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Hot Casimir Wormholes

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Abstract. In this paper, we have for the first time considered the consequences of finite temperature contributions to a traversable wormhole. This was done by using finite temperature generalization of the Casimir effect as a source of a hot traversable wormhole. To include finite temperature effects, we have considered the plates positioned either parametrically fixed or radially varying. Such results have been obtained in both high and low-temperature regimes. We explicitly investigate the effect of such finite temperature corrections on the size of a traversable wormhole.

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Contents

1 Introduction

A wormhole is a solution of the Einstein Field Equations (EFE) that acts as a tunnel from one region of spacetime to another. The first wormhole solutions considered were unstable, and it was not possible for observers to pass through them [1, 2]. Unlike these unstable wormholes, it is possible to construct stable wormhole solutions, violating the weak energy condition [3–5]. Such a stable wormhole solution is called a traversable wormhole. As classical solutions obey the weak energy conditions, the construction of a traversable wormhole has to crucially depend on quantum effects, which can violate the weak energy conditions [6–8]. A substance where the quantum effects violate weak energy conditions is known as exotic matter [9, 10]. It is known that the Casimir effect is a form of vacuum energy representing a good candidate to describe exotic matter. The Casimir effect appears between two plane parallel, closely spaced, uncharged, metallic plates in vacuum. It was predicted theoretically in 1948[11] and experimentally confirmed in the Philips laboratories[12, 13]. However, only in recent years further reliable experimental investigations have confirmed such a phenomenon[14, 15]. At zero temperature, the Casimir effect predicts a force of the form

$$F(d) = -\frac{3\hbar c\pi^2 S}{720d^4}, \tag{1.1}$$

where S is the surface of the plates and d is the separation between them. The force $F(d)$ is also responsible for producing a pressure,

$$P(d) = \frac{F(d)}{S} = -\frac{3\hbar c\pi^2}{720d^4}. \tag{1.2}$$

Both of them are obtained with the help of the renormalized energy

$$E^{\mathrm{Ren}}(d) = -\frac{\hbar c\pi^2 S}{720d^3}. \tag{1.3}$$

The energy density is obtained by dividing $E^{\mathrm{Ren}}(d)$ by the volume $V = Sd$, yielding

$$\rho_C(d) = -\frac{\hbar c\pi^2}{720d^4}. \tag{1.4}$$

It is important to observe that this effect has a strong dependence on the geometry of the boundaries. Indeed, Boyer[16] proofed the positivity of the Casimir effect for a conducting spherical shell of radius r. The same positivity has been proofed also in Ref.[17], by means of heat kernel and zeta regularization techniques. We can see that a relationship exists between the energy density $\rho_C(d)$ and the pressure $P(d)$. This is described by an Equation of State (EoS) of the form $P = \omega\rho$ with $\omega = 3$. Exotic matter violates the Null Energy Condition (NEC), namely for any null vector k^μ, we have $T_{\mu\nu}k^\mu k^\nu \geq 0$. Such exotic matter can produce traversable wormhole solutions to the EFE [6, 7]. The quantum effects in this case are only needed to produce the exotic matter, and the rest of the treatment remains classical [6, 8]. As the geometry remains classical, the EFE must be replaced with the semiclassical EFE, namely $G_{\mu\nu} = \kappa \langle T_{\mu\nu}\rangle^{\mathrm{Ren}}$and $\kappa = 8\pi G/c^4$, where $\langle T_{\mu\nu}\rangle^{\mathrm{Ren}}$ describes the renormalized quantum contribution of some matter fields. The first connection between a traversable wormhole and the Casimir energy was investigated in Ref.[7]. Only recently such a connection has been reconsidered to understand the possible profile of a traversable wormhole obtained with the help of the Casimir energy. Such wormholes have been dubbed Casimir wormholes [18]. Casimir wormholes have been also considered including an additional electromagnetic field to study the effects of an electrovacuum on a traversable wormhole [20]. The traversable wormhole produced by a Casimir source, with a scalar field has been studied both with and without a potential term [21]. The Einstein-Maxwell theory coupled to charged massless fermions, has also been used to construct traversable wormholes [22]. It has been argued that such solutions can be embedded in the standard model by making the overall size of the wormhole smaller than the electroweak scale. This solution is viewed as a pair of entangled near-extremal black holes, and there is an interaction term generated by the fermionic fields. It is known that wormholes can be viewed as a pair of entangled black holes [24, 25]. This has even motivated the study of traversable wormholes as quantum processors [26]. This can be done by analyzing the holographic dual to a traversable wormhole. Specifically, the holographic dual has been analyzed using the SYK many-body system. The Casimir wormholes in modified theories of gravity have also been studied. The modifications to such wormholes in gravity with higher scalar curvature and torsion terms have been investigated [27, 28]. The effect of noncommutativity on Casimir wormholes in higher dimensional Gauss-Bonnet gravity has also been discussed [29]. It was observed that both noncommutativity and higher curvature terms modify the behavior of Casimir wormholes. It is possible to analyze low energy collective excitations of a magnetically charged black hole using a large number of Alfven wave modes [30]. The Casimir energy of the Alfven wave modes has been used to construct a traversable wormhole [31]. It is possible to generalize the usual Casimir effect to a non-abelian Casimir effect, which is produced by Yang-Mills theory [32]. This non-abelian Casimir effect has also been used to construct traversable wormhole solutions [33]. The effect of the minimal length on Casimir wormholes has also been considered [34–36]. Such minimal length occurs due to quantum gravitational modifications to the low energy quantum mechanics [37] and quantum field theory [38, 39]. The generalization of Casimir wormholes to gravity's rainbow has also been studied [40]. In gravity's rainbow, the spacetime geometry depends on the energy of the probe, and this energy dependence can modify the behavior of the wormhole. Thus, the modification of gravity does modify the behavior of Casimir wormholes. However, in all these cases, the magnitude of the Casimir effect remains small, and so the size of the wormhole also remains small. This is because the wormhole in all these modifications to gravity is still produced by quantum fluctuations, and their magnitude at large scales remains small. The Casimir effect assumes perfectly conducting surfaces.

