Option Pricing in Non-Gaussian Ornstein-Uhlenbeck Markets

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Abstract

Non-Gaussian Ornstein-Uhlenbeck processes allow to model several distributional features of assets’ returns, including volatility clustering, fat tails and leverage. The most common specifications however do not allow to model long range dependence in the volatility process or self-exciting dynamics. Here we focus on the recently introduced class of Volatility Modulated non-Gaussian Ornstein-Uhlenbeck (VMOU) processes, that introduce a Stochastic Volatility of Volatility (SVV) component, allowing for richer dynamics for the processes, while maintaining good analytical properties. We present the framework, showing how to introduce SVV and how to compute structure preserving equivalent martingale measures. We also recall the Fourier transform option pricing setting, showing an implementation based on Non-Gaussian Ornstein-Uhlenbeck processes. Finally, we run a simulation study to highlight the empirical properties of VMOU processes, with particular attention to the clustering of volatility of volatility.

Key words

VMOU Processes, volatility of volatility, self-exciting dynamics

JEL Classification: G12, G13

1. Introduction

Stochastic volatility is a well recognized stylized fact in equity markets, and has been widely studied in the financial literature. Probably the most known stochastic volatility models is the one introduced by Heston (1993) that, building from the Black-Scholes and Merton framework, models the instantaneous variance as a Cox–Ingersoll–Ross (CIR) process. Another approach is the one proposed by Barndorff-Nielsen and Shephard (2001), that model stock as diffusion process and volatility as a non-Gaussian Ornstein–Uhlenbeck (OU) process driven by a Lévy subordinator (Barndorff-Nielsen and Shephard, 2001) hereafter these will be referred to as BNS models. A third approach for the introduction of stochastic volatility is to change the clock time $t$ to a random time with a non-decreasing process, typically a Lévy subordinator (Barndorff-Nielsen and Nicolato, 2003).

Here we focus on the BNS framework, that is characterized by good empirical performances and high analytical tractability. BNS models can explain several stylized facts exhibited by equity prices, including heavy tails, volatility clustering and leverage effects (i.e. the negative correlation between asset returns and volatility) (Barndorff-Nielsen and Shephard, 2001). Generally, they also allow easy option pricing using Fourier transform techniques (see Eberlein et Al., 2010). One of the main drawback is that most of the specifications of this framework do not allow to capture long range dependence. Moreover, empirical analyses highlighted that the volatility in the market has itself a stochastic volatility, and that the market presents self-excitation and cross-exciting dynamics, in particular when considering intra-day frequencies (Aït-Sahalia et Al., 2015). A possible extension of BNS models that includes naturally self-exciting dynamics may be the use of Hawkes processes in the volatility process. This extension

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however would lead to relevant problems concerning option pricing, since Hawkes process are not semimartingales. An alternative and promising approach, consists in the introduction of a stochastic volatility of volatility SVV component, as proposed by (Barndotff-Nielsen and Veraart (2013)) with the Volatility modulated non-Gaussian Ornstein-Uhlenbeck (MVOU) processes. MVOU are an extension of the BNS model in which the volatility process has a stochastic volatility itself. In particular, the SVV effect can be obtained in two ways: by introducing a stochastic volatility for the variance process or by changing the time of the Lévy subordinator of the variance process.

In this work we present some properties of the VMOU processes and we lay out an option pricing methodology based on Fourier transform.

Section 2 gives the theoretical description of the VMOU model, Section 3 discusses the change of measure and option pricing, Section 4 shows an application of option pricing, Section 5 shows a simulation study aimed at highlighting the empirical properties of VMOU processes, Section 5 concludes and presents future research goals.

