COGARCH models: some applications in finance

Ilia Negri*

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Abstract

COGARCH models are continuous time version of the well known GARCH models of financial returns. The aim of this paper is to show how the method of Prediction-Based Estimating Functions can be applied to estimate the parameters of a COGARCH(1,1) model from observations taken from real data set. In particular the General Motors tick-by-tick data of the Trades and Quotes database of the New York Stock Exchange (NYSE) are considered. A comparison between the results obtained with the method of moments estimator is done in terms of stability of the estimates over different windows of observations and in terms of analysis of residuals.

Keywords: Prediction based estimating function - Method of moment estimator - General Motors returns series

1 Introduction

COGARCH, that stands for COntinuous (in time), Generalized, Auto Regressive, Conditionally Heteroscedastic models were introduced in 2004 by Klüppelberg, Lindner and Maller (see [7]). In the last years COGARCH models have been widely studied and applied. For example Buchmann, and Müller in [3] look at COGARCH as limit experiment of GARCH processes, Klüppelberg, Maller and Szimayer in [8] present a COGARCH option pricing model including the possibility of default, Swishchuk and Couch in [13] present volatility and variance swaps valuations for the COGARCH (1,1) model Müller, Durand, Maller in [4] make an analysis of Merton's hypothesis for COGARCH models. One of the reason that suggest to use these models to fit financial log-return data is due to the fact that they are able to capture the so called stylized facts observed in real data: uncorrelated log-returns but correlated absolute log-return, time varying volatility, conditional heteroscedasticity, cluster in volatility, heavy tailed and asymmetric unconditional distributions, leverage effects. The aims of this paper is to apply the COGARCH (1,1) model to some real financial data sets, estimate the parameters of the model

*Department of Management, Information and Production Engineering, University of Bergamo. Viale Marconi, 5, 24044, Dalmine (BG), Italy. ilia.negri@unibg.it.

via the prediction based estimating functions as presented in [1] by Negri and Bibbona, and to look at the performance of these estimates. In particular the General Motors tick-by-tick data of the Trades and Quotes database of the New York Stock Exchange (NYSE) are considered. A comparison between the results obtained with the method of moments estimator is done in terms of stability of the estimates over different windows of observations and in terms of analysis of residuals. The paper is organized as follow. In Section 2 the COGARC(1,1) model is presented. The estimation procedure based on Prediction-Based Estimating Functions is introduced in Section 3 and in Section 4 the application to the real data set is presented. Finally in Section 5 some conclusions and further developments are given.

2 The model

The COGARCH(1,1) model is defined as the solution $(G, \sigma^2) = (G_t, \sigma_t^2)_{t\geq 0}$ to the following stochastic differential equations

$$\begin{cases} dG_t = \sigma_{t-} dL_t \\ d\sigma_t^2 = (\beta - \eta\sigma_{t-}^2)dt + \phi\sigma_{t-}^2 d[L]_t^d \end{cases} \tag{1}$$

where $[L]_t^d$ is the discrete part of the quadratic variation $[L]_t = \tau^2 t + [L]_t^d$ of the Lévy process $L = (L_t)_{t\geq 0}$, defined as

$$[L]_t^d = \sum_{0<s\leq t} \Delta L_s^2$$

with $\Delta L_s = L_s - L_{s-}$. We define $G_0 = 0$ and σ_0^2 independent of L. We assume $\mathbb{E}(L_1) = 0$ and $\mathbb{E}(L_1^2) = 1$ so that the volatility of the component G_t is given solely by σ_t.

The parameter space $\Theta \subset \mathbb{R}^3$ is defined as the set of those $\theta = (\beta, \eta, \phi)$ such that $\beta > 0$, $\eta > 0$ and $\phi > 0$.

