of the Yukawa–Casimir profile. The combination of an attractive and repulsive potential is suggested also by the potential (108) together with the option (109) . Note that the option (109) is relevant only if one wishes to reproduce an ABTW. If such an option is not adopted the existence of a traversable wormhole is guaranteed the same. For the profile (124), it is immediate to calculate the form of the shape function

$$b(r) = r_0 + \frac{r_0(1+\nu r_0)(\alpha-1)\,\rho_C\kappa}{\nu^2} + \frac{\alpha r_0\kappa\rho_C(1+\mu r_0)}{\mu^2} - \frac{r_0\kappa\rho_C(\nu r+1)(\alpha-1)\,e^{-\nu(r-r_0)}}{\nu^2} - \frac{\alpha r_0\kappa\rho_C(\mu r+1)\,e^{-\mu(r-r_0)}}{\mu^2} \tag{125}$$

and for $r \to \infty$, we find

$$b(r) \underset{r\to\infty}{\simeq} r_0 + \frac{r_0(1+\nu r_0)(\alpha-1)\rho_C\kappa}{\nu^2} + \frac{\alpha r_0\kappa\rho_C(1+\mu r_0)}{\mu^2} \tag{126}$$

which can be set to zero if

$$\alpha = \frac{\mu^2\left(\kappa\nu r_0\rho_C + \rho_C\kappa - \nu^2\right)}{\left(\mu^2\nu r_0 + \mu\nu^2 r_0 + \mu^2 + \nu^2\right)\rho_C\kappa}. \tag{127}$$

Plugging Eq. (127) into (125), one finds

$$b(r) = r_0\frac{\left(e^{-\nu(r-r_0)}\left(\rho\kappa(1+\mu r_0)+\mu^2\right)(\nu r+1) - e^{-\mu(r-r_0)}\left(\rho\kappa(1+\nu r_0)-\nu^2\right)(\mu r+1)\right)}{(\nu r_0+1)\mu^2+\mu\nu^2 r_0+\nu^2} \tag{128}$$

which is useful to compute the function of the EoS $\omega(r)$

$$\omega(r) = \frac{\left(\rho\kappa(\mu r_0+1)+\mu^2\right)e^{-\nu(r-r_0)}(\nu r+1) - \left((\nu r_0+1)\rho\kappa-\nu^2\right)(\mu r+1)e^{-\mu(r-r_0)}}{r^2\left(\nu^2\left(\rho(\mu r_0+1)\kappa+\mu^2\right)e^{-\nu(r-r_0)} - \mu^2\left((\nu r_0+1)\rho\kappa+\nu^2\right)e^{-\mu(r-r_0)}\right)}. \tag{129}$$

This time the function $\omega(r)$ goes to zero for large values of r, while on the throat one gets

$$\omega(r_0) = \frac{(1+\nu r_0)\mu^2 + (1+\mu r_0)\nu^2}{2\nu^2 r_0^2\mu^2 + r_0^3\rho\kappa\nu\mu(\nu-\mu) + r_0^2\rho\kappa\left(\nu^2-\mu^2\right)}. \tag{130}$$

Even in this case, if we desire to extract information on the throat size, we need to compare $\omega(r_0)$ with a physical source like the Casimir source. To do this, we assume that

$$\omega(r_0) = 1. \tag{131}$$

To do calculations in practice, it is useful the following setting

$$\mu = \frac{m}{r_0}; \qquad \nu = \frac{n}{r_0} \qquad \text{and} r_0 = \frac{x}{\sqrt{\rho\kappa}}; \qquad m, n \in \mathbb{R}_+ \tag{132}$$

and Eq. (131) becomes

$$\frac{(1+n)m^2 + (1+m)n^2}{\left(2n^2m^2 + x^2nm(n-m) + x^2\left(n^2-m^2\right)\right)} = 1, \tag{133}$$

whose solution is

$$x = \frac{\sqrt{\left(2n^2-n-1\right)m^2 - n^2m - n^2}}{\sqrt{((n+1)m+n)(m-n)}}. \tag{134}$$