2. The VMOU model

2.1 The BNS model

We first briefly introduce the BNS model, and then we show how to expand it to the VMOU framework. We assume that the logarithmic asset price $Y = (Y_t)_{t \geq 0}$ is an Itô’s semimartingale defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$:

$$dY_t = a_t dt + \sigma_t dW_t + dJ_t,$$

$$dY_t = \left(\mu - \frac{1}{2} \sigma_t^2\right) dt + \sigma_t dW_t + \rho dL_{\lambda t},$$

$$d\sigma_t^2 = -\lambda \sigma_t^2 dt + dL_{\lambda t},$$

where $a = (a_t)_{t \geq 0}$ is a predictable drift process, $\sigma = (\sigma_t)_{t \geq 0}$ is a càdlàg process, and $J = (J_t)_{t \geq 0}$ is the pure jump component. In particular, in the BNS model, the volatility $\sigma$ follows a non-Gaussian OU process driven by a Lévy subordinator (i.e. a Lévy process with positive increments) and the jump component of the log-asset price is the same that drives the volatility process. The dynamics for the model can therefore be described by the following SDEs:

$$dY_t = \left(\mu - \frac{1}{2} \sigma_t^2\right) dt + \sigma_t dW_t + \rho dL_{\lambda t},$$

$$d\sigma_t^2 = -\lambda \sigma_t^2 dt + dL_{\lambda t},$$

where the Background Driving Lévy process (BDLP) $L$ has no drift, $L = \{L_t, t \geq 0\}$ is a subordinator $\mu \in \mathbb{R}$ and $\rho \in \mathbb{R}_+$. Note that the Brownian motion $W$ and the BDLP $L$ are independent and the filtration is the one generated by the pair $(W, L)$. The leverage effect is obtained thanks to the coefficient $\rho$ in Equation 2.

The model is characterized by a high level of analytical tractability. In particular, for the special choice of a Gamma-OU and IG-OU processes, it is possible to derive in closed form the characteristic function of the log price of the assets under the structure preserving martingale measure, allowing for fast and reliable option pricing (Nicolato and Venardos, 2003).

2.2 The VMOU model

Volatility modulated non-Gaussian OU models allow to introduce a stochastic volatility of volatility (SVV) component in the BNS framework outlined in the previous section. In particular SVV can be obtained in two ways: the first is to introduce a stochastic integrand $\nu$ independent from $L$ which scales the jump size of the subordinator. The dynamics of the variance can then be expressed as:
$$d\sigma_t^2 = -\lambda \sigma_t^2 dt + \nu_{\lambda t}^{-}dL_{\lambda t}, \quad (4)$$

where $\nu = (\nu_t)_{t \geq 0}$ is a stationary positive, càdlàg process.

An alternative approach to introduce SVV is to perform a stochastic time change on the subordinator. The dynamics for the instantaneous variance are then the following:

$$d\sigma_t^2 = -\lambda \sigma_t^2 dt + dL_{\tau_t}, \quad (5)$$

where $\tau_t = \int_0^t \xi_s ds$ and $\xi = (\xi_t)_{t \geq 0}$ is a stationary positive, càdlàg process, independent from $L$. The process $\xi$ is chosen to be absolutely continuous. As an alternative, $\xi$ could have been modeled as a Lévy subordinator, although in this case we would return to the original BNS framework, since a time-change performed using a Lévy subordinator performed on another Lévy subordinator is itself a Lévy subordinator (Barndorff-Nielsen and Veraart (2013)).

The two approaches for the introduction of SVV (stochastic integrand and time-change) could in theory be used together, although in practical applications it may be more suitable to use only one of the two.

### 3. Martingale Measures and Option Pricing

In order to use VMOU model to price derivatives it is necessary to identify an equivalent martingale measure. The stochastic volatility market that we are modelling is free of arbitrage but not complete, therefore the martingale measure is not unique and we have to select one. We consider the class of structure preserving martingale measures, discussed in the context of BNS model in Nicolato and Venardos (2003) and Hubalek and Sgarra (2009).

The peculiarity of these equivalent martingale measures is that under such measures the evolution of the underlying assets follows a stochastic differential equation with the same structure, although with possibly different parameters (Hubalek and Sgarra (2009)). This in turn allows to use Fourier-transform techniques for option pricing.