We remember that the The GARCH(1,1) model is defined, for $n \in \mathbb{N}$, as follows

$$\begin{cases} Y_n = \sigma_n \varepsilon_n \\ \sigma_n^2 = \beta + \lambda Y_{n-1}^2 + \delta\sigma_{n-1}^2 \end{cases}$$

For GARCH(1,1) model we can write

$$\sigma_i^2 = \beta \sum_{k=0}^{i-1} \prod_{j=k+1}^{i-1} (\delta + \lambda\varepsilon_j^2) + \sigma_0^2 \prod_{j=0}^{i-1} (\delta + \lambda\varepsilon_j^2)$$

The previous formula can be rewritten as

$$\sigma_i^2 = \beta \int_{k=0}^{i-1} exp\left[\sum_{j=\lfloor u\rfloor+1}^{i-1} log(\delta + \lambda\varepsilon_j^2)\right] du + \sigma_0^2 exp\left[\sum_{j=0}^{i-1} log(\delta + \lambda\varepsilon_j^2)\right]$$

The idea is to replace the noise $(\varepsilon_i)_{i\geq 0}$ by increments of a Lévy process. It can be proved (Klüppelberg *et al.* 2004) after changing the parameters, and some computation that the process $(\sigma_t^2)_{t\geq 0}$ satisfy (1).

The continuous volatility process $(\sigma_t^2)_{t\geq 0}$ can be defined via an auxiliary Lévy process:

$$X_t = \eta t - \sum_{0<s\leq t} \log(1+\phi\Delta L_s^2)$$

as

$$\sigma_t^2 = \beta \mathrm{e}^{-(X_t - X_u)} \int_u^t \mathrm{e}^{-(X_u - X_s)}\, ds + \mathrm{e}^{-(X_t - X_u)} \sigma_u^2.$$

σ_t^2 is the instantaneous volatility or spot volatility, which is assumed to be stationary and latent. In contrast to classical stochastic volatility models, driven by two independent source of noise, in COGARCH(1,1) models L drives both, the volatility and the price process. So as in GARCH(1,1) models there is only one source of noise. The process G jumps at the same times as L does, with jump size $\Delta G_t = \sigma_t \Delta L_t$. Large change in the Lévy process results in an increase of the volatility and at the same time, in an increase or decrease of the process G.

3 Statistical Estimation

The estimation of the model parameters are based on a sample of equally spaced returns $G_{ir,r} = G_{(i+1)r} - G_{ir}$. In [5] Haug *et al.* explicit estimators are derived from a method of moments. In [9] Maller *et al.* proposed a pseudo maximum likelihood method that allows also for non equally spaced observations. Kim and Lee in [6] the asymptotic properties of the PML estimator proposed in [9] are proved. In [10] Müller et al an MCMC-based estimation method has been proposed for the model driven by a compound Poisson process. In [1] the Prediction Based Estimating Functions (PBEFs) method introduced in (Sorensen, 2000) is applied to the COGARCH(1,1) model. Let us present briefly the problem. We have to estimate the $p = 3$-dimensional parameter $\theta = (\beta, \eta, \phi)$, $\beta > 0$, $\eta \geq 0$ and $\phi \geq 0$. Let $\mathcal{H}_i^\theta$ be the Hilbert space of all square integrable real functions of the observations $\{G_{jr,r}\}_{j=0}^i$ endowed with the usual inner product

$$\langle h, g\rangle = \mathbb{E}_\theta \left(h(G_{0,r}, \ldots, G_{ir,r}) g(G_{0,r}, \ldots, G_{ir,r})\right),$$

where $\mathbb{E}_\theta$ denotes the expectation under the model with parameter θ. Let us fix an integer q. For any $i = q+1, \ldots, n$ we introduce the closed subspaces $\mathcal{P}_i^\theta$ of $\mathcal{H}_i^\theta$ spanned by the q observations that come before the i-th, i.e. $\mathcal{P}_i^\theta = \mathrm{span}(1, G^2_{(i-q)r,r}, \ldots, G^2_{(i-1)r,r})$, where 1 denotes the constant function with unit value. Provided that $\mathbb{E}_\theta(G^2_{ir,r}) < \infty$ for every $\theta \in \Theta$ and every $i = 1, \ldots, n$, the *prediction-based estimating functions* is

$$S_n(\theta) = \sum_{i=q+1}^{n} w^{i-1}(\theta, n)(G^2_{ir,r} - \pi^{i-1}(\theta)) \tag{2}$$