To constraint r to be very small, we observe that the r.h.s. of Eq. (134) vanishes when

$$m_\pm = n\frac{1 \pm \sqrt{9n^2-4n-4}}{2\left(2n^2-n-1\right)} \qquad n \geq \frac{2}{9}\left(1+\sqrt{10}\right) \simeq 0.92495. \tag{135}$$

m_- will be discarded because is the negative root. Note that it is not necessary to have a vanishing x, rather we need an x with a value of the order of 10^{-10} or greater. This is due to the rescaling in (132) setting the size of the wormhole throat to be of the order of

$$r_0 \simeq x \times 10^{17} m. \tag{136}$$

Therefore we can conclude that with a linear combination of two Yukawa–Casimir profiles, actually a difference of them, one finds a traversable wormhole with a throat that can be fine tuned with respect to the original Casimir source.

5 Conclusions

In this paper we have taken under examination the modification of the Casimir wormhole examined in Ref. [17] which uses, as a source, the negative energy density of the Casimir device. Differently from Ref. [17], this time we have imposed the ZTF condition to obtain different solutions. We have found that the ZTF condition can be imposed only if we modify the form of the energy density or the form of the shape function. To this purpose we have considered Yukawa type modifications of the original profile. The motivation for this choice stands in the attempt to detect signals of variations of the ordinary gravitational field even for TW. In this context, it appears interesting to note that, recently, investigations on how it is possible to see signals of a TW have been considered, especially in the framework of extended theories of gravitation [37–40]. To further proceed, we have divided the paper in two parts: the first part is devoted to the analysis of the modification of the shape function with a Yukawa term and the second part is devoted to the modification of the original Casimir energy density with an appropriate Yukawa term. In the first part we have examined three different profiles having in common the Casimir wormhole shape function, namely

$$b(r) = \frac{2}{3}r_0 + \frac{r_0^2}{3r} \tag{137}$$

and we have included a Yukawa modification of the type $\exp(-\mu(r-r_0))$ acting on every single term and globally. Two of the three profiles have shown features compatible with the throat size estimated in Ref. [36], while one of them has developed a divergent behavior for large values of the radial variable r. I recall to the reader that, in this paper, we have examined the Casimir energy density with the plates separation considered as a parameter and not as a variable. This choice has led to have a huge throat size instead of a Planckian one like in Ref. [17]. As a further analysis, we have also considered a superposition of different categories of TW. Even in this case the resulting size of the wormholes throat is huge and compatible with the size of a GABTW described in Ref. [36]. I recall again to the reader that the huge size of the wormhole throat has been found by imposing that the inhomogeneous function of the EoS (11) at the throat has a constant value compatible with the ordinary Casimir relationship $p = 3\rho$. In the second part, we have fixed the form of the energy density and we have deduced the form of the shape function with the help of the first of the EFE (5). Even in this case, we have analyzed three different profiles. Since every of these profiles has produced a correction to the size of the throat at infinity, we have considered the possibility of taken another generalization for the ABTW. In particular we have found a value of the radial variable, located at $r = \bar{r}$ where the shape function vanishes and we have truncated the region outside $r = \bar{r}$. In this way the pressure outside the region $r = \bar{r}$ vanishes. However, for the profile of Sect. 4.1, the energy density does not vanish for $r \geq \bar{r}$, it is small because of the exponential but not nought. Therefore the structure of an ABTW or GABTW cannot be reproduced. On the contrary, for the profiles discussed in Sects. 4.2 and 4.3, it is possible to reproduce an ABTW in a generalized form different by the GABTW at the price of introducing a repulsive potential, namely we have the difference of two Yukawa profiles: one attractive and one repulsive. I recall the reader that an ABTW is defined by the following shape and redshift functions

$$\begin{aligned} & b(r) = r_0\left(1-\left(\frac{r-r_0}{a}\right)\right)^2, \qquad \Phi(r) = 0; \\ & \quad r_0 \leq r \leq r_0 + a \\ & \quad b(r) = 0, \qquad \Phi(r) = 0; \qquad r \geq r_0 + a. \end{aligned} \tag{138}$$