In particular, Barndorff-Nielsen and Veraart (2013) show that a structure preserving change of measure for VMOU process can be performed by constructing the product measure of a structure preserving change of measure for the BNS model with a structure preserving change of measure for the SVV components. We also highlight an interesting property of VMOU models proved in Barndorff-Nielsen and Veraart (2012): the authors show that in VMOU models, the variance risk premium dynamics under a structure preserving equivalent martingale measure is fully determined by the SVV component.

Once obtained the characteristic function of the log-price process, it is possible to price options and other derivatives using Fourier-transform techniques, such as the ones proposed by Carr and Madan (1999) and Eberlein et al. (2010). In particular, in case of plain vanilla call options, we can use the method of Carr and Madan (1999). Let $k$ denote the log of the strike price $K$ and $C_T(k)$ the value of an European call option with maturity $T$ and strike $K$. Indicating with $q_T(s)$ the risk neutral density of the log price and $\phi_T(u)$ the corresponding characteristic function, we can denote the value of the call option as:

$$C_T(k) \equiv \int_k^\infty e^{-rt} (e^s - e^k) q_T(s) ds \quad (6)$$

As proved in Carr and Madan (1999), we can then express the value of the option as follows:

$$C_T(k) = \frac{e^{-ak}}{\pi} \int_0^\infty e^{-itk} \phi_T(u) du, \quad (7)$$

with
\[
\psi_T(u) = \frac{e^{iT\varphi_T(u-(\alpha+1)i)}}{a^2 + a - u^2 + iu(2\alpha + 1)},
\]

where \( \alpha \) is a damping factor. Equation 7 can be solved numerically using fast Fourier transform or, as proposed by Chourdakis (2004), fractional Fourier transform.

The technique can be applied to VMOU processes, for which the characteristic function can be obtained in semi-analytical form. We can express the Laplace transform of a VMOU process under the structure preserving equivalent martingale measure and to implement Carr transform of \( Y_t \).

It is possible to determine the Laplace transform or, as proposed by Madan (1999) Fourier pricing. In particular, the parameters of the model, \( \sigma_t \), are obtained in semi-analytical form. Moreover, the process \( \sigma_t^2 \) is characterized by both stochastic integrand and change of time (see Equations 4 and 5).

The discounted process is a martingale if and only if \( \mu + \left( \beta + \frac{1}{2} \right) \sigma_t^2 = r \) (Nicolato and Vernandos, 2003). If the martingale condition holds, it is possible to determine the Laplace transform as follows, as proved in Barndorff-Nielsen and Veraart (2013), Proposition 2.1:

Let \( \phi(u) = \mathbb{E}_t^\mathbb{Q} \left[ e^{iuY_{t+h}} \right] \) for \( h \geq 0 \). Then:

\[
\phi(u) = \exp \left( uY_t + u\mu h + \left( \beta u + \frac{\sigma_t^2}{2} \right) \epsilon(h) \sigma_t^2 \right) \mathbb{E}_t^\mathbb{Q} \left[ \exp \left( \lambda \int_t^{t+h} \chi_L \left( f(s,u) \right) ds - ds \right) \right],
\]

where \( f(s,u) := \left( \beta u + \frac{\sigma_t^2}{2} \right) \epsilon(t) \) and \( \chi_L \) denotes the log-transformed Laplace transform of \( L \).

In the VMOU case the integral has to be valued numerically using Monte Carlo techniques, while the traditional BNS admits, in some specifications, a closed form for the characteristic transform of \( Y_{t+h} \). In the following sub-Section, we present a brief application of Fourier pricing technique with the BNS model.