However, actual plates, are never perfectly conducting. Rather they are characterized by a complex permittivity, with a material-dependent function of frequency. This consideration modifies the original zero-temperature Casimir force, to a finite-temperature Casimir effect. In this finite-temperature effect, the effects of thermal corrections are considered besides the effects produced by quantum fluctuations. This can be done using the Lifshitz theory, in which the electromagnetic field stress tensor is obtained by considering the correlated fluctuating charges and currents in the plates [41]. This finite temperature generalization of the conventional Casimir effect is called the thermal Casimir effect [42, 43]. It has been observed that the force produced due to thermal fluctuations dominates over the force produced due to zero temperature quantum fluctuation at separations greater than a critical value [44, 45]. Thus, it seems possible that they could become important in scaling up the size of traversable wormholes. Even though it seems natural to consider the thermal Casimir effect for this, this has never been considered. Hence, we for the first time study the effect of temperature on traversable wormhole and construct hot Casimir wormholes. We have to observe that, by assuming the $T = 0$ form of the energy density (1.4), a consistent calculation can be obtained if we adopt the configuration considered in Ref.[7], where the plates are of spherical form. As pointed out in Ref.[7], this approximation introduces an error which can be small if we are very close to the throat, which is exactly what we need in this paper. To further proceed, we introduce the following spacetime metric

$$ds^2 = -e^{2\Phi(r)}\,dt^2 + \frac{dr^2}{1-b(r)/r} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{1.5}$$

representing a spherically symmetric and static wormhole. Here, $\Phi(r)$ and $b(r)$ are arbitrary functions of the radial coordinate $r \in [r_0, +\infty)$, and they represent the redshift function, and the shape function, respectively [6, 8]. An important property of a traversable wormhole is the flare-out condition, given by $(b - b'r)/b^2 > 0$. It must be satisfied along with the condition $1 - b(r)/r > 0$. Furthermore, at the throat $b(r_0) = r_0$ is satisfied, and the condition $b'(r_0) < 1$ is imposed to obtain wormhole solutions. It is also fundamental that there are no horizons present, which are identified as the surfaces with $e^{2\Phi(r)} \to 0$, so that $\Phi(r)$ must be finite everywhere. With the help of the line element (1.5), we can write the EFE in an orthonormal reference frame, leading to the following set of equations

$$\frac{b'(r)}{r^2} = \kappa\rho(r), \tag{1.6}$$

$$\frac{2}{r}\left(1-\frac{b(r)}{r}\right)\Phi'(r) - \frac{b(r)}{r^3} = \kappa p_r(r), \tag{1.7}$$

$$\left(1-\frac{b(r)}{r}\right)\left[\Phi''(r) + \Phi'(r)\left(\Phi'(r)+\frac{1}{r}\right)\right] - \frac{b'(r)\,r - b(r)}{2r^2}\left(\Phi'(r)+\frac{1}{r}\right) = \kappa p_t(r),$$

in which $\rho(r)$ is the energy density[1], $p_r(r)$ is the radial pressure, and $p_t(r)$ is the lateral pressure. We can complete the EFE with the expression of the conservation of the stress-energy tensor which can be written in the same orthonormal reference frame

$$p_r'(r) = \frac{2}{r}\left(p_t(r) - p_r(r)\right) - \left(\rho(r) + p_r(r)\right)\Phi'(r). \tag{1.8}$$

[1]However, if $\rho(r)$ represents the mass density, then we have to replace $\rho(r)$ with $\rho(r)\,c^2$.

