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Long-term individual financial planning under stochastic dominance constraints

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Abstract We analyse an optimal households' *asset-liability management* (ALM) problem formulated as a goal-based problem in which the family aims over an extended planning horizon at achieving an investment goal, in the form of a real-estate investment and a retirement goal at the end of the planning horizon. The problem is formulated as a *multistage stochastic program* (MSP) and we evaluate in this article the impact of *second order stochastic dominance* (SSD) constraints on different specifications of the family objective functions and with respect to three alternative benchmark policies adopted to generate the SSD constraint set. The problem is formulated as a linear stochastic program and, following Kopa et al. (2018) the SSD constraints are based on a simple permuation matrix, whose effectiveness in determining the decision maker strategies is confirmed in a case study developed in the second part of the article. We show that depending on the adopted benchmark policy, SSD feasibility even if far away on the planning horizon may influence root node decisions and affect both the adopted investment and the liability optimal policies. Interestingly, SSD feasibility, depending benchmark policy may also imply *first-oder stochastic dominance* (FSD). Finally we analyse in the article from a qualitative viewpoint the relationship between a minimum shortfall

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with respect to the goals objective and the introduced SSD constraints at the terminal horizon.

Keywords dynamic stochastic programming, stochastic dominance, asset-liability management, goal-based investing, life cycle policy, consumption-investment trade-off

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JEL classification C44, C61, C63, D15, G11, G22, G41

Introduction

We consider in this work a family's dynamic asset-liability management (ALM) problem over a long-term planning horizon formulated as a *stochastic programming* (DSP) problem with an individual objective to minimise the shortfall with respect to a terminal retirement goal, after at an intermediate stage, investing in the real estate market. Both the intermediate and terminal goals are stochastic and the decision problem is formulated relying on estimates of average salaries, living costs and of a liquidity buffer for precautionary reasons. The attainability of each goal is subject to an uncertainty generated by asset returns and liability costs and we formulate the optimization problem with a *second order stochastic dominance* (SSD) constraints. The asset universe includes mutual funds, pension funds, unit-linked contracts and annuities plus cash, while libilities are limited to living costs and fixed or floating rate mortgages for the real estate investment.

The decision maker faces a financial planning problem whose effective solution is however not easy due to its' long-term nature and the uncertainties affecting most of the economic and financial variables over such horizon. Consistently with the life-cycle hypothesis in Modigliani and Ando (1957), we assume a rational investor who wishes to preserve living and consumption standards over her/his lifetime. Retirement goals are set accordingly so to keep current consumption levels during retirement, that depending on individuals current ages may be very far out in the future: when not otherwise specified both investment and retirement goals are set in nominal terms and then inflation-adjusted. Since Modigliani and Ando (1957), this class of decision problems, at the very heart of households' economics, has naturally attracted a rather extensive and diversified interests by the scientific community resulting into several possible optimization approaches Merton (1969); Samuelson (1969); Hakansson (1974); Kahneman and Tversky (1979); Kallberg and Ziemba (1979); Mulvey and Ziemba (1998); Levy (2006); Consigli (2007); Medova et al. (2008). Indeed the problem is *dynamic* and *stochastic* since not only living costs depend on consumer price inflation, but also the labor income is random particularly over a long horizon, as are portfolio returns and borrowing costs. Upon retirement depending on the pension scheme, defined benefit or defined contribution, see Consigli et al. (2012b) and Kopa et al. (2018), longevity risk may also be beared by the individual. In this work we aim at clarifying the implications of stochastic dominance constraints on optimal dynamic goal-based investment policies as the investors' risk attitudes change within a realistic setting.

Already in previous developments Consigli and Dempster (1998); Medova et al. (2008); Consigli et al. (2011, 2012b) we **(toglierei riferimenti personali)** emphasised the potentials of DSP formulations in the area of individual asset-liability management: almost all features and modeling complexities considered above can be accommodated within a DSP model allowing for long term plannning of households life projects. Here we extend previous studies on optimal retirement plans Consigli et al. (2012b) and Kopa et al. (2018) to analyse specifically the interaction between medium and long-term targets under *stochastic dominance* (SD) constraints and for varying agents' risk attitude.