### 3.1 Fourier Pricing with BNS Model

When the variance process of the BNS model is specified with an OU-gamma or a OU-NIG process (see Schoutens, 2004), it is possible to obtain the characteristic function of the log-price process under the structure preserving equivalent martingale measure and to implement Carr Madan (1999) Fourier pricing. In particular, in the case of OU-gamma variance, Equation 8 can be written as:

\[
\phi(u) = \mathbb{E}_t^\mathbb{Q} \left[ \exp(\text{i}uY_t | S_0, \sigma_0) \right] = \mathbb{E}_t^\mathbb{Q} \left[ \exp(\text{i}u \log(S_0) + (r-q-a\lambda\rho(b-\rho)^{-1})t) \right] \times \exp \left( -\frac{1}{2} \lambda^{-1}(u^2 + iu)(1 - \exp(-\lambda t)) \sigma_0^2 \right) \times \exp \left( a(b - f_2(u))^{-1} \left( b \log \left( \frac{b-f_1(u)}{b-iu\rho} \right) + f_2(u) \lambda t \right) \right),
\]

where \( f_1(u) = iu\rho - \frac{1}{2} (u^2 + iu)(1 - \exp(-\lambda t)) \) and \( f_2(u) = iu\rho - \frac{1}{2} (u^2 + iu) \), \( r \) is the risk-free rate, \( q \) the dividend yield of the asset and \( a, b, \lambda, \rho \) the parameters of the model, as specified in Schoutens (2004, Chapter 7.1.1). We used the US 3-months treasury bill as the risk free rate and the historical dividend yield of the S&P 500. The calibration of the model is performed on 167 European call options written on the S&P 500 index priced at the closing time of 12/08/2017. The dataset is available upon request by the authors. The calibration has
been performed by minimizing the mean square error (MSE) of the option price obtained by the model and the market price:

\[
\sum_{\text{options}} \frac{(\text{market price} - \text{model price})^2}{\text{number of options}}
\]  

(10)

Figure 1 shows the market prices of the options and the fitted values, as well as the implied volatilities. Table 1 reports the value of the RMSE and other statistics and compares them to the results of pricing done with Black, Scholes and Merton model. We see that the BNS model proves to be far superior in fitting option prices compared to the Black, Scholes and Merton model. Notice however that the calibration has been performed using the market options in a single day. In theory the model parameters should remain constant over time, however in practice they typically change and the market practice is to recalibrate the model daily (see for instance da Fonseca and Grasselli, 2011). The usage of a more suitable model, such as the VMOU, may in theory reduce the instability over time of the parameters, allowing more accurate pricing and forecasting. We will address this issue in future investigations.
Table 1: Statistical measures for quality of fit – AAE is Average Absolute Error, RMSE is Root Mean Square Error, APE is Average Percentage Error. The indicators are specified as in (Schoutens, 2004, Chapter 1.2.2).

<table>
<thead>
<tr>
<th>Model</th>
<th>AAE</th>
<th>RMSE</th>
<th>APE (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BNS-OU-gamma</td>
<td>1.52</td>
<td>2.19</td>
<td>2.86</td>
</tr>
<tr>
<td>Black, Scholes and Merton</td>
<td>7.82</td>
<td>10.64</td>
<td>14.74</td>
</tr>
</tbody>
</table>

4. Simulation of VMOU Processes

In this Section, we conduct a simulation study with two specifications of the VMOU process. The goal is to describe the empirical properties of the models and to highlight their modeling flexibility, specifically in relation to long range dependence and volatility clustering of the volatility process. The first specification is constructed using the stochastic integral \( v \) (hereafter defined as the stochastic integral model):

\[
\begin{align*}
    dY_t &= \left( \mu - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dW_t + \rho dL_t, \\
    d\sigma_t^2 &= -\lambda \sigma_t^2 dt + v_{\lambda t} dL_t, \\
    dv_{\lambda t}^2 &= -\lambda_t^0 \lambda_t v_{\lambda t}^2 dt + dL_{\lambda t}^0,
\end{align*}
\]

where \( L_t \) and \( L_t^0 \) are independent gamma processes with parameters \( a = 1, s = 0.08 \) and \( a^0 = 0.1, s^0 = 5 \), respectively. Moreover, \( \mu = 0.05, \lambda = 10, \lambda^0 = 1 \) and \( \rho = -0.01 \).