The rest of the paper is structured as follows: in section 2, we examine the low-temperature contribution to the Casimir energy source for a traversable wormhole. This is done with the plates positioned at a distance either parametrically fixed or radially varying. In section 3, we examine the high-temperature contribution to the Casimir energy source for a traversable wormhole. This is again done with the plates positioned at a distance either parametrically fixed or radially varying. We summarize and conclude in section 4. We will use SI units in this paper.

2 Low-Temperature corrections to the Casimir Wormhole

The thermal Casimir effect predicts the following corrections to the free energy at low-temperature[46, 47]

$$E(d,T) = -\frac{\hbar c \pi^2 S}{720 d^3}\left[1 + \frac{45\zeta(3)}{\pi^3}\left(\frac{T}{T_e}\right)^3 - \left(\frac{T}{T_e}\right)^4\right]; \qquad T \ll T_e, \tag{2.1}$$

where we have defined

$$T_e = \frac{\hbar c}{2 d k_B}. \tag{2.2}$$

To have an order of magnitude on T_e, we can use estimated value for different quantities, and obtain

$$T_e = \frac{\hbar c}{2k_B d} \simeq \frac{\left(10^{-34}\text{J s}\right)\left(10^{8} m\ \text{s}^{-1}\right)}{2d\left(10^{-23}\text{J K}^{-1}\right)} \simeq \frac{10^{-3}}{d} m\ K. \tag{2.3}$$

Therefore for plates separated by a distance of the order $10^{-6}m$, we obtain $T_e \simeq 5\times 10^2 K$. Note that the cubic term is independent on d in Eq.(2.1). Now for low-temperatures, we obtain

$$P(d,T) = -3\frac{\hbar c\pi^2}{720 d^4}\left[1 + \frac{1}{3}\left(\frac{T}{T_e}\right)^4\right] \qquad T \ll T_e. \tag{2.4}$$

Eq.(2.1) will be the cornerstone of this section. Following the previous work [18], we can consider the plates either parametrically fixed or radially varying. We have two possible configurations:

1. We divide $E(d,T)$ with a volume term of the form $V = Sd$, leading to the following form of the energy density

$$\rho_{L,1}(d,T) = -\frac{\hbar c\pi^2}{720 d^4} f(T,d), \tag{2.5}$$

   where the temperature function is given by

$$f(T,d) = 1 + \frac{45\zeta(3)}{\pi^3}\left(\frac{T}{T_e}\right)^3 - \left(\frac{T}{T_e}\right)^4. \tag{2.6}$$

2. Following [18], we promote the distance between the plates d to a radial variable r. We divide $E(r,T)$ with a volume term of the form $V = Sr$, and obtain the energy density

$$\rho_{L,2}(r,T) = -\frac{\hbar c\pi^2}{720 r^4} f(T,r). \tag{2.7}$$

Here we have substituted the distance d with the radial variable r into the Eq.(2.6). Due to the thermal corrections, the EoS between $\rho_C(d,T)$ and the pressure $P(d,T)$ gets modified. Indeed, for low-temperatures, we obtain $P_L(d,T) = \omega_L(d,T)\rho_L(d,T)$ with

$$\omega_L(d,T) = \frac{3}{f(T,d)}\left(1+\frac{1}{3}\left(\frac{T}{T_e}\right)^4\right). \tag{2.8}$$

This relationship is true even when we substitute d with r.

2.1 Constant Plates distance separation at Low Temperature

If we give a look at the energy density (2.5), we can observe that this is constant with respect to the radial variable r. Therefore the first EFE leads to the following shape function

$$b(r) = r_0 - \frac{8\pi G}{c^4}\left(\frac{\hbar c\pi^2}{720d^4}f(T,d)\right)\left(r^3 - r_0^3\right) = r_0 - \frac{r_1^2(T)}{3d^4}\left(r^3 - r_0^3\right), \tag{2.9}$$

where we have defined

$$r_1^2(T) = \frac{\hbar G}{c^3}\frac{\pi^3}{90}f(T,d) = \frac{\pi^3\ell_P^2}{90}f(T,d) = r_1^2 f(T,d). \tag{2.10}$$

Even if the flare-out condition is always satisfied,

$$b'(r_0) = -\frac{r_0^2 r_1^2(T)}{d^4} < 1 \tag{2.11}$$

the shape function (2.9) does not represent strictly a traversable wormhole, because it is not asymptotically flat. However, we can observe that there exists

$$\bar{r} = r_0\sqrt[3]{1+\frac{3d^4}{r_1^2(T)\,r_0^2}} \tag{2.12}$$

such that $b(\bar{r}) = 0$. Since $T \ll T_e$, from Eq.(2.6), we can restrict the range of $f(T)$ to the values $[1, 1.12]$. At $T = 0$, the energy density (2.5) has been taken under consideration in the picture of a generalized-absurdly-benign-traversable wormhole [23]. Nevertheless, we can see that there exist other possible configurations generated by the shape function (2.9). Indeed, plugging Eq.(2.9) into Eq.(1.7), we obtain

$$\Phi'(r) = \frac{\left(r_0^3 - (3\omega_L+1)r^3\right)r_1^2(T) + 3r_0d^4}{\left(2r^4 - 2rr_0^3\right)r_1^2(T) + 6r\,d^4(r-r_0)}, \tag{2.13}$$

where we have used the EoS $P_L(d,T) = \omega_L\rho_{L,1}(d,T)$. Close to the throat, we obtain

$$\Phi'(r) \simeq \frac{r_0^2\left(1-(3\omega_L+1)\right)r_1^2(T) + 3d^4}{6d^4(r-r_0)}. \tag{2.14}$$

