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# Does phantom energy contribute to self sustained traversable wormholes?

**Remo Garattini**

**Abstract.** We compute the graviton one loop contribution to a classical energy in a *traversable* wormhole background. The form of the shape function considered is obtained by the equation of state $p = \omega\rho$. We investigate the size of the wormhole as a function of the parameter $\omega$. The investigation is evaluated by means of a variational approach with Gaussian trial wave functionals. A zeta function regularization is involved to handle with divergences. A renormalization procedure is introduced and the finite one loop energy is considered as a *self-consistent* source for the traversable wormhole.The case of the phantom region is briefly discussed.

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## 1. Introduction

The discovery that our universe is undergoing an accelerated expansion[1] leads to reexamine the Friedmann-Robertson-Walker equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}\left(\rho + 3p\right), \tag{1}$$

to explain why the scale factor obeys $\ddot{a} > 0$. Indeed, it is evident from the previous formula, that a sort of *dark energy* is needed to cause a negative pressure with equation of state

$$p = \omega\rho. \tag{2}$$

A value of $\omega < -1/3$ is required for the accelerated expansion, while $\omega = -1$ corresponds to a cosmological constant. A specific form of dark energy, denoted *phantom energy* has also been proposed with the property of having $\omega < -1$. It is interesting to note that the phantom energy violates the null energy condition, $p + \rho < 0$, necessary ingredient to sustain the traversability of wormholes. A wormhole can be represented by two asymptotically flat regions joined by a bridge. To exist, it must satisfy the Einstein field equations: one example is represented by the Schwarzschild solution. One of the prerogatives of a wormhole is its ability to connect two distant points in space-time. In this amazing perspective, it is immediate to recognize the possibility of traveling crossing wormholes as a short-cut in space and time. Unfortunately, although there is no direct evidence, a Schwarzschild wormhole does not possess this property.

It is for this reason that in a pioneering work Morris and Thorne[2] and subsequently Morris, Thorne and Yurtsever[3] studied a class of wormholes termed "*traversable*". Unfortunately, the traversability is accompanied by unavoidable violations of null energy conditions, namely, the matter threading the wormhole's throat has to be "*exotic*". It is clear that the existence of dark and phantom energy supports the class of exotic matter. In this direction, Lobo[4], Kuhfittig[5] and Sushkov[6] have considered the possibility of sustaining the wormhole traversability with the help of phantom energy. In a previous work, we explored the possibility that a wormhole can be sustained by its own quantum fluctuations[7]. In practice, it is the graviton propagating on the wormhole background that plays the role of the "*exotic*" matter. This has not to appear as a surprise, because the computation involved, namely the one loop contribution of the graviton to the total energy, is quite similar to compute the Casimir energy on a fixed background. It is known that, for different physical systems, Casimir energy is negative and this is exactly one of the features that the exotic matter should possess. In particular, we conjectured that quantum fluctuations can support the traversability as effective source of the semiclassical Einstein's equations. However in Ref.[7], we limited the analysis in the region where the equation of state(2) assumes the particular value $\omega = 1$. In this paper, we will consider $\omega \in (0, +\infty)$, although the semiclassical approach can be judged suspicious because of the suspected validity of semiclassical methods[1] at the Planck scale[8].

## 2. The effective Einstein equations and the traversable wormhole metric

We begin with a look at the classical Einstein equations

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3}$$

where $T_{\mu\nu}$ is the stress-energy tensor, $G_{\mu\nu}$ is the Einstein tensor and $\kappa = 8\pi G$. Consider a separation of the metric into a background part, $\bar{g}_{\mu\nu}$, and a perturbation, $h_{\mu\nu}$,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{4}$$