Therefore outside the location $r = r_0 + a$, the spacetime is Minkowski. The same happens for a GABTW, where one finds

$$\begin{aligned} & b(r) = r_0\frac{(1-\mu\,(r-r_0))^{\alpha}}{(1-\nu\,(r-r_0))^{\beta}}, \\ & \quad \Phi(r) = 0; \qquad r_0 \leq r \leq r_0 + 1/\mu \\ & \quad b(r) = 0, \qquad \Phi(r) = 0; \qquad r \geq r_0 + 1/\mu. \end{aligned} \tag{139}$$

Of course $1/\mu$ plays the rôle of a and viceversa. Note that it is the exponent in Eq. (138) and in Eq. (139) that plays a key rôle to determine the Minkowski structure for $r \geq r_0 + a$ or $r \geq r_0 + 1/\mu$. Such a property is completely absent for the profiles discussed in Sects. 4.1, 4.2 and 4.3 and one must build a profile that potentially can develop such a property. This is the reason why a repulsive Yukawa profile is necessary to have a vanishing value outside a certain region. Of course this is related to the attempt to reproduce some of the features of an ABTW. If one abandons this request, the repulsive potential is not fundamental. However, our insistence to reproduce the features of an ABTW is justified by the fact the negative energy density is concentrated in a very small region of the space and there is no redshift. Coming back to the profile (124) in Sect. 4.3, it represents again a difference of two Yukawa profiles, and behaves "*like*" a ABTW, because for a sufficiently large values of r, $\rho(r)$, $p_r(r)$ and $p_t(r)$ vanish. Nevertheless, because of the exponentials, it is not necessary that the radial value r needs to be really large. Another interesting feature of the profile (124) is that, this time, we can fine tune the throat size down to acceptable values, which is exactly what one needs. Always on the side of phantom energy I proposed the idea of *Self-Sustained Traversable Wormholes*, namely TW sustained by their own quantum fluctuations [41–45]. Even in this case, because the quantum fluctuation carried by the graviton behaves like the ordinary Casimir effect, we found that no need for phantom contribution is necessary. On this context, in a next paper we will explore how behaves a system formed by the *Casimir TW*, here analyzed, and the corresponding self-sustained TW version. To conclude, we have also to point out that in the context of *Self-Sustained Traversable Wormholes*, namely TW sustained by their own quantum fluctuations [41–45], could be interesting to consider how the Casimir source and the quantum fluctuation carried by the graviton combine to see if the GABTW can be self-sustained in this context. In this picture, the Casimir source could be interpreted as the switch on to power the traversability of the wormhole.

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Appendix A: Features of the superposition of traversable wormholes

In this section we are going to explore some of the features of the profile (79) which is quite general to include the other profiles discussed in Sect. 2 of this paper. We begin to examine the proper length which is defined as

$$b(r) = r_0 \left(\alpha \exp\left(-\mu\left(r-r_0\right)\right) + (1-\alpha)\left(\frac{r_0}{r}\right)^c \times \exp\left(-\nu\left(r-r_0\right)\right)\right), \tag{A1}$$

$$l\left(r\right) = \pm \int_{r_0}^{r} \frac{dr'}{\sqrt{1-\frac{b(r')}{r'}}}, \tag{A2}$$

where the "±" depends on the wormhole side we are. In the case of the shape function (79), we know that it vanishes exponentially for $r \to +\infty$. This is true for $\mu \neq 0$ and $\nu \neq 0$. For instance for $\mu = \nu = 0$ and $c = 1$, one finds that the shape function is represented by the EB wormhole (48) , whose proper length is

$$l\left(r\right) = \pm\sqrt{r^2 - r_0^2}. \tag{A3}$$

For this reason, for the other cases it is sufficient to consider what happens close to the throat. In general, we can write

$$\frac{\sqrt{r}}{\sqrt{r-b\left(r\right)}} \underset{r\to r_0}{\simeq} \frac{\sqrt{r}}{\sqrt{1-b'\left(r_0\right)}\sqrt{r-r_0}}. \tag{A4}$$