In order to avoid a horizon, we assume that

$$\omega_L = \frac{d^4}{r_0^2 r_1^2(T)}. \tag{2.15}$$

Comparing Eq.(2.8) and Eq.(2.15), we also obtain

$$r_0 = r_0\left(T,d\right) = \frac{d^2}{\sqrt{3\left(1+\frac{1}{3}\left(\frac{T}{T_e}\right)^4\right)r_1}}, \tag{2.16}$$

The size of the throat has a dependence on the temperature T. However, since $T \ll T_e$, the correction with respect the zero temperature case is negligible. From Eq. (2.16), it is possible to estimate location of the boundary. Thus, we obtain

$$\bar{r} = \frac{d^2}{\sqrt{3\left(1+\frac{1}{3}\left(\frac{T}{T_e}\right)^4\right)r_1}}\sqrt[3]{1+\frac{9}{f\left(T,d\right)}\left(1+\frac{1}{3}\left(\frac{T}{T_e}\right)^4\right)}. \tag{2.17}$$

Coming back to the redshift function, if we solve Eq.(2.13) and we plug Eq.(2.15) inside the solution, we get

$$\begin{aligned}\Phi\left(r\right) = & -\frac{\sqrt{3}\,d^4}{2r_0\sqrt{r_0^2 r_1^2\left(T\right)+4d^4}\,r_1\left(T\right)}\arctan\left(\frac{\left(2r+r_0\right)r_1\left(T\right)}{\sqrt{3r_0^2 r_1^2\left(T\right)+12d^4}}\right) \\ & -\frac{3d^4\ln\left(r_1^2\left(T\right)r^2+r_0 r_1^2\left(T\right)r+r_0^2 r_1^2\left(T\right)+3d^4\right)}{4r_0^2 r_1^2\left(T\right)}-\frac{\ln\left(r\right)}{2}+C.\end{aligned} \tag{2.18}$$

Note that $\Phi\left(r\right) \to -\infty$ when $r \to +\infty$. However, we have to remember that there exists $\bar{r}$ such that $b\left(\bar{r}\right) = 0$. This means that we can choose C in such a way that is is possible to impose $\Phi\left(\bar{r}\right) = 0$. To complete the evaluation of this configuration, we need to compute $p_t\left(d,T\right)$. Differently from the $T = 0$ case, the form of the stress-energy tensor (SET) is not known. It is possible to assume that even at $T \neq 0$, the SET can be traceless. Thus, we obtain

$$\rho_{L,1}\left(d,T\right) - \omega_L\left(d,T\right)\rho_{L,1}\left(d,T\right) + 2p_t\left(d,T\right) = 0, \tag{2.19}$$

which implies

$$p_t\left(d,T\right) = \rho_{L,1}\left(d,T\right)\frac{\left(\omega_L\left(d,T\right)-1\right)}{2}. \tag{2.20}$$

So, $p_t\left(d,T\right)$ is in agreement with the $T = 0$ case. On the other hand, from the third EFE, we obtain

$$p_t\left(d,T\right) = -\omega_t\left(r,d,T\right)\frac{r_1^2\left(T\right)}{\kappa d^4} = \omega_t\left(r,d,T\right)\frac{\rho_{L,1}\left(d,T\right)}{\kappa}, \tag{2.21}$$

where

$$\begin{aligned}&\omega_t\left(r,d,T\right) \\ &= -\frac{\left(3r_1^2\left(T\right)\omega_L^2 r^3+3r_1^2\left(T\right)\omega_L r_0^3-12d^4\omega_L r+9d^4\omega_L r_0+r_1^2\left(T\right)r^3-r_1^2\left(T\right)r_0^3-3r_0 d^4\right)}{4\left(r_1^2\left(T\right)r^3-r_1^2\left(T\right)r_0^3+3r\,d^4-3r_0 d^4\right)}.\end{aligned} \tag{2.22}$$

It is clear that we have the same problem that occurred in previous work on zero-temperature Casimir wormholes [18]. Therefore the final SET, with finite temperature corrections, can be written as

$$T_{\mu\nu} = \frac{\rho_{L,1}(d,T)}{\kappa}\left[diag\left(-1, -\omega_L(d,T), \frac{(\omega_L(d,T)-1)}{2}, \frac{(\omega_L(d,T)-1)}{2}\right) + \left(\omega_t(r,d,T) - \frac{(\omega_L(d,T)-1)}{2}\right) diag(0,0,1,1)\right], \tag{2.23}$$

where the first term of $T_{\mu\nu}$ is traceless.