The Einstein tensor $G_{\mu\nu}$ can also be divided into a part describing the curvature due to the background geometry and that due to the perturbation,

$$G_{\mu\nu}\left(g_{\alpha\beta}\right) = G_{\mu\nu}\left(\bar{g}_{\alpha\beta}\right) + \Delta G_{\mu\nu}\left(\bar{g}_{\alpha\beta}, h_{\alpha\beta}\right), \tag{5}$$

where, in principle $\Delta G_{\mu\nu}\left(\bar{g}_{\alpha\beta}, h_{\alpha\beta}\right)$ is a perturbation series in terms of $h_{\mu\nu}$. In the context of semiclassical gravity, Eq.(3) becomes

$$G_{\mu\nu} = \kappa \left\langle T_{\mu\nu}\right\rangle^{ren}, \tag{6}$$

where $\left\langle T_{\mu\nu}\right\rangle^{ren}$ is the renormalized expectation value of the stress-energy tensor operator of the quantized field. If the matter field source is absent, nothing prevents us from defining an effective stress-energy tensor for the fluctuations as[2]

$$\left\langle T_{\mu\nu}\right\rangle^{ren} = -\frac{1}{\kappa}\left\langle \Delta G_{\mu\nu}\left(\bar{g}_{\alpha\beta}, h_{\alpha\beta}\right)\right\rangle^{ren}. \tag{7}$$

From this point of view, the equation governing quantum fluctuations behaves as a backreaction equation. If we fix our attention to the energy component of the Einstein field equations, we

[1] To this purpose, see also paper of Hochberg, Popov and Sushkov[9] and the paper of Khusnutdinov and Sushkov[10].
[2] Note that our approach is very close to the gravitational *geon* considered by Anderson and Brill[12]. The relevant difference is in the averaging procedure.

need to introduce a time-like unit vector $u^{\mu}$ such that $u \cdot u = -1$. Then the semi-classical Einstein's equations (6) projected on the constant time hypersurface $\Sigma$ become

$$G_{\mu\nu}\left(\bar{g}_{\alpha\beta}\right)u^{\mu}u^{\nu}=\kappa\left\langle T_{\mu\nu}u^{\mu}u^{\nu}\right\rangle^{ren}=-\left\langle \Delta G_{\mu\nu}\left(\bar{g}_{\alpha\beta},h_{\alpha\beta}\right)u^{\mu}u^{\nu}\right\rangle^{ren}. \tag{8}$$

To further proceed, it is convenient to consider the associated tensor density and integrate over $\Sigma$. This leads to

$$\frac{1}{2\kappa}\int_{\Sigma}d^{3}x\sqrt{{}^{3}\bar{g}}G_{\mu\nu}\left(\bar{g}_{\alpha\beta}\right)u^{\mu}u^{\nu}$$
$$=-\int_{\Sigma}d^{3}x\mathcal{H}^{(0)}=-\frac{1}{2\kappa}\int_{\Sigma}d^{3}x\sqrt{{}^{3}\bar{g}}\left\langle \Delta G_{\mu\nu}\left(\bar{g}_{\alpha\beta},h_{\alpha\beta}\right)u^{\mu}u^{\nu}\right\rangle^{ren}, \tag{9}$$

where $\mathcal{H}^{(0)}$ is the background field super-hamiltonian. Thus the fluctuations in the Einstein tensor are, in this context, the fluctuations of the hamiltonian. To compute the expectation value of the perturbed Einstein tensor in the transverse-traceless sector, we use a variational procedure with gaussian wave functionals. In practice, the right hand side of Eq.(9) will be obtained by expanding

$$E_{wormhole}=\frac{\left\langle \Psi\left|H_{\Sigma}\right|\Psi\right\rangle}{\left\langle \Psi|\Psi\right\rangle}=\frac{\left\langle \Psi\left|H_{\Sigma}^{(0)}+H_{\Sigma}^{(1)}+H_{\Sigma}^{(2)}+\ldots\right|\Psi\right\rangle}{\left\langle \Psi|\Psi\right\rangle} \tag{10}$$

and retaining only quantum fluctuations contributing to the effective stress energy tensor. $H_{\Sigma}^{(i)}$ represents the hamiltonian approximated to the $i^{th}$ order in $h_{ij}$ and $\Psi$ is a *trial wave functional* of the gaussian form. Then Eq.(9) becomes

$$H_{\Sigma}^{(0)}=\int_{\Sigma}d^{3}x\mathcal{H}^{(0)}=-\frac{\left\langle \Psi\left|H_{\Sigma}^{(1)}+H_{\Sigma}^{(2)}+\ldots\right|\Psi\right\rangle}{\left\langle \Psi|\Psi\right\rangle}. \tag{11}$$