Thus the approximated proper length becomes

$$\begin{aligned} l\left(r\right) &\underset{r\to r_0}{\simeq} \pm\frac{1}{\sqrt{1-b'\left(r_0\right)}} \int_{r_0}^{r} \frac{\sqrt{r'}dr'}{\sqrt{r'-r_0}} \\ &= \pm\frac{r_0}{\sqrt{1-b'\left(r_0\right)}} \left[\sqrt{\frac{r}{r_0}}\sqrt{\frac{r}{r_0}-1} + \ln\left(\sqrt{\frac{r}{r_0}} + \sqrt{\frac{r}{r_0}-1}\right)\right]. \end{aligned} \tag{A5}$$

A further approximation leads to

$$l\left(r\right) \underset{r\to r_0}{\simeq} \pm\frac{r_0\sqrt{r/r_0}\sqrt{r/r_0-1}}{\sqrt{1-b'\left(r_0\right)}} \tag{A6}$$

and, in the case of the shape function (79) , (A6) assumes the form

$$l\left(r\right) = \pm\frac{r_0\sqrt{r/r_0}\sqrt{r/r_0-1}}{\sqrt{1+r_0\left(\alpha\mu+\nu\left(1-\alpha\right)\right)+c\left(1-\alpha\right)}}. \tag{A7}$$

The argument of the denominator is positive if and only if the flare-out condition is satisfied. However, we can use the constraint (86) and (90) to have a better estimate. We find

$$l\left(r\right) = \frac{6\sqrt{r-r_0}\sqrt{r_0}\sqrt{5}d}{\sqrt{\pi^{3/2}\sqrt{30}l_p r_0 + 30\,d^2}} \tag{A8}$$

or by means of the constraint (87), we can write the proper length only in terms of the plates separation

$$l\left(r\right) = \frac{\sqrt{3}d\sqrt[4]{10}}{\pi^{\frac{3}{2}}l_p}\sqrt{l_p\pi^{\frac{3}{2}}\sqrt{3}r - 3d^2\sqrt{10}}. \tag{A9}$$

In a similar way, to compute the embedded surface, we need to evaluate

$$z\left(r\right) = \pm\int_{r_0}^{r}\frac{dr'}{\sqrt{\frac{r'}{b(r')}-1}}, \tag{A10}$$

which, close to the throat, becomes

$$\begin{aligned} z\left(r\right) &= \pm\int_{r_0}^{r}\frac{\sqrt{b\left(r'\right)}dr'}{\sqrt{r'-b\left(r'\right)}} \underset{r\to r_0}{\simeq} \pm\frac{\sqrt{r_0}}{\sqrt{1-b'\left(r_0\right)}}\int_{r_0}^{r} \times\frac{dr'}{\sqrt{r'-r_0}} \\ &= \pm\frac{2\sqrt{r_0}\sqrt{r-r_0}}{\sqrt{1+r_0\left(\alpha\mu+\nu\left(1-\alpha\right)\right)+c\left(1-\alpha\right)}}. \end{aligned} \tag{A11}$$

To further investigate the properties of the Traversable Wormholes described by the shape function (79), we consider the computation of the total gravitational energy for a wormhole, defined as [31]

$$E_G\left(r\right) = \int_{r_0}^{r}\left[1-\sqrt{\frac{1}{1-b\left(r'\right)/r'}}\right]\rho\left(r'\right)dr'r'^2 + \frac{r_0}{2G} = M - M_{\pm}^{P}, \tag{A12}$$

where

$$M = \int_{r_0}^{r} 4\pi\rho\left(r'\right)r'^2dr' + \frac{r_0}{2G} \tag{A13}$$

is the total mass and

$$M_{\pm}^{P} = \pm\int_{r_0}^{r}\frac{4\pi\rho\left(r'\right)r'^2}{\sqrt{1-b\left(r'\right)/r'}}dr'. \tag{A14}$$