The vector $w^{i-1}(\theta, n) = (w_k^{i-1}(\theta, n))_{k=1}^p$ has components $w_k^{i-1}(\theta, n) \in \mathcal{P}_{i-1}^\theta$ where $\pi^{i-1}(\theta)$ is the minimum mean square error predictor of $G_{ir,r}^2$ in $\mathcal{P}_{i-1}^\theta$, that is the orthogonal projection of $G_{ir,r}^2$ on $\mathcal{P}_{i-1}^\theta$. An estimator is obtained solving the equation $S_n(\theta) = 0$ providing that this solution exists. The vector $w^{i-1}(\theta, n) = (w_k^{i-1}(\theta, n))_{k=1}^3$ has components $w_k^{i-1}(\theta, n) \in \mathcal{P}_{i-1}^\theta$ and $\pi^{i-1}(\theta)$ is the minimum mean square error predictor of $G_{ir,r}^2$ in $\mathcal{P}_{i-1}^\theta$, that is the orthogonal projection of $G_{ir,r}^2$ on $\mathcal{P}_{i-1}^\theta$. Such projection exist and it is uniquely determined by the normal equations

$$\mathbb{E}_\theta \left(\pi(G_{ir,r}^2 - \pi^{i-1}(\theta)) \right) = 0 \quad \forall \pi \in \mathcal{P}_{i-1}^\theta.$$

Define $C(\theta)$ the covariance matrix of the q vector $(G_{(i-1)r,r}^2, \ldots, G_{(i-q)r,r}^2)^T$ and $b(\theta)$ the vector whose components are $b_j(\theta) = \mathrm{Cov}_\theta(G_{(i-j)r,r}^2, G_{ir,r}^2)$ for $j = 1, \ldots, q$.

As the increment process $G_{ir,r}$ is stationary the matrix $C(\theta)$ and the vector $b(\theta)$ do not depend on i. We define the vector $a(\theta) = C(\theta)^{-1} b(\theta)$ whose components are denoted by $a_j(\theta)$ for $j = 1, \ldots, q$ and the scalar $a_0(\theta) = \mathbb{E}_\theta G_{ir,r}^2 - \sum_{j=1}^q a_j(\theta) \mathbb{E}_\theta (G_{(i-j)r,r}^2)$. Moreover we denote by $\tilde{a}(\theta)$ the $q+1$ vector $\tilde{a}(\theta) = (a_0(\theta), a_1(\theta), \ldots, a_q(\theta))^T$.

An explicit expression for the predictors is given by Sørensen in [11] (see also of the same author [12]) by

$$\pi^{i-1}(\theta) = a_0(\theta) + \sum_{j=1}^q a_j(\theta) G_{(i-j)r,r}^2 \tag{3}$$

As the components $w_k^{i-1}(\theta, n)$ of the vector $w^{i-1}(\theta, n)$ are elements of $\mathcal{P}_{i-1}^\theta$, they can be decomposed as $w_k^{i-1}(\theta, n) = w_{k0}^{i-1}(\theta, n) + \sum_{j=1}^q w_{kj}^{i-1}(\theta, n) G_{(i-j)r,r}^2$ for some scalars $w_{k0}^{i-1}(\theta, n)$ and $w_{kj}^{i-1}(\theta, n)$ $j = 1, \ldots, q$ that we collect into the $p \times (q+1)$ matrices $W_n^{i-1}(\theta)$ whose elements are $w_{kl}^{i-1}(\theta, n)$ for $1 \le k \le p$ and $0 \le l \le q$.

With these notations the estimating function (2) can be written as

$$S_n(\theta) = \sum_{i=1}^n W_n^{i-1}(\theta) H^i(\theta)$$

where $H^i(\theta)$, $i = 1, \ldots, n$, are $(q+1)-$vectors whose components are

$$H_0^i(\theta) = G_{ir,r}^2 - a_0(\theta) - a_1(\theta) G_{(i-1)r,r}^2 - \cdots, -a_q(\theta) G_{(i-q)r,r}^2$$

and for $k = 1, \ldots, q$,

$$H_k^i(\theta) = G_{(i-k)r,r}^2 (G_{ir,r}^2 - a_0(\theta) - a_1(\theta) G_{(i-1)r,r}^2 - \cdots - a_q(\theta) G_{(i-q)r,r}^2).$$

Since the increment process $G_{ir,r}$ is stationary, so is the vector $H^i(\theta)$ and there is no reason to give different weights for different i, thus we restrict our PBEFs to those that can be written in the form

$$S_n(\theta) = W_n(\theta) \sum_{i=1}^n H^i(\theta). \tag{4}$$

Let us introduce the vector $Z_i = (1, G^2_{(i-1)r,r}, \dots, G^2_{(i-q)r,r})^T$, the matrix $\tilde{C}(\theta) = \mathbb{E}\big(Z^i(Z^i)^T\big)$, and the matrix $D(\theta) = -W(\theta)\tilde{C}(\theta)\partial_{\theta^T}\tilde{a}(\theta)$. In terms of such quantities we state the following conditions.