The chosen background to compute the quantity contained in Eq.(9) will be that of a traversable wormhole. In Schwarzschild-like coordinates, the traversable wormhole metric can be cast into the form

$$ds^{2}=-\exp\left(-2\phi\left(r\right)\right)dt^{2}+\frac{dr^{2}}{1-\frac{b(r)}{r}}+r^{2}\left[d\theta^{2}+\sin^{2}\theta d\varphi^{2}\right]. \tag{12}$$

where $\phi\left(r\right)$ is called the redshift function, while $b\left(r\right)$ is called the shape function. Using the Einstein field equation (3), in an orthonormal reference frame, we obtain the following set of equations

$$\rho\left(r\right)=\frac{1}{8\pi G}\frac{b'}{r^{2}}, \tag{13}$$

$$p_{r}\left(r\right)=\frac{1}{8\pi G}\left[\frac{2}{r}\left(1-\frac{b\left(r\right)}{r}\right)\phi'-\frac{b}{r^{3}}\right], \tag{14}$$

$$p_{t}\left(r\right)=\frac{1}{8\pi G}\left(1-\frac{b\left(r\right)}{r}\right)\left[\phi''+\phi'\left(\phi'+\frac{1}{r}\right)\right]-\frac{b'r-b}{2r^{2}}\left(\phi'+\frac{1}{r}\right), \tag{15}$$

in which $\rho\left(r\right)$ is the energy density, $p_{r}\left(r\right)$ is the radial pressure, and $p_{t}\left(r\right)$ is the lateral pressure. Using the conservation of the stress-energy tensor, in the same orthonormal reference frame, we get

$$p_{r}'=\frac{2}{r}\left(p_{t}-p_{r}\right)-\left(\rho+p_{r}\right)\phi'. \tag{16}$$

The Einstein equations can be rearranged to give

$$b'=8\pi G\rho\left(r\right)r^{2}, \tag{17}$$

$$\phi' = \frac{b + 8\pi G p_r r^3}{2r^2\left(1 - \frac{b(r)}{r}\right)}. \tag{18}$$

Now, we introduce the equation of state $p_r = \omega\rho$, and using Eq.(13), Eq.(18) becomes

$$\phi' = \frac{b + \omega b' r}{2r^2\left(1 - \frac{b(r)}{r}\right)}. \tag{19}$$

The redshift function can be set to a constant with respect to the radial distance, if

$$b + \omega b' r = 0. \tag{20}$$

The integration of this simple equation leads to

$$b(r) = r_t\left(\frac{r_t}{r}\right)^{\frac{1}{\omega}}, \tag{21}$$

where we have used the condition $b(r_t) = r_t$. Thus, the line element (12) becomes

$$ds^2 = -Adt^2 + \frac{dr^2}{1 - \left(\frac{r_t}{r}\right)^{1+\frac{1}{\omega}}} + r^2\left[d\theta^2 + \sin^2\theta d\varphi^2\right], \tag{22}$$

where $A$ is a constant coming from $\phi' = 0$ which can be set to one without loss of generality. The parameter $\omega$ is restricted by the following conditions

$$b'(r_t) < 1 \tag{23}$$

and

$$\frac{b(r)}{r} \to 0 \qquad when \qquad r \to +\infty. \tag{24}$$

This implies that $\omega \in (-\infty, -1) \cup (0, +\infty)$. Proper radial distance is related to the shape function by