SD preferences are well studied in decision theory and they are rigorously established from a mathematical perspective in optimization theory. SD was introduced more than 50 years ago and it was firstly applied to economics and finance in Quirk and Saposnik (1962), Hadar and Russell (1969) and Hanoch and Levy (1969). Then, the SSD constraints were applied to static stochastic programs in Dentcheva and Ruszczynski (2003) and Luedtke (2008) and to portfolio efficiency analysis, see e.g. Kuosmanen (2004), Dupačová and Kopa (2012) and, more recently, Kopa and Post (2015). Similarly, the FSD constraints were used in Kuosmanen (2004), Dentcheva and Ruszczyński (2004) and Dupačová and Kopa (2014). In multistage stochastic optimization, the SSD constraints were applied to asset-liability modeling in Yang et al. (2010) and in an individual pension allocation problem in Kopa et al. (2018).

The investment-consumption problem is formulated in discrete time: we consider a data tree process with randomness affecting interest rates, investment returns, goals values, family income and living costs. The decision maker faces annual living costs and we assume that family goals are placed only at pre-specified decision points over the decision horizon. The first goal is represented by the real estate: depending on the evolution of the house market, several years from now, the investment might require different levels of borrowing: accordingly, since outstanding debt reduces the overall family wealth, the retirement goal achievement might be jeopardised. The investor wishes to generate through a dynamic investment policy a family wealth capturing both goals and *stochastically dominating* to the second order wealth processes generated by benchmark policies. We consider in this article three such kind of policies.

To clarify the article's contribution we depart from two relevant previous works: in Kopa et al. (2018) a computationally efficient extension of an individual pension problem under stochastic dominance constraints was presented with respect uniquely to a final retirement target. In Consigli et al. (2012a) we solved a household financial planning problem without any stochastic dominance constraints but as here below including a sequence of goals within a relatively standard problem formulation. Here we extend those works in several directions:

- Second order stochastic dominance **(sostituirei con SSD)** is analysed with respect to a range of possible specifications of the investor's risk profile and a specific focus on the trade-off between a retirement and a real estate goals is developed;
- The objective function is formulated as a risk-reward function in which risk is measured by the drawdown with respect to the input, inflation-adjusted targets at different time horizons;
- The SSD principle is defined with respect to a set of realistic investment benchmarks, and the resulting optimal policy **(manca il verbo)** back-tested out-of-sample against the associated policies.
- For given benchmark portfolios, we consider the implications on *first-order stochastic dominance* (FSD) relationship of imposing SSD constraints at different stages of the dynamic problem.

The main article's contribution may be found in the extensive set of evidences on the optimal investment policies and the assessment of the SD constraints effectiveness for different problems specifications. Of specific interest is the sensitivity of the root node investment and scenario-dependent liability decisions under SSD constraints and varying risk-return trade-offs over a long-term planning horizon. The inclusion of SSD constraints in an individual ALM problem with time- and state-dependent targets is novel and leads to a set of relevant financial evidences: we analyse in the case study the impact of SSD feasibility on a representative family's optimal wealth distribution at the end of a 20 year horizon. An extended set of evidences confirms that SSD conditions limit significantly the lower tail of the wealth distributions without jeopardising the optimal policy upside. The interaction between the stochastic dominance constraints and different specifications of an agent risk-reward function is also analysed in detail.

The article is structured as follows: in section 1 we analyse the goals' trade-off and introduce the key elements of an individual asset-liability management (ALM) problem from an economic perspective to clarify the rationale of a decision model based on SSD preferences. The ALM problem's formulation is developed in section 2, while in subsections 2.1 and 2.2 we introduce the optimization problem and discuss the model's mathematical implications, the scenario generation approach and the implementation details. The section 3 is devoted to a case study and an extended set of evidences to highlight the implications of the adopted modeling approach on the individuals' investment strategies and present a qualified set of comparative evidences.