The second specification is constructed using a time change (hereafter time change model), in which the process \( \xi \) follows a CIR process driven by a Brownian motion \( B \) independent from \( W \) (see Equation 5):

\[
\begin{align*}
    dY_t &= \left( \mu - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dW_t + \rho dL_t, \\
    d\sigma_t^2 &= -\lambda \sigma_t^2 dt + 0.5 dL_{\tau t}, \\
    d\tau_t &= \xi_t dt, \\
    d\xi_t &= \alpha (\beta - \xi_t) dt + \gamma \sqrt{\xi_t} dB_t,
\end{align*}
\]

where \( \alpha = 2, \beta = 1, \gamma = 1 \) and the parameters for \( dY_t \) and \( d\sigma_t^2 \) are the same as in the stochastic integral model (Equation 11).

Finally, we also simulate a traditional BNS model with OU-gamma stochastic volatility as a reference (hereafter called BNS-OU-gamma). The specification is the following:

\[
\begin{align*}
    dY_t &= \left( \mu - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dW_t + \rho dL_t, \\
    d\sigma_t^2 &= -\lambda \sigma_t^2 dt + dL_t,
\end{align*}
\]

where the parameters are set as in Equations 12 and 13. Notice that the parameters chosen make sure that in the three specifications the mean of \( \sigma_t^2 \) is equal. For each setting we simulate 3000 observations of length 1 trading day. Figure 2 shows simulated trajectories of \( \sigma^2 \) computed using the three specifications, the \( \xi \) process in the time change model process and the \( v^2 \) in the stochastic integral model. We can see that the processes of \( \sigma^2 \) exhibit richer dynamics in the two specifications of MVOU model compared to the traditional BNS-OU-gamma model. In particular, the processes display volatility clustering and sharper rises in volatility compared to the BNS-OU-gamma. The two MVOU models obtain the results in rather different ways, in the stochastic integral case, the volatility dynamics are driven by sharp jumps in the process \( v^2 \), while in the time change model they are influenced by the diffusion process \( \xi \).

The presence of an SVV component can also increase the memory of the process, as shown by the autocorrelation function of the variance function and of the absolute returns for the three
specification of $\sigma^2$, as presented in Figure 3, that is larger for the VMOU models compared to the BNS-OU-gamma model. Finally, we notice that by changing the parameters of the SVV it is possible to obtain a large variety of specifications, in particular concerning the volatility clustering of the $\sigma^2$ processes, allowing for large model flexibility. This last point is illustrated in Figure 4, where we simulate the stochastic integral model with different values for the parameters $\lambda^{(v)}$, that governs the level of serial correlation of the process. This model flexibility may be particularly useful to better capture market features and to model self-exciting dynamics.

Figure 2: Simulation study – simulated instantaneous $\sigma$ process for stochastic integral model, time change model and the BNS-OU-gamma model, respectively (Panels a,b,c), $v^2$ in the stochastic integral model (d) and $\xi_t$ process for the time change model (e).

Figure 3: Simulation study – Autocorrelation function of process $V$ and of absolute returns for different model specifications.

Figure 4: Simulation study – Simulations of stochastic integrand model with different values of $\lambda^{(v)}$. 
5. Conclusions and Possible Extensions

VMOU processes constitute a natural extension of BNS random volatility models. The flexibility of the framework and the ability to describe stylized facts of financial markets, make them promising modeling tools. In this work we present the main features of the framework, describing how the stochastic volatility of volatility component is introduced, and showing how to compute the structure preserving equivalent martingale measure. We then recall the Fourier transform technique by Carr and Madan (1999) and we present an implementation with BNS-OU-gamma model, calibrating the process parameters using S&P500 option prices. Finally, we present a simulation study where we visualize the features of VMOU processes, comparing them with a traditional BNS model. We conclude saying that the model shows promising features, especially in capturing the volatility clustering of the volatility itself, showing features similar to self-exciting dynamics. Further investigations are required to implement Fourier option pricing with this model and to test the performances on real data, especially at intra-day frequency.

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