$$\begin{aligned} l(r) &= \pm\int_{r_t}^{r}\frac{dr'}{\sqrt{1 - \frac{b_\pm(r')}{r'}}} \\ &= \pm r_t\frac{2\omega}{\omega+1}\sqrt{\rho^{\left(1+\frac{1}{\omega}\right)} - 1}\,{}_2F_1\left(\frac{1}{2}, \frac{1-\omega}{2\omega+2}; \frac{3}{2}; 1 - \rho^{\left(1+\frac{1}{\omega}\right)}\right), \end{aligned} \tag{25}$$

where the plus (minus) sign is related to the upper (lower) part of the wormhole or universe and where ${}_2F_1(a, b; c; x)$ is a hypergeometric function. Two coordinate patches are required, each one covering the range $[r_t, +\infty)$. Each patch covers one universe, and the two patches join at $r_t$, the throat of the wormhole defined by

$$r_t = \min\{r(l)\}. \tag{26}$$

When $\omega = 1$, we recover the special case where $b(r) = r_t^2/r$. To concretely compute the r.h.s of Eq.(11), we note that the correct setting is

$$\int_\Sigma d^3x\sqrt{{}^3\bar{g}}\left\langle \Delta G_{\mu\nu}\left(\bar{g}_{\alpha\beta}, h_{\alpha\beta}\right) u^\mu u^\nu\right\rangle^{ren} = \int_\Sigma d^3x\sqrt{{}^3\bar{g}}\frac{\left\langle \Psi\left|\mathcal{H}^{(2)} - \sqrt{{}^3g}^{(2)}\mathcal{H}^{(0)}\right|\Psi\right\rangle}{\langle\Psi|\Psi\rangle}, \tag{27}$$

where we have considered perturbations of the line element (22) of the type $g_{ij} = \bar{g}_{ij} + h_{ij}$.The linear term disappears because of the Gaussian integration. Following the same procedure of Refs.[14, 15], we arrive at the following relevant expression of the one-loop-like Hamiltonian form for TT (traceless and transverse) deformations

$$H_{\Sigma}^{\perp} = \frac{1}{4}\int_{\Sigma} d^3x\sqrt{g}G^{ijkl}\left[(16\pi G)\,K^{-1\perp}(x,x)_{ijkl} + \frac{1}{(16\pi G)}(\triangle_2)^a_j\,K^{\perp}(x,x)_{iakl}\right]. \tag{28}$$

The propagator $K^{\perp}(x,x)_{iakl}$ comes from a functional integration and it can be represented as

$$K^{\perp}(\overrightarrow{x},\overrightarrow{y})_{iakl} := \sum_{\tau}\frac{h_{ia}^{(\tau)\perp}(\overrightarrow{x})\,h_{kl}^{(\tau)\perp}(\overrightarrow{y})}{2\lambda(\tau)}, \tag{29}$$

where $h_{ia}^{(\tau)\perp}(\overrightarrow{x})$ are the eigenfunctions of $\triangle_2$, whose eigenvalues will be denoted with $\tilde{E}^2(\tau)$. $\tau$ denotes a complete set of indices and $\lambda(\tau)$ are a set of variational parameters to be determined by the minimization of Eq.(28). The expectation value of $H^{\perp}$ is easily obtained by inserting the form of the propagator into Eq.(28)

$$E(\lambda_i) = \frac{1}{4}\sum_{\tau}\sum_{i=1}^{2}\left[(16\pi G)\,\lambda_i(\tau) + \frac{\tilde{E}_i^2(\tau)}{(16\pi G)\,\lambda_i(\tau)}\right]. \tag{30}$$

By minimizing with respect to the variational function $\lambda_i(\tau)$, we obtain the total one loop energy for TT tensors

$$E^{TT} = \frac{1}{4}\sum_{\tau}\left[\sqrt{\tilde{E}_1^2(\tau)} + \sqrt{\tilde{E}_2^2(\tau)}\right]. \tag{31}$$