1 Goal-based investments under a stochastic dominance perspective

We consider a household whose financial allocation decisions evolve over a long-time working lifespan: family planning is determined by current savings and future investment and consumption plans as reflected in a finite number of family *goals*. In sub-section 3 we consider a canonical 4-member family and analyse how an intermediate real estate and an horizon retirement goals will affect optimal investment policies over a 20-year planning horizon and to which extent leverage decisions on early targets may affect the likelihood to achieve goals later on. To clarify the modeling process and convey the issues involved by the formulation of a household *asset-liability management* (ALM) problem under stochastic dominance constraints, we introduce in this section some notation and summarize the random factors affecting this class of financial management problems. The rationale **(rationale?)** and implications of an SSD approach are also clarified.

In a DSP setting, asset and liability scenarios are assumed to be generated by an extended set of random coefficients whose evolution along a scenario tree provides the fundamental model for discrete risk evaluation within the optimization problem, see Dupačová et al. (2000); Consigli et al. (2011); Bertocchi et al. (2011). We consider a generic finite decision horizon $\mathcal{T} := \{t_0, t_1, ..., t_H\}$, with typically $t_0 = 0$, current time and $t_H = T$ retirement age, and a family wealth $W(t, \omega)$, $t \in \mathcal{T}$, that given an initial investment portfolio X_0 will evolve in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ Consigli et al. (2012a) **(spiegare la referenza)** along random trajectories ω. Two goals are considered in this study: an intermediate real estate goal $\tilde{W}^f_\tau$ with $t_0 < \tau < T, \tau \in \mathcal{T}$ based on current, time 0, market values and a terminal retirement goal $\tilde{W}^r_T$: this latter is determined relying on current living standards and a pension conversion factor consistent with economic practice. Both goals are defined at $t = 0$ relying on current information and they will evolve along random inflation and real estate revaluation paths, thus generating stochastic values at the specified times: the resulting probability distributions at τ and T respectively are relevant in the definition of the SD constraints. For $t \in \mathcal{T}$, before rebalancing, the individual's wealth is determined by the investment portfolio value X_{t_j}, by the investment goals $\tilde{W}^f_{t_j}$ acquired in $\tau < t_j$, the oustanding debt y_{t_j} and by a cash surplus z_{t_j}:

$$W(t_j, \omega) = X_{t_{j-1}} r^X_{t_j}(\omega) + \sum_{k<j} \tilde{W}^f_{t_k,t_j}(\omega) - y_{t_j}(\omega) + z_{t_j}(\omega). \tag{1}$$

In (1), $r^X_{t_j}(\omega)$ is the portfolio value random variation over the time step $[t_{j-1}, t_j]$ and $\tilde{W}^f_{t_j}(\omega)$ defines the value in t_j of investment goals acquired before t_j. The portfolio evolution can be captured through buying and selling decisions: again for $t \in \mathcal{T}$ we have $X(t_j, \omega) = X_{t_{j-1}} r^X_{t_j}(\omega) + X^+_{t_j}(\omega) - X^-_{t_j}(\omega)$ indicating with $+$ and $-$ aggregate buying and selling decisions along the random paths ω. Similarely we denote with $y^+_{t_j}(\omega)$ and $y^-_{t_j}(\omega)$ debt borrowing and reimboursement decisions taken in t_j along path ω, while $b_{t_j}(\omega)$ is the interest rate on the debt at time t_j. The cash position evolves in each stage $t \in \mathcal{T}$, trajectory ω, as:

$$\begin{aligned} z_{t_j}(\omega) = z_{t_{j-1}}(1 + r_{z,t_{j-1}}) + I_{t_j}(\omega) - C_{t_j}(\omega) - \tilde{W}^f_{t_j} \mathbb{I}^f_{t_j} + \\ + X^-_{t_j}(\omega) - X^+_{t_j}(\omega) - y_{t_{j-1}} b_{t_j}(\omega) + y^+_{t_j} - y^-_{t_j} \end{aligned} \tag{2}$$