2.2 Variable Plates distance separation at Low Temperature

In this subsection, we consider the energy density (2.7). The first modification we have to consider is

$$T_e = \frac{\hbar c}{2dk_B} \longrightarrow \frac{\hbar c}{2rk_B}. \tag{2.24}$$

In order to follow closely with the parametrically fixed case, we write

$$\frac{\hbar c}{2dk_B} = \frac{\hbar c}{2dk_B}\frac{d}{r} = T_e\frac{d}{r}. \tag{2.25}$$

Then $f(T,d)$ becomes

$$f(T,r) = 1 + \frac{45\zeta(3)}{\pi^3}\left(\frac{T}{T_e}\frac{r}{d}\right)^3 - \left(\frac{T}{T_e}\frac{r}{d}\right)^4. \tag{2.26}$$

We cannot include $f(T,r)$ directly, but we have to consider its effects term by term. This implies that the energy density (2.7) can be written as

$$\rho_{L,2}(r,T) = \hbar c\left[-\frac{\pi^2}{720r^4} - \frac{\zeta(3)}{16d^3\pi r}\left(\frac{T}{T_e}\right)^3 + \frac{\pi^2}{720d^4}\left(\frac{T}{T_e}\right)^4\right]. \tag{2.27}$$

Despite the complicated expression, the first EFE can be solved

$$\begin{aligned} b(r) &= r_0 - \frac{8\pi G}{c^4}\int_{r_0}^{r}\rho_{L,2}(r',T)\, r'^2 dr' \\ &= r_0 - \frac{\pi^3}{90}\ell_P^2\left[\left(\frac{1}{r} - \frac{1}{r_0}\right) + \ln\left(\frac{r}{r_0}\right)\frac{45\zeta(3)}{\pi^3 d^3}\left(\frac{T}{T_e}\right)^3 - \frac{1}{d^4}\left(r^3 - r_0^3\right)\left(\frac{T}{T_e}\right)^4\right]. \end{aligned} \tag{2.28}$$

As we can see, the only convergent term corresponds to the $T = 0$ Casimir energy density, which has been found in previous work on zero temperature Casimir wormholes [18]. The last term is such that the following property of a traversable wormhole $1 - b(r)/r > 0$ is violated. Therefore this case will be discarded.

3 High Temperature corrections to the Casimir Wormhole

For the high-temperature case, namely $T \gg T_e$, the corrected thermal Casimir energy is given by[46, 47]

$$E(d,T) = -\frac{k_B T S}{8\pi d^2}\left[\zeta(3) + \left(\frac{4\pi T}{T_e} + 2\right)\exp\left(-\frac{2\pi T}{T_e}\right) + O\left(\exp\left(-\frac{4\pi T}{T_e}\right)\right)\right], \tag{3.1}$$

where T_e has been defined in Eq. (2.2) and $\zeta(x)$ is the Riemann zeta function. Its related pressure is defined by

$$P(d,T) = -\frac{k_B T}{4\pi d^3}\zeta(3)\,. \tag{3.2}$$

Following the previous section, we can consider the plates either parametrically fixed or radially varying. For high-temperature corrections, it is reasonable to take into account only the leading order corrections, since the other terms are exponentially suppressed. Therefore, from Eq.(3.1), we consider

$$E(d,T) = -\frac{k_B T S}{8\pi d^2}\zeta(3)\,. \tag{3.3}$$

As in the low-temperature approximation, we can consider two possible configurations:

1. We divide $E(d,T)$ with a volume term of the form $V = Sd$, leading to the following form of the energy density

$$\rho_{H,1}(d,T) = -\frac{k_B T}{8\pi d^3}\zeta(3)\,. \tag{3.4}$$

   The related pressure is represented by Eq.(3.2) leading to the following EoS

$$\frac{P(d,T)}{\rho_{H,1}(d,T)} = \omega_H = 2. \tag{3.5}$$

   Note the difference with respect to the $T = 0$ case where $\omega = 3$.

2. Using previous work on zero temperature Casimir wormholes [18], we promote the distance separating plates d to a radial variable r, and we divide $E(r,T)$ with a volume term of the form $V = Sr$. Thus, we obtain the energy density as

$$\rho_{H,2}(r,T) = -\frac{k_B T}{8\pi r^3}\zeta(3)\,. \tag{3.6}$$

   Even in this case, we have that the EoS leads to the same value of Eq.(3.5), namely $\omega_H = 2$.