The above expression makes sense only for $\tilde{E}_i^2(\tau) > 0$, $i = 1, 2$. The meaning of $\tilde{E}_i^2$ will be clarified in the next section. Coming back to Eq.(11), we observe that the value of the wormhole energy on the chosen background is

$$\int_{\Sigma} d^3x\mathcal{H}^{(0)} = -\frac{1}{16\pi G}\int_{\Sigma} d^3x\sqrt{\bar{g}}R^{(3)} = A(\omega)\,\frac{r_t}{G}, \qquad \omega > -1 \tag{32}$$

where

$$A(\omega) = \frac{1}{1+\omega}B\left(\frac{1}{2},\frac{1}{1+\omega}\right) = \frac{\sqrt{\pi}}{(1+\omega)}\frac{\Gamma\left(\frac{1}{1+\omega}\right)}{\Gamma\left(\frac{3+\omega}{2+2\omega}\right)}. \tag{33}$$

$B(x,y)$ is the Beta function and $\Gamma(x)$ is the gamma function. Then the one loop the self-consistent equation for TT tensors becomes

$$A(\omega)\,\frac{r_t}{G} = -E^{TT}. \tag{34}$$

Note that for the special value of $\omega = 1$, we get

$$\frac{\pi r_t}{2G} = -E^{TT}, \tag{35}$$

in agreement with the result of Ref. [7]. Note also that the self-consistency on the hamiltonian as a reversed sign with respect to the energy component of the Einstein field equations. This means that an eventual stable point for the hamiltonian is an unstable point for the effective energy momentum tensor and vice versa.

## 3. Phantom energy and the traversable wormhole

The key point to establish the possible role of phantom energy is the following eigenvalue problem

$$\left(\triangle_2 h^{TT}\right)_i^j = \tilde{E}^2 h_i^j \tag{36}$$

where $\tilde{E}^2$ is the eigenvalue of the corresponding equation, where $\triangle_2$ is the associated Lichnerowicz operator computed on the background of Eq.(22). By following the method of Regge and Wheeler in analyzing the equation as modes of definite frequency, angular momentum and parity[18], we are led to study the following system of PDE's

$$\begin{cases} \left(-\triangle_l + 2\left(\frac{b'(r)}{r^2} - \frac{b(r)}{r^3} - \frac{b'(r)}{r^2}\right)\right) H(r) = \tilde{E}_{1,l}^2 H(r) \\ \\ \left(-\triangle_l + 2\left(\frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3} - \frac{b'(r)}{r^2}\right)\right) K(r) = \tilde{E}_{2,l}^2 K(r) \end{cases}, \tag{37}$$

where $\triangle_l$ is

$$\left(1 - \frac{b(r)}{r}\right)\frac{d^2}{dr^2} + \left(\frac{4r - b'(r)\,r - 3b(r)}{2r^2}\right)\frac{d}{dr} - \frac{l(l+1)}{r^2} - \frac{6}{r^2}\left(1 - \frac{b(r)}{r}\right). \tag{38}$$

Defining reduced fields and passing to the proper geodesic distance from the *throat* of the bridge, the system (37) becomes

$$\begin{cases} \left[-\frac{d^2}{dx^2} + V_1(r)\right] f_1(x) = \tilde{E}_{1,l}^2 f_1(x) \\ \\ \left[-\frac{d^2}{dx^2} + V_2(r)\right] f_2(x) = \tilde{E}_{2,l}^2 f_2(x) \end{cases} \tag{39}$$

where we have defined $r \equiv r(x)$ and

$$\begin{cases} V_1(r) = \frac{l(l+1)}{r^2} + U_1(r) \\ \\ V_2(r) = \frac{l(l+1)}{r^2} + U_2(r) \end{cases}, \tag{40}$$

with

$$\begin{cases} U_1(r) = c_1(r) + \left(\frac{1}{\omega} - 3\right) c_2(r), \\ U_2(r) = c_1(r) + 3\left(\frac{1}{\omega} + 1\right) c_2(r), \\ c_1(r) = \frac{6}{r^2}\left(1 - \left(\frac{r_t}{r}\right)^{1+\frac{1}{\omega}}\right) \qquad c_2(r) = \frac{1}{2r^2}\left(\frac{r_t}{r}\right)^{1+\frac{1}{\omega}} \end{cases}. \tag{41}$$