1. There exist a constant $\delta > 0$ such that $\mathbb{E}_\theta(G_1^{8+\delta}) < \infty$.
2. The vector $\tilde{a}(\theta)$ and the matrix $W_n(\theta)$ are continuously differentiable with respect to θ.
3. There exist a non-random matrix $W(\theta)$ such that for every compact set $K \subset \Theta$
$$W_n(\theta) \xrightarrow{\mathbb{P}_{\theta_0}} W(\theta) \qquad\qquad \partial_\theta W_n(\theta) \xrightarrow{\mathbb{P}_{\theta_0}} \partial_\theta W(\theta)$$
uniformly for $\theta \in K$ as $n \longrightarrow \infty$.
4. The matrix $D(\theta_0)$ has full rank 3.
5. We have $W(\theta)\,\mathbb{E}_{\theta_0}\big(H^i(\theta)\big) \neq 0$ for any $\theta \neq \theta_0$.

Under the previous condition a solution $\hat{\theta}_n$ of (2) exist is consistent and asymptotically Normal. See Bibbona and Negri [1] and Sørensen [11]. Knowing all simple and joint moments up to the order *four* $\mathbb{E}\big(G^2_{jr,r}G^2_{ir,r}\big), \mathbb{E}\big(G^4_{jr,r}\big), \mathbb{E}\big(G^2_{jr,r}\big)$ for any integer i, j is essential to calculate the predictors and hence to calculate any estimating function in the form (4). Such explicit expressions for the COGARCH(1,1) model are given in [5]. However the asymptotic variance of the estimates involves the matrix M which depends on all the simple and joint moments up to the order *eight*, e.g. $\mathbb{E}\big(G^8_{jr,r}\big)$, $\mathbb{E}\big(G^6_{jr,r}\big)$, $\mathbb{E}\big(G^2_{jr,r}G^6_{ir,r}\big)$, $\mathbb{E}\big(G^2_{ir,r}G^2_{jr,r}G^2_{kr,r}G^2_{hr,r}\big)$ and similar. Such explicit expressions for the COGARCH(1,1) model are given in [1]. According to the general theory (see for example Sørensen [11] and [12]), among all the PBEFs in the form (4) it is possible to select an *optimal* one. The optimal PBEF will be such that the corresponding estimator has the smallest possible asymptotic variance. The weight matrix of the optimal PBEF is

$$W_n^* = \partial_\theta \tilde{a}^T(\theta)\tilde{C}(\theta)M_n^{-1}(\theta) \tag{5}$$

where the form of the matrix $M_n(\theta)$ can be found in for example in Bibbona and Negri [1]. Also the optimal weight matrix W^* depends on all the simple and joint moments up to the order *eight*. In term of the existence of higher moments, the condition requested for the asymptotic normality of the estimators obtained via PBEF and via the MM is the same (Condition 1. above $\mathbb{E}_\theta(G_1^{8+\delta}) < \infty$ for some $\delta > 0$).

4 Applications and examples in finance

Taking the of log-returns of a financial series we estimate the parameters in the model through the functions implemented in the COGARCH package, based on the PBES's method. Following the flow chart in Figure 1 the function eCOGARCH

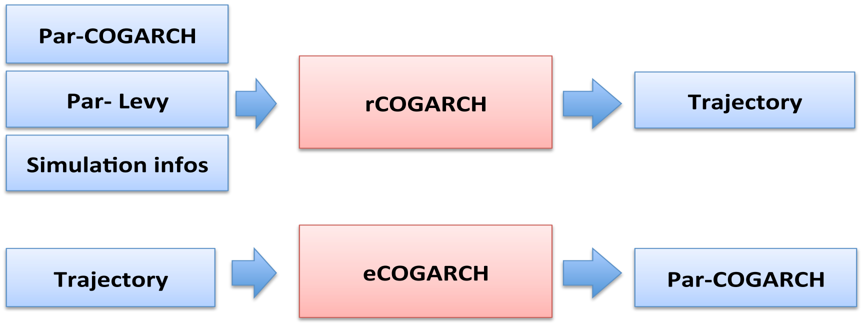

Figure 1: Flow chart of the Cogarch package. First line: the simulation part. Second line: the estimation part.

is able, given a trajectory, to estimate the parameters via the prediction-based estimating functions and the method of moment estimator. Then the rCOCARCH function is able to simulate a COGARCH model given the value of the parameters, the Lévy process driving the COGARCH models and some informations about the trajectory we want to simulate, such as the step between two observations. The package is available at `https://r-forge.r-project.org/projects/cogarch/`. The COGARCH(1,1) model will be fitted to 5 minutes log returns of three different time series. The data-sets is one of the same presented by in Haug *et al.* in [5].