3.1 Constant Plates Separation

In this section, we consider the profile (3.4), which produces a shape function of the form

$$b(r) = r_0 - \frac{8\pi G}{c^4}\left(\frac{k_B T}{8\pi d^3 3}\right)\zeta(3)\left(r^3 - r_0^3\right) = r_0 - \frac{l_P^2\zeta(3)}{6d^4}\left(\frac{T}{T_e}\right)\left(r^3 - r_0^3\right). \tag{3.7}$$

Except for the coefficient in front of the cubic term. which is different, the shape function is formally the same as Eq.(2.9). This implies that, even in this case, the flare-out condition is always satisfied,

$$b'(r_0) = -\frac{l_P^2\zeta(3)}{2d^4}\left(\frac{T}{T_e}\right)r_0^2 < 1. \tag{3.8}$$

Now by looking at its analytic form, we can observe that there exists an $\bar{r}$, such that $b(\bar{r}) = 0$. Indeed, we obtain

$$\bar{r} = r_0\sqrt[3]{1 + \frac{l_1^2(d,T)}{r_0^2}} \tag{3.9}$$

where we have defined

$$l_1(d,T)=\frac{d^2}{l_P}\sqrt{\frac{6}{\zeta(3)}\left(\frac{T_e}{T}\right)}\simeq\frac{5\times10^{24}}{\sqrt{T}}m \tag{3.10}$$

Here we have considered distance between the plates of order $d\simeq10^{-6}m$ and $T_e\simeq5\times10^2\,^\circ K$. Note that for very high T, $l(d,T)\to0$. Plugging Eq.(3.7) into Eq.(1.7), we obtain

$$\Phi'(r)=\frac{r_0^3+r_0l_1^2(d,T)-(3\omega_H+1)r^3}{2r\left(r\left(r^2+l_1^2(d,T)\right)-r_0\left(l_1^2(d,T)+r_0^2\right)\right)}. \tag{3.11}$$

Close to the throat, we obtain

$$\Phi'(r)\simeq\frac{l_1^2(d,T)-3\omega r_0^2}{2\left(l_1^2(d,T)+r_0^2\right)(r-r_0)}. \tag{3.12}$$

Therefore, the horizon is not formed, if we choose

$$\omega_H=\frac{l_1^2(d,T)}{3r_0^2}. \tag{3.13}$$

Moreover, from Eq.(3.5), we can determine the size of the throat. Indeed, we find

$$r_0=\frac{\sqrt{6}l_1(d,T)}{6}\simeq\frac{5.6\times10^{19}}{\sqrt{T}}m. \tag{3.14}$$

Thus, the effect of high-temperature corrections on the Casimir energy is a reduction of the throat size. Nevertheless, we have to observe that in a laboratory $T\simeq10^8\ K$ as an ideal result and this implies that $r_0\simeq10^{11}m$, which is an order of magnitude bigger than the solar system size. To understand if a traversable wormhole can form from a finite temperature Casimir source, we have to solve Eq.(3.11) with the condition (3.13). Thus, we obtain

$$\begin{aligned}\Phi(r)=&-\frac{\ln(r)}{2}-\frac{l_1^2(d,T)}{4r_0^2}\ln\left(l_1^2(d,T)+r^2+rr_0+r_0^2\right)\\&-\frac{l_1^2(d,T)}{2r_0\sqrt{4l_1^2(d,T)+3r_0^2}}\arctan\left(\frac{2r+r_0}{\sqrt{4l_1^2(d,T)+3r_0^2}}\right)+C.\end{aligned} \tag{3.15}$$

The redshift function is divergent for $r\to\infty$. However, with the help of Eq.(3.9), we can remove such a divergence by imposing that $\Phi(\bar{r})=0$. Then, we obtain

$$\begin{aligned}\Phi(r)=&-\frac{1}{2}\ln\left(\frac{r}{\bar{r}}\right)-\frac{l_1^2(d,T)}{4r_0^2}\ln\left(\frac{l_1^2(d,T)+r^2+rr_0+r_0^2}{l_1^2(d,T)+\bar{r}^2+\bar{r}r_0+r_0^2}\right)\\&-\frac{l_1^2(d,T)}{2r_0\sqrt{4l_1^2(d,T)+3r_0^2}}\left(\arctan\left(\frac{2r+r_0}{\sqrt{4l_1^2(d,T)+3r_0^2}}\right)-\arctan\left(\frac{2\bar{r}+r_0}{\sqrt{4l_1^2(d,T)+3r_0^2}}\right)\right).\end{aligned} \tag{3.16}$$

It is interesting to observe that for $T\ggg T_e$, the redshift function reduces to

$$\Phi(r)\simeq-\frac{1}{2}\ln\left(\frac{r}{\bar{r}}\right) \tag{3.17}$$

with $\bar{r} \to r_0$. This implies that the variable r collapses to $r = 0$, for both $\Phi(r)$ and $b(r)$. To complete the calculations, we need to compute $p_t(r)$. Nevertheless, before going on we can gain information on $p_t(r)$ by assuming that even for high-temperature the Casimir SET is traceless. This implies that

$$p_t(d,T) = -\frac{1}{2}\rho_{H,1}(d,T) \tag{3.18}$$

So, the corresponding SET can be written as

$$T_{\mu\nu} = \rho_{H,1}(d,T)\left[diag\left(1,2,-1/2,-1/2\right)\right]. \tag{3.19}$$