In order to use the WKB approximation, we define two r-dependent radial wave numbers $k_1\left(x, l, \tilde{E}_{1,nl}\right)$ and $k_2\left(x, l, \tilde{E}_{2,nl}\right)$

$$\begin{cases} k_1^2\left(x, l, \tilde{E}_{1,nl}\right) = \tilde{E}_{1,nl}^2 - \frac{l(l+1)}{r^2} - U_1(r) \\ \\ k_2^2\left(x, l, \tilde{E}_{2,nl}\right) = \tilde{E}_{2,nl}^2 - \frac{l(l+1)}{r^2} - U_2(r) \end{cases}. \tag{42}$$

The number of modes with frequency less than $\tilde{E}_i$, $i = 1, 2$, is given approximately by

$$\tilde{g}\left(\tilde{E}_i\right) = \int \nu_i\left(l, \tilde{E}_i\right)(2l+1)\,dl, \tag{43}$$

where $\nu_i\left(l,\omega_i\right)$, $i=1,2$ is the number of nodes in the mode with $\left(l,\omega_i\right)$, such that

$$\nu_i\left(l,\omega_i\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dx\sqrt{k_i^2\left(x,l,\omega_i\right)}. \tag{44}$$

Here it is understood that the integration with respect to $x$ and $l$ is taken over those values which satisfy $k_i^2\left(x,l,\tilde{E}_i\right)\geq 0$, $i=1,2$. Thus the total one loop energy for TT tensors is given by (recall that $r\equiv r\left(x\right)$)

$$E^{TT}=\frac{1}{4}\sum_{i=1}^{2}\int_0^{+\infty}\tilde{E}_i\frac{d\tilde{g}\left(\tilde{E}_i\right)}{d\tilde{E}_i}d\tilde{E}_i=\sum_{i=1}^{2}\int_{-\infty}^{+\infty}dxr^2\left[\frac{1}{4\pi}\int_{\sqrt{U_i(r)}}^{+\infty}\tilde{E}_i^2\sqrt{\tilde{E}_i^2-U_i\left(r\right)}d\tilde{E}_i\right].$$

We use the zeta function regularization method to compute $E^{TT}$. To this purpose, we introduce the additional mass parameter $\mu$ in order to restore the correct dimension for the regularized quantities. Such an arbitrary mass scale emerges unavoidably in any regularization schemes. Then we have

$$\rho_i\left(\varepsilon\right)=\frac{1}{4\pi}\mu^{2\varepsilon}\int_{\sqrt{U_i(r)}}^{+\infty}d\tilde{E}_i\frac{\tilde{E}_i^2}{\left(\tilde{E}_i^2-U_i\left(r\right)\right)^{\varepsilon-\frac{1}{2}}} \tag{45}$$

If one of the functions $U_i\left(r\right)$ is negative, then the integration has to be meant in the range where $\tilde{E}_i^2+U_i\left(r\right)\geq 0$. In both cases the result of the integration is

$$=-\frac{U_i^2\left(r\right)}{64\pi^2}\left[\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{U_i\left(r\right)}\right)+2\ln 2-\frac{1}{2}\right], \tag{46}$$

where the absolute value has been inserted to take account of the possible change of sign. Then the total regularized one loop energy is

$$E^{TT}\left(r_t,\varepsilon;\mu\right)=4\pi\left\{2\int_{r_t}^{+\infty}dr\frac{r^2}{\sqrt{1-\frac{b(r)}{r}}}\left[\left(\rho_1\left(\varepsilon\right)+\rho_2\left(\varepsilon\right)\right)\right]\right\}, \tag{47}$$

where the factor $4\pi$ comes from the angular integration, while the factor 2 in front of the integral appears because we have come back to the original radial coordinate $r$: this means that we have to double the computation because of the upper and lower universe. Therefore the self consistent equation (34) can be written in the form