The stocks time series is General Motors (GM). We consider tick-by-tick data of the Trades and Quotes database of the New York Stock Exchange (NYSE). The general data set spans over 4 months starting in February 2002. We considered only the prices between 9.35 a.m. and 4 p.m. to compute the five minutes log returns based on previous tick interpolation. There were 83 trading days between the beginning of February and the end of May 2002. Hence, each of the series has a total length of 6391 data points. For our application we consider only a subset of this general data set.

Here we follow a different approach than the one in [5]. Precisely we estimate the parameter of the COGARCH(1,1) model for different windows of 500 data. This is a reasonable number for the estimation procedure. At any estimation step we move one step ahed the observation window and we compute the value of the three parameters for 350 times. The estimated values are reported in Figure 4. We can see how the values of the parameter are reasonably stable. This stability is better understand if you have a look to Figure 2 and to Figure 3. The values of the three parameter are reported for 500 times ahed instead of 350. This is just to show that after observation 850 we have to investigate better how this instability for both the two estimator procedures arise. The estimator based on PBEF seems to be more stable, as we can deduce from Figure 2 and from Figure 3

The mean values and the standard deviation of the three parameters are reported in Table 1.

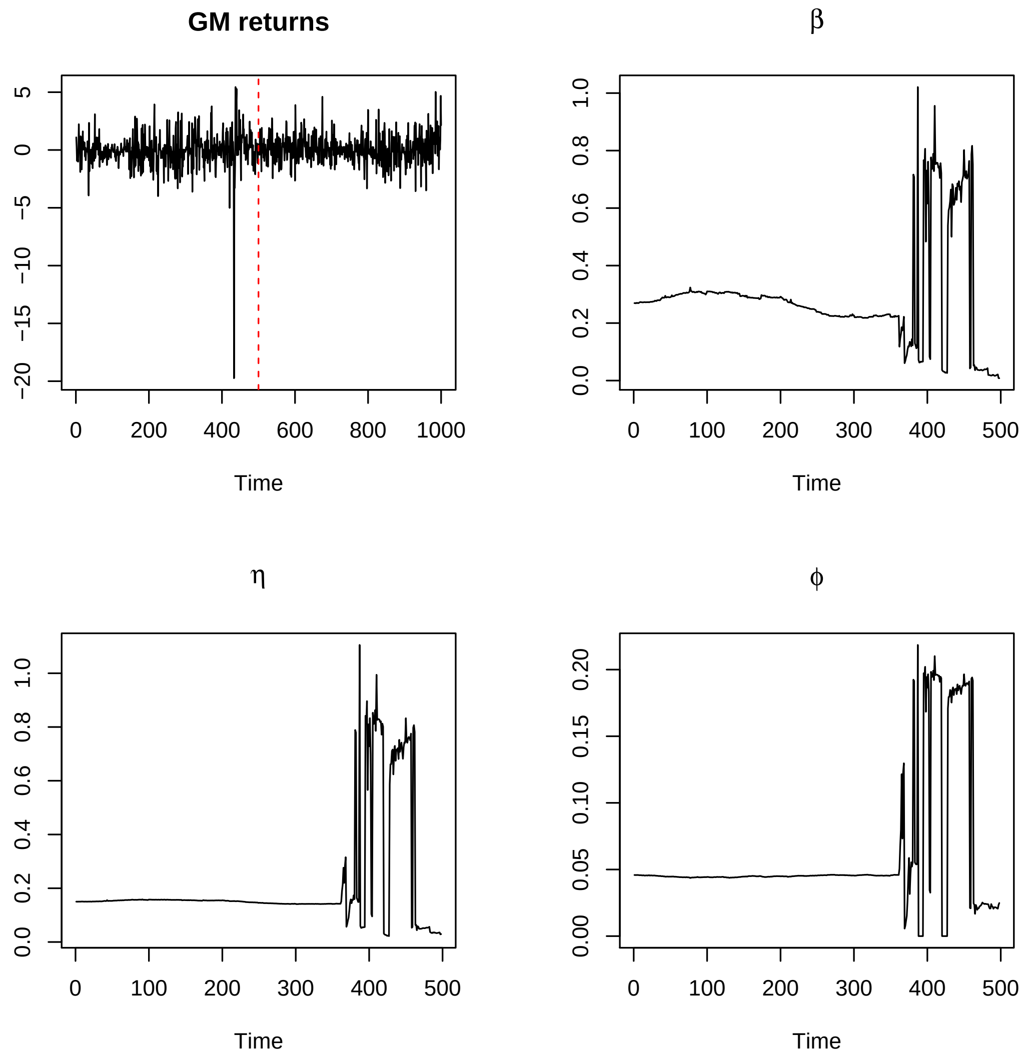

Figure 2: The GM return values (Top left) and the values of the three estimated parameter by PBEF

Table 1: Mean and standard deviation of the parameter η, β and ϕ for the GM return series. The values are evaluated on 350 windows of 500 observations each.

	Mean PBEF	Std PBEF	Mean MME	Std MME
β	0.27	0.0322	0.02	0.0136
η	0.15	0.0058	0.02	0.0130
ϕ	0.05	0.0006	0.01	0.0099

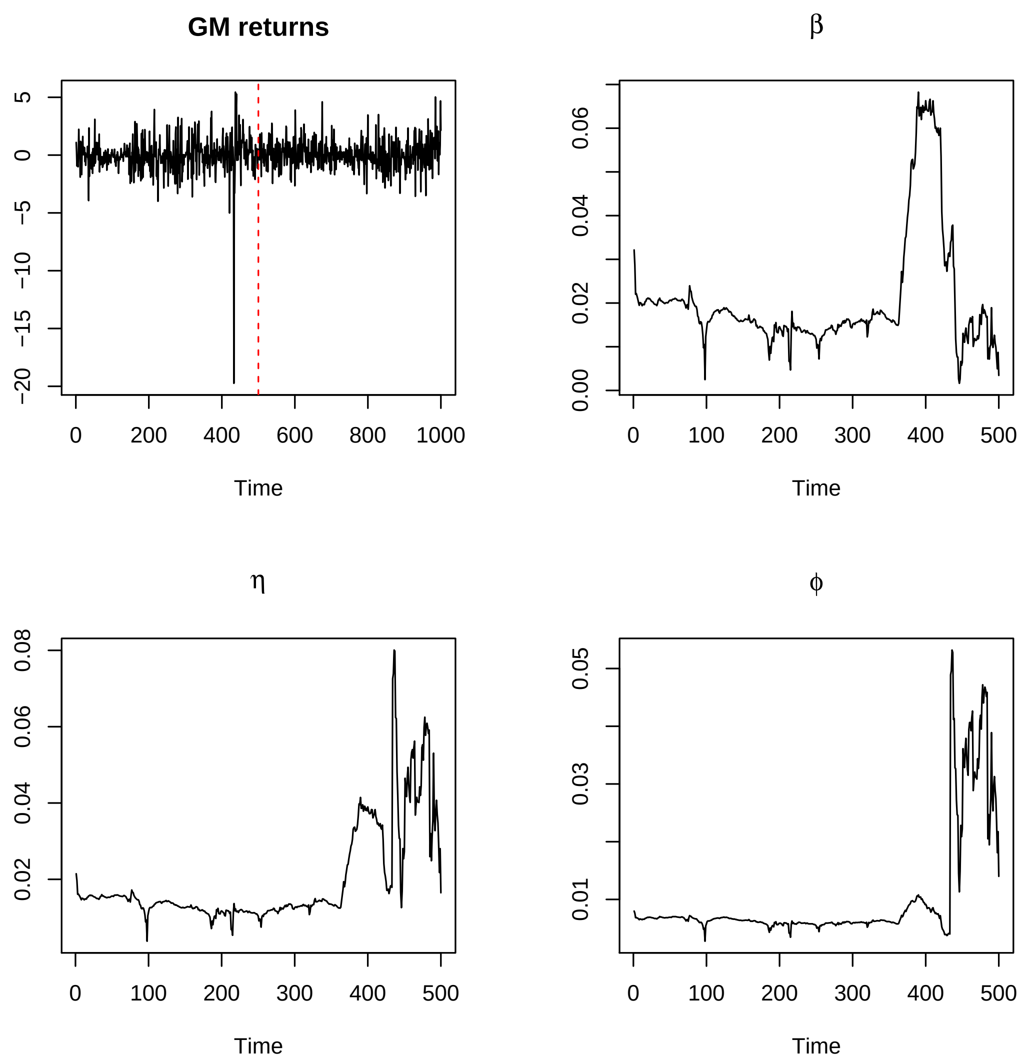

Figure 3: The GM return values (Top left) and the values of the three estimated parameter by MME