However, from the third EFE, we obtain

$$p_t(d,T) = -\omega_t^H(r,d,T)\frac{3}{\kappa l_1^2(d,T)} = \omega_t^H(r,d,T)\frac{\rho_{H,1}(d,T)}{\kappa}, \tag{3.20}$$

where

$$\omega_t^H(r,d,T) = -\frac{l_1^4(d,T)\left(r^2+rr_0-3r_0^2\right)+3r_0^4\left(r^2+rr_0+r_0^2\right)}{4r_0^4\left(l_1^2(d,T)+r^2+rr_0+r_0^2\right)}. \tag{3.21}$$

Just as previously done in Eq.(2.23), we can separate the SET into a traceless term and a trace part. We observe that

$$T_{\mu\nu} = \frac{\rho_{H,1}(d,T)}{\kappa}\left[diag\left(1,2,-\frac{1}{2},-\frac{1}{2}\right)\right.$$
$$\left.+\left(\omega_t^H(r,d,T)+\frac{1}{2}\right)diag\left(0,0,1,1\right)\right], \tag{3.22}$$

where the first term of $T_{\mu\nu}$ is traceless.

3.2 Variable Plates separation

In this subsection, we investigate the energy density (3.6), representing the Casimir source with high-temperature corrections. The first EFE leads to

$$b(r) = r_0 - \frac{8\pi G}{c^4}\left(\frac{k_BT}{8\pi}\right)\zeta(3)\int_{r_0}^{r}\frac{dr'}{r'} = r_0 - l(T)\ln\left(\frac{r}{r_0}\right), \tag{3.23}$$

where we have defined

$$l(T) = \zeta(3)\, l_P^2\frac{k_BT}{\hbar c}. \tag{3.24}$$

From the shape function (3.23), we observe that the flare-out condition is always satisfied, since

$$b'(r_0) = -\frac{l(T)}{r_0} < 1. \tag{3.25}$$

Moreover from the condition $1 - b(r)/r > 0$, it is possible to show that there exists $r = \bar{r}$, such that $b(\bar{r}) = 0$, where

$$\bar{r} = r_0\exp\left(\frac{r_0}{l(T)}\right). \tag{3.26}$$

Finally, to obtain a traversable wormhole, we need to compute the redshift function. From Eq.(1.7), we obtain

$$\Phi'(r) = -\frac{r_0 - l(T)\left(\omega + \ln(r/r_0)\right)}{2r\left(l(T)\left(\ln(r/r_0)\right)+(r-r_0)\right)}, \tag{3.27}$$

where we have imposed the EoS $p_r(r) = \omega\rho(r)$. Close to the throat, we can use the approximation given by

$$\Phi'(r) \simeq \frac{r_0 - \omega l(T)}{2r_0(r-r_0)}. \tag{3.28}$$

We can choose ω, such that $\Phi'(r) = 0$, avoiding the appearance of a horizon. This can be done if

$$\omega = \frac{r_0}{l(T)}. \tag{3.29}$$

Note that from Eq.(3.2), it is possible to fix the value of ω. Indeed, we observe that

$$\frac{P(r,T)}{\rho_{H,2}(r,T)} = \omega = 2. \tag{3.30}$$

This is different from the zero temperature value, $\omega = 3$. From Eq.(3.30) and Eq.(3.29), we obtain

$$r_0 = 2l(T) \simeq \frac{10^{-67}m}{K}T. \tag{3.31}$$

Now even with a temperature much larger than the Planck Temperature

$$T_P = \sqrt{\frac{\hbar c^5}{Gk_B^2}} \simeq 1.416784 \times 10^{32}K, \tag{3.32}$$

the solution has a physical meaning. Therefore this solution will not be considered. However, another possibility comes from the following observation. The energy density (3.6) can be rewritten in the following way

$$\rho_{H,2}(r,T) = -\frac{\hbar c}{16\pi r^3}\frac{\zeta(3)}{\lambda_C(T)}. \tag{3.33}$$