$$A\left(\omega\right)\frac{r_t}{G}=\frac{1}{16\pi}\left[\frac{a}{\varepsilon r_t}+\frac{b}{r_t}+\frac{2a}{r_t}\ln\left(\frac{\sqrt{8}r_t\mu}{\sqrt[4]{e}}\right)\right], \tag{48}$$

where the coefficients $a$ and $b$ come from the integration over the $r$ coordinate. Following the same steps of Ref.[7] of renormalizing the Newton's constant, we get

$$\frac{A\left(\omega\right)}{G_0\left(\mu_0\right)}=\frac{1}{16\pi}\left[\frac{b}{r_t^2}+\frac{2a}{r_t^2}\ln\left(\frac{\sqrt{8}r_t\mu_0}{\sqrt[4]{e}}\right)\right], \tag{49}$$

where we have used a renormalization group-like equation. In order to have only one solution[3], we find the extremum of the r.h.s. of Eq.(49) and we get

$$\frac{a-b}{2a}=\ln\left(\frac{\sqrt{8}\bar{r}_t\mu_0}{\sqrt[4]{e}}\right)\qquad\Longrightarrow\qquad\bar{r}_t=\frac{\sqrt[4]{e}}{\sqrt{8}\mu_0}\exp\left(\frac{a-b}{2a}\right)\tag{50}$$

and

$$\frac{1}{G_0\left(\mu_0\right)}=\frac{a\mu_0^2}{2\pi A\left(\omega\right)\sqrt{e}}\exp\left(-\frac{a-b}{a}\right).\tag{51}$$

We fix firstly our attention on the following choice: $G_0\left(\mu_0\right)\equiv l_p^2$, then the wormhole radius becomes

$$\bar{r}_t=\sqrt{\frac{a\left(\omega\right)}{16\pi A\left(\omega\right)}}l_p,\tag{52}$$

where we have reestablished the $\omega$ dependence of the coefficient $a$. It is useful to write the expression for $\omega\to\pm\infty$ and for $\omega\to 0$. We get

$$\begin{cases}\bar{r}_t\simeq\left[\frac{\sqrt{105}}{5\sqrt{\pi}}\left(1+\frac{1}{\omega}\left(\frac{449}{420}-2\ln\left(2\right)\right)+O\left(\omega^{-2}\right)\right)\right]l_p & \omega\to+\infty\\ \bar{r}_t\simeq\left[\frac{\sqrt{30}}{12\sqrt{\pi}\sqrt{\omega}}+O\left(\omega^{1/2}\right)\right]l_p & \omega\to 0\end{cases}.\tag{53}$$

The following plots show the behavior of $\bar{r}_t$ as a function of $\omega$. It is visible the presence of a

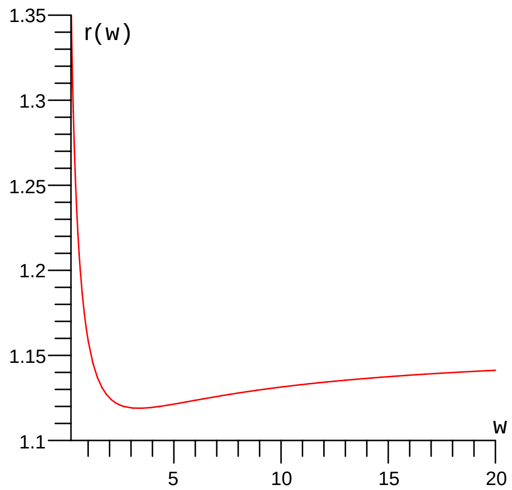


**Figure 1.** Plot of the wormhole throat $\bar{r}_t$ as a function of $\omega$ in the positive range with a fixed $G_0\left(\mu_0\right)$.