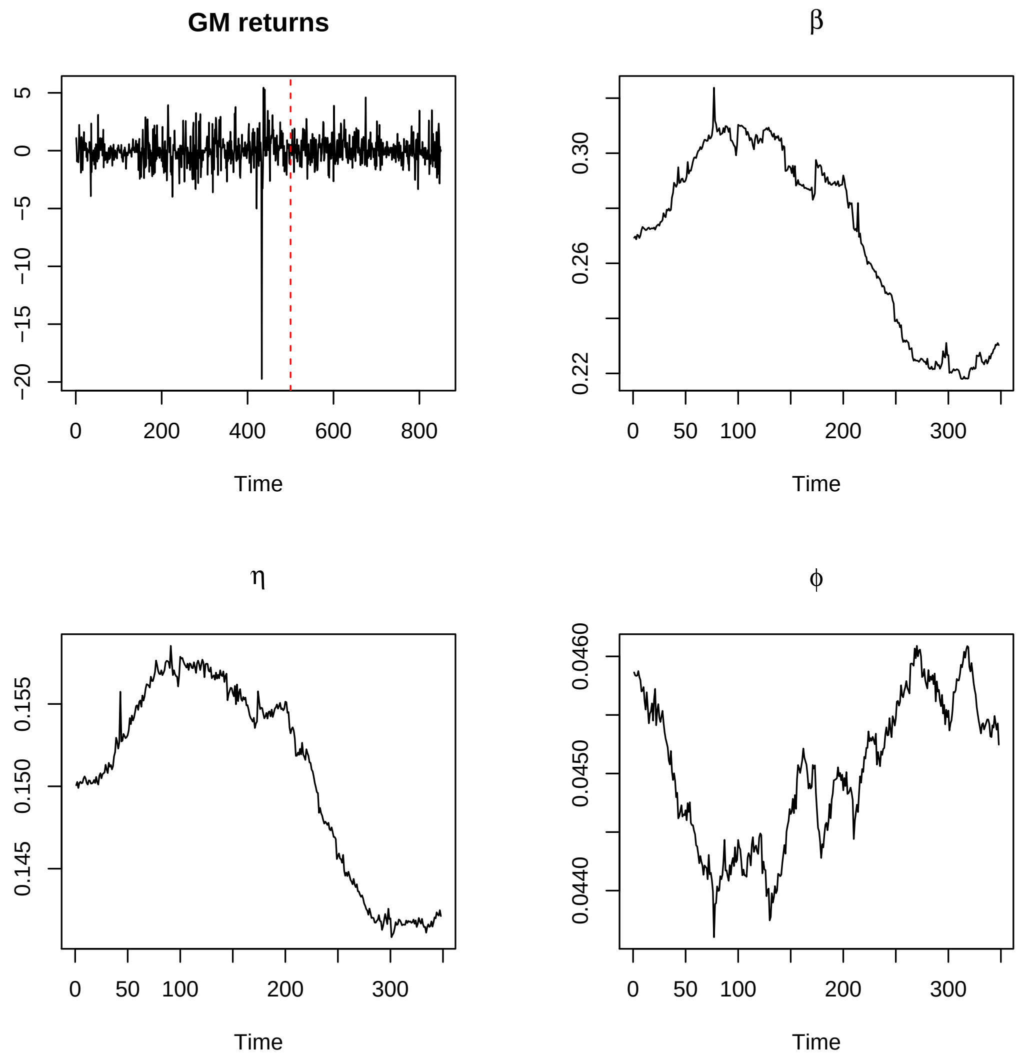

Figure 4: The GM return values (Top left) and the values of the three estimated parameter by PBEF for 350 step in windows of 500 observations