Here we have introduced the Casimir thermal wavelength

$$\lambda_C(T) = \frac{\hbar c}{2k_B T}. \tag{3.34}$$

In the high-temperature approximation, or long-distance approximation, the following inequality is satisfied

$$\frac{2\pi}{\lambda_C(T)}d \gg 1, \tag{3.35}$$

where d is the distance separating the plates. By promoting the d distance to a variable distance r, we can write

$$\frac{1}{\lambda_C(T)} = \frac{A}{2\pi r}, \qquad A \gg 1. \tag{3.36}$$

Thus, $\rho_{H,2}(r,T)$ can be cast into the form

$$\rho_{H,2}(r,T) = -A\frac{\hbar c}{32\pi^2 r^4}\zeta(3). \tag{3.37}$$

Thus the first EFE can be written as

$$b(r) = r_0 - \frac{\hbar G}{4\pi c^3}A\zeta(3)\int_{r_0}^{r}\frac{dr'}{r'^2} = r_0 + r_T^2\left(\frac{1}{r} - \frac{1}{r_0}\right), \tag{3.38}$$

where we have defined

$$r_T^2 = \frac{A}{4\pi}\zeta\left(3\right) l_P^2. \tag{3.39}$$

As we can see, we have the same formal expression found in previous work on zero temperature Casimir wormholes [18]. Therefore, if we solve the second EFE (1.7), we get

$$\Phi'\left(r\right) = \frac{\left(\left(-\omega+1\right) r_0 - r\right) r_T^2 + r r_0^2}{2r\left(r-r_0\right)\left(r r_0 + r_T^2\right)} \tag{3.40}$$

and close to the throat we find that a horizon can be avoided if

$$\omega = \frac{r_0^2}{r_T^2}. \tag{3.41}$$

However, this time the value of ω is given by

$$\frac{P\left(r,T\right)}{\rho_{H,2}\left(r,T\right)} = \omega = 2 \tag{3.42}$$

and not $\omega = 3$, as in previous work on zero temperature Casimir wormholes [18]. So, we can now estimate the size of the throat as

$$r_0 = \sqrt{\frac{A}{2\pi}} l_P. \tag{3.43}$$

Compared to the zero temperature result [18], we can observe that the effect of the temperature is to enlarge the size of the throat.

4 Conclusions

In this paper, we have investigated how finite temperature corrections to the Casimir Energy can affect a Casimir Wormhole[18]. To do calculations in practice, we have considered both the low and high-temperature approximations. We have done this using two configurations for the plates. In one configuration, the plates have been held parametrically fixed and in the other one they have been taken radially varying. The transition from a configuration having the plates parametrically fixed to a configuration having the plates radially varying is based on this simple assumption

$$\rho_C\left(d\right) = -\frac{\hbar c \pi^2}{720 d^4} \qquad \rightarrow \qquad \rho_C\left(r\right) = -\frac{\hbar c \pi^2}{720 r^4}. \tag{4.1}$$

As long as the Casimir apparatus remains unchanged, the previous assumption is a simple relabelling of the variables. Indeed, even the inclusion of a tilting of the plates dramatically changes the results. Thus, keeping in mind the standard configuration, when the plates are parametrically fixed, for both low and high-temperature approximations, one finds the same predictions obtained using the zero temperature Casimir effect [23]. Thus, for the parametrically fixed case, the size of the throat is really huge. In particular, for the low-temperature, we obtain approximately the same result as was obtained using the zero temperature limit. For the high-temperature, we have the possibility of reducing the throat size because of the presence of the $\sqrt{T}$ in the denominator. However, to obtain a physically relevant result, we have to produce a temperature so large that it is not realizable with current technology.

Indeed, the maximum realizable temperature is of the order of $T \simeq 10^8\ ^\circ K$ and this predicts a traversable wormhole with a throat size bigger than the solar system. On the other hand, when the plates are considered radially varying, in the low-temperature limit, one finds no solutions compatible with the traversability, and for the high-temperature approximation, we need to consider a twofold approach. The first one deals with the energy density, and it is predicting a solution that is not traversable. The second one introduces the Casimir thermal wavelength and, for this reason, one is forced to introduce an extra radial component. The final result looks like the original Casimir wormhole, but with an additional large constant appearing in front of the energy density. This allows the throat of the wormhole to have a small size but much more bigger compared to the Planckian one predicted at zero temperature. Therefore, we either have a Planckian traversable wormholes or we have giant traversable wormholes, which has a throat that cannot be considerably reduced. Nevertheless, to summarize, we can certainly claim that the thermal effects do not destroy the traversability of the hot Casimir wormhole. From this point of view, it is interesting to note that in some Casimir experiments, if the plates enter in a superconductive phase, it is possible to show an increase of negative energy [48, 49]. This means that the throat size can be increased. An important question related to the possibility of changing the throat size is about its stability in time. One possible approach could be on the line of introducing quantum gravitational corrections to the original classical background following Refs.[50–55], where the one-loop graviton contribution to a classical energy in a traversable wormhole background has been computed. Such a quantum correction appears closely related also to other forms of gravitational Casimir effect[56–58]. However this formalism avoids the introduction of the notion of time since the beginning and to this purpose will be discarded. Another possibility comes from a generalization of the metric (1.5) to the following one

$$ds^2 = -e^{2\Phi(r,t)}\, dt^2 + \frac{dr^2}{1 - b(r,t)/r} + R^2\,(r,t)\,\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{4.2}$$

where an explicit dependence on time has been introduced. Nevertheless, this is beyond the scope of this paper and it will be discussed elsewhere.

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