is the proper mass. In particular we find for the total mass

$$Mc^2 = \int_{r_0}^{r} 4\pi\rho\left(r'\right)r'^2dr' + \frac{r_0}{2G} = \frac{4\pi}{\kappa}\int_{r_0}^{r} b'\left(r'\right)dr' + \frac{r_0}{2G} = \frac{4\pi}{\kappa}b\left(r\right) \underset{r\to\infty}{\to} 0 \tag{A15}$$

where we have used the relationship (91) and we have momentarily reintroduced the speed of light. For $M_{\pm}^{P}$, we can estimate the value of the integral close to the throat, following what has been done for the proper length and the embedded surface. We can write

$$M_{\pm}^{P}c^2 = \pm\int_{r_0}^{r}\frac{4\pi\rho\left(r'\right)r'^2}{\sqrt{1-b\left(r'\right)/r'}}dr' \underset{r\to r_0}{\simeq} \pm\frac{b'\left(r_0\right)}{2G\sqrt{1-b'\left(r_0\right)}} \times\int_{r_0}^{r}\frac{dr'}{\sqrt{r-r_0}}$$

$$\simeq \pm \frac{(\alpha-1)(\nu r_0+c)-r_0\mu\alpha}{G\sqrt{1+r_0(\alpha\mu+\nu(1-\alpha))+c(1-\alpha)}}\sqrt{r-r_0}, \tag{A16}$$

where the "±" depends one the wormhole side we are. Thus the total gravitational energy (A12) becomes

$$E_G(r) \simeq \begin{matrix} \frac{r_0}{2G} & r \to r_0 \\ 0 & r \to \infty \end{matrix} . \tag{A17}$$

Even for the total energy, this is true for $\alpha \neq 0$, $\nu \neq 0$ and $c \neq 1$. Indeed for $\alpha = \nu = 0$ and $c = 1$ we can write the total gravitational energy of the EB wormhole (48) which reduces to

$$E_G(r) = \frac{r_0}{3G}\left(1 \mp \frac{\sqrt{3}\pi c^4}{6}\right). \tag{A18}$$

It is interesting to note that the total energy is concentrated completely on the throat and at infinity vanishes showing a screening mechanism: in other words, the "*imprint at infinity*" disappears [16]. Another important traversability condition is that the acceleration felt by the traveller should not exceed Earth's gravity $g_\oplus \simeq 980\ cm/s^2$. In an orthonormal basis of the traveller's proper reference frame, we can find

$$|\mathbf{a}| = \left|\sqrt{1-\frac{b(r)}{r}}e^{-\Phi(r)}\left(\gamma e^{\Phi(r)}\right)'\right| \leq \frac{g_\oplus}{c^2} \tag{A19}$$

and in this case, because $\Phi(r) = 0$, the traveller has no acceleration, which is in agreement with Ref. [15]. As regards the lateral tidal forces, we find

$$\begin{aligned} &\left|\frac{\gamma^2c^2}{2r^2}\left[\frac{v^2(r)}{c^2}\left(b'(r)-\frac{b(r)}{r}\right)+2r(r-b(r))\Phi'(r)\right]\right||\eta| \\ &= \left|\frac{\gamma^2c^2}{2r^3}\left[-\frac{v^2(r)b(r)}{c^2}\left(\frac{1}{\omega(r)}+1\right)\right]\right||\eta| \leq g_\oplus, \end{aligned} \tag{A20}$$

where we have used the relationship (11). This is a constraint about the velocity with which observers traverse the wormhole. η represents the size of the traveller which can be fixed approximately equal, at the symbolic value of 2 m [15]. If we assume a constant speed v and $\gamma \simeq 1$, close to the throat, the lateral tidal constraint becomes

$$\begin{aligned} &\left|\frac{\gamma^2c^2}{2r_0^2}\left[-\frac{v^2(r_0)}{c^2}(1+[r_0\mu\alpha+(1-\alpha)(\nu r_0+c)])\right]\right||2| \\ &\simeq \left|\left[\frac{v^2(r_0)}{r_0^2}\right](1+[r_0\mu\alpha+(1-\alpha)(\nu r_0+c)])\right| \lesssim g_\oplus \\ &\implies v \lesssim r_0\sqrt{(1+[r_0\mu\alpha+(1-\alpha)(\nu r_0+c)])g_\oplus}. \end{aligned} \tag{A21}$$