In equation (2) we denote with I_{t_j} and C_{t_j} the family income and living costs during the period and we denote the investment goals with $\tilde{W}_t^f$ where $\mathbb{I}_t^f$ takes value 1 or 0 depending on the targets' time input to the problem. From a modeling viewpoint, a final relevant issue refers to the evaluation of the financial position at time T, end of the planning horizon, assumed here to coincide with the retirement age Consigli et al. (2012b) **(spiegare la referenza)**. Some authors consider a lifetime planning horizon Medova et al. (2008). In the case study in section 3 we consider among the assets different pension funds and the family by setting a retirement goal $\tilde{W}_T^\tau$ will determine a desirable family income upon retirement **(questa frase mi suona male)**. Buying and selling decisions on individual assets $i \in \mathcal{A}$ at time t are specified by $x_{i,t}^+$ and $x_{i,h,t}^-$, thus $X_t = \sum_{i\in\mathcal{A}} \sum_{h<t} x_{i,h,t}$, $X_t^+ = \sum_{i\in\mathcal{A}} x_{i,t}^+$ and $X_t^- = \sum_{i\in\mathcal{A}} \sum_{h<t} x_{i,h,t}^-$. Following Consigli et al. (2012b), in T for given pension fund value and prevailing interest rated we introduce a terminal adjustment, a so-called *end effect* E_T that will also determine the terminal wealth distribution: $W(T,\omega) = X_{t_{n-1}} r_T^X(\omega) + \sum_{h<T} \tilde{W}_{h,T}^f(\omega) - y_T(\omega) + z_T(\omega) + E_T(\omega)$. We go more in detail on this issue in section 2.

Let now $F_{W_t}(w) = \mathbb{P}(W_t \leq w)$ **(nelle figure la notazione delle distribuzioni cumulate e' leggeremenete diversa, in particolare la W e' ad apice, e per il benchmark non c'e' la B a pedice)** be the cumulative probability distribution of the individual wealth at time t. We wish to evaluate the SD relationships between $F_{W_t}(w)$ and the wealth dstributions generated by alternative benchmark policies: $F_{k,t}(w) = \mathbb{P}(B_{k,t} \leq w)$ is the cumulative distribution generated by the k-th benchmark portfolio at time t. Relying on normalised portfolio and benchmark values (see below) SD relationships can be easily specified with respect to the wealth distributions $F_{W_t}(w)$ and $F_{B_{k,t}}(w)$ or the associated compound return distributions as w varies. They can also be assessed with respect to the quantiles of the distributions, for $\alpha \in (0,1)$, relying on $F_{W_t}^{-1}(\alpha)$ and $F_{k,t}^{-1}(\alpha)$. Here below we consider a discrete probability space generated by wealth and benchmarks tree processes. Under such assumption, following Kopa et al. (2018), we show that indeed first and second order SD constraints can be enforced through appropriate linear operators leading respectively to a mixed integer or a linear programming problem formulation. We analyse first the economic implications of the SD constraints when applied to a dynamic framework.

Consider the real estate target-time τ: before the acquisition, let this be $\tau-$, the individual's wealth would not include any mortgage nor debt position and $W_{\tau-}(\omega) = \sum_{i\in\mathcal{A}} x_{i,\tau-}(\omega) + z_{\tau-}$. Upon acquisition we may record either or both a liability increase and an asset portfolio reduction aimed at funding the investment: the family wealth along the random path ω after the investment will be

$$W_\tau(\omega) = \sum_{i\in\mathcal{A}} \sum_{h<\tau} x_{i,h,\tau}(\omega) + \tilde{W}_\tau^f - y_\tau^+ + z_\tau,$$

where

$$\sum_{i\in\mathcal{A}