Table 2: Mean, standard deviation, skewness and kurtosis of residuals from the PBEF and the MME for the financial series GM

	Mean	Std	Skewness	Kurtosis
PBEF	-0.05	0.72	-0.02	4.43
MME	-0.05	0.73	0.00	4.42

We investigate the performance of the estimation procedure based on PBEF by a residuals analysis. The residuals are given by

$$r_t = \frac{G_t}{\hat{\sigma}_{t-1}}$$

where the estimated volatility $\hat{\sigma}_t^2$ is given by

$$\hat{\sigma}_t^2 = \hat{\beta} + (1 - \hat{\eta})\hat{\sigma}_{t-1} + \hat{\phi} G_t^2.$$

The details are given by Haug *et al.* in [5]. Table 2 reports the value of the mean the standard deviation, the skewness and the kurtosis of the residuals obtained with the two estimator procedure. We can see how the indexes are very similar both for PBEF and MME. In Bibbona and Negri [1] a small gain of the optimal estimator was observed in the case of simulated data. Obtaining the optimal prediction based estimating function is very expensive in computational time, so we don't apply this estimation procedure as in a test set the gain was very pour both in term of stability of the estimates and in the analysis of the residuals.

5 Conclusion and further developments

This study is a first attempt to apply two statistical estimations procedure for COGARCH(1,1) models to real data. In the study given by Haug *et al.* in [5] they estimate the value of the parameters on a unique trajectory. Here we try to estimate the parameters on multiple and contiguous windows of the series of data to understand how the estimated values of the parameter vary. Both the estimator methods have some stability problems that have to be investigate better. Anyway both estimator procedure are good as the analysis of the residuals reveals but the procedure based on PBEF seems to give more stable estimates of the parameters. In a next study the results for some other data set could be included and also the result based on the pseudo maximum likelihood method introduced by Maller *et al.* in [9]. In terms of forecasting it would be very interesting to understand how we can model the returns and compare the various method of estimation in terms of prediction errors. One possibility that we want to investigate is to model returns with CARMA(1,1) model. CARMA(p.q) models were introduced by Brockwell in [2].

References

[1] Bibbona, E., Negri, I. (2016). Higher Moments and Prediction-Based Estimation for the COGARCH(1,1) Model, *Scandinavian Journal of Statistics*, Vol. 42: 891 910.

[2] Brockwell, P.J. Lévy-driven CARMA processes, (2001). *Ann. Inst. Statist. Math.*, Vol. 53, No.1, 113-124.

[3] Buchmann, Boris; Mller, Gernot. (2012). Limit experiments of GARCH. *Bernoulli*, 18, no. 1, 64–99.

[4] Müller, G., Durand, R., Maller, R. (2011). The riskreturn tradeoff: A COGARCH analysis of Merton's hypothesis, *Journal of Empirical Finance*, Vol. 18, 2, p. 306-320.

[5] Haug, S., Klüppelberg, C., Lindner, A., Zapp, M. (2007). Method of moment estimation in the COGARCH(1,1) model. *Econom. J.*, 10:320341, 2007.

[6] Kim, M., Lee, S. (2013). On the maximum likelihood estimation for irregularly observed data from COGARCH(1,1) models. *REVSTAT, Statistical Journal* 11(2), 135-168.

[7] Klüppelberg, C., Lindner, A., and Maller, R. (2004). A continuous-time GARCH process driven by a Lévy process: stationarity and second-order behaviour. J. Appl. Prob.

[8] Klüppelberg, Maller and Szimayer (2011). The COGARCH: a review, with news on option pricing and statistical inference, Surveys in stochastic processes, EMS Ser. Congr. Rep. p. 29–58

[9] Maller, R.A., Müller, G., Szimayer, A. (2009). GARCH modelling in continuous time for irregularly spaced time series data. *Bernoulli*, 14:519542, 2009.

[10] Müller, G. (2010). MCMC estimation of the COGARCH(1,1) model. *J. Financial Economectrics* 8(4):481510.

[11] Sørensen, M. (2000). Prediction-based estimating functions. *Econom. J.*, 3(2), 123147.

[12] Sørensen, M. (2011). Prediction-based estimating functions: review and new developments. *Braz. J. Probab. Stat.*, 25(3), 362391.

[13] Swishchuk, A. and Couch, M. (2010), Volatility and Variance Swaps for the COGARCH(1,1) Model. *Wilmott J.*, 2: 231246.