If the observer has a vanishing v, then the tidal forces are null. Note that the total time defined by

$$\Delta t = \int_{r_0}^{r}\frac{e^{-\phi(r')}dr'}{v\sqrt{1-\frac{b(r')}{r'}}} \tag{A22}$$

and the total proper time given by

$$\Delta\tau = \int_{r_0}^{r}\frac{dr'}{v\sqrt{1-\frac{b(r')}{r'}}} \tag{A23}$$

are the same for the profile (79) because the redshift function is nought. Assuming that v is approximately constant, we can use the estimate (A21) to complete the evaluation of the crossing time which approximately is

$$\Delta\tau = \Delta t \simeq \frac{2\times 10^4}{(1+r_0(\alpha\mu+\nu(1-\alpha))+c(1-\alpha))\sqrt{g_\oplus}}. \tag{A24}$$

However, we can use the constraint (84), (86), (87) and (90) to have a better estimate of the crossing time which becomes

$$\Delta\tau = \Delta t \simeq \frac{3\times 10^4}{2\sqrt{g_\oplus}} \simeq 4.79\times 10^3 s, \tag{A25}$$

where we have considered a possible time trip in going from one station located in the lower universe, say at $l = -l_1$, and ending up in the upper universe station, say at $l = l_2$. Following Ref. [15], we have located l_1 and l_2 at a value of the radius such that

$$l_1 \simeq l_2 \simeq \frac{10^4 r_0}{\sqrt{1-b'(r_0)}} \tag{A26}$$

that it means $1-b(r)/r \simeq 1$ which is in agreement with the estimates found in Ref. [15]. The last property we are going to discuss is the " total amount" of ANEC violating matter in the spacetime [27] which is described by Eq. (A27). For the metric (79), one obtains

$$\begin{aligned} I_V &= \frac{1}{\kappa}\int_{r_0}^{r_0+\varepsilon}(r-b(r))\left[\ln\left(\frac{e^{2\phi(r)}}{1-\frac{b(r)}{r}}\right)\right]'dr \\ &= \frac{1}{\kappa}\int_{r_0}^{r_0+\varepsilon}\frac{b(r)}{r}\left(1+\frac{1}{\omega(r)}\right)dr \qquad \text{(A27)} \\ &\simeq \frac{1}{\kappa}\left[\int_{r_0}^{r_0+\varepsilon}\alpha\mu r_0-\beta\nu r_0+1+c(r)(r-r_0)\right]dr, \qquad \text{(A28)} \end{aligned}$$

where we have approximated the expression close to the throat and where we have defined

$$\begin{aligned} c(r) = &-\alpha\mu - r_0^{-1} + \beta\nu - \alpha^2\mu^2 r_0 + \alpha\mu r_0\beta\,\nu + \alpha\mu^2 r_0 \\ &-(-\alpha\mu r_0+\beta\nu r_0)\beta\nu - \beta\nu^2 r_0. \end{aligned} \tag{A29}$$

After the integration, we find

$$I_V = \frac{1}{\kappa}\left(\frac{3}{2}\alpha r_0 - \frac{1}{2}\alpha^2 r_0 + \frac{\nu r_0 \beta\alpha}{\mu} - \frac{\beta\nu r_0}{\mu} - \frac{\alpha}{2\mu} + \frac{1}{\mu} - \frac{\beta\nu^2 r_0}{2\mu^2} - \frac{\beta^2\nu^2 r_0}{2\mu^2} + \frac{\beta\nu}{2\mu^2} - \frac{1}{2\mu^2 r_0}\right) \underset{\mu r_0 \gg 1}{\simeq} \frac{1}{\kappa}\left(\frac{3}{2}\alpha r_0 - \frac{1}{2}\alpha^2 r_0\right), \tag{A30}$$

and the result is finite. Therefore we can conclude that, in proximity of the throat the ANEC can be arbitrarily small as it should be.

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