[3] Note that in the paper of Khusnutdinov and Sushkov[10], to find only one solution, the minimum of the ground state of the quantized scalar field has been set equal to the classical energy. In our case, we have no external fields on a given background. This means that it is not possible to find a minimum of the one loop gravitons, in analogy with Ref.[10]. Moreover the renormalization procedure in Ref.[10] is completely independent by the classical term, while in our case it is not. Indeed, thanks to the self-consistent equation (34), we can renormalize the divergent term.

minimum for $\bar{\omega} = 3.35204$, where $\bar{r}_t(\bar{\omega}) = 1.11891$. As we can see, from the expression (53) and from the Fig.1, the radius is divergent when $\omega \to 0$. At this stage, we cannot establish if this is a physical result or a failure of the scheme. When $\omega \to +\infty$, $\bar{r}_t$ approaches the value $1.15624 l_p$, while for $\omega = 1$, we obtained $\bar{r}_t = 1.15882 l_p$. It is interesting to note that when $\omega \to +\infty$, the shape function $b(r)$ in Eq.(21) approaches the Schwarzschild value, when we identify $\bar{r}_t$ with $2MG$. In this sense, it seems that also the Schwarzschild wormhole is traversable. Secondly, we identify $\mu_0$ with the Planck scale and we get from Eq.(50) the following plot Note the absence

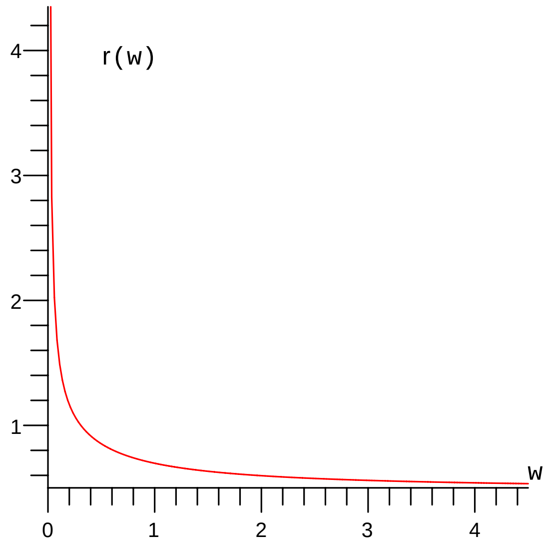


**Figure 2.** Plot of the wormhole throat $\bar{r}_t$ as a function of $\omega$ in the positive range with a fixed $\mu_0$.

of a minimum.

## 4. Summary and Conclusion

In this paper, we have generalized the analysis of self-sustaining wormholes[7] by looking how the equation of state (2) can affect the traversability, when the sign of the parameter $\omega$ is positive. The paper has been motivated by the work of Lobo[4], Kuhfittig[5] and Sushkov[6] , where the authors search for classical traversable wormholes supported by phantom energy. Since the phantom energy must satisfy the equation of state, but in the range $\omega < -1$, we have investigated the possibility of studying the whole range $-\infty < \omega < +\infty$. Unfortunately, evaluating the classical term we have discovered that such a term is well defined in the range $-1 < \omega < +\infty$. The interval $-1 < \omega < 0$ should be interesting for the existence of a "*dark*" energy support. Once again, the "*dark*" energy domain lies outside the asymptotically flatness property. So, unless one is interested in wormholes that are not asymptotically flat, i.e. asymptotically de Sitter or asymptotically Anti-de Sitter, we have to reject also this possibility. Therefore, the final stage of computation has been restricted only to positive values of the parameter $\omega$. In this context, it is interesting to note that also the Schwarzschild wormhole is traversable, even if in the limiting procedure of $\omega \to +\infty$. Despite of this, the obtained "*traversability*" has to be regarded as in "*principle*" rather than in "*practice*" because the wormhole radius has a Planckian size. We do not know , at this stage of the calculation, if a different approach for a self sustained wormhole can give better results. On the other hand, the positive $\omega$ sector seems to corroborate the Casimir process of the quantum fluctuations supporting the opening of the wormhole. Even in this region, we do not know what happens approaching directly the point $\omega = 0$, because it seems that this approach is ill defined. Nevertheless, in this paper we have studied the behavior of the energy. Work in progress seems to show that dealing with energy density one can get more general results even in the "*phantom*" sector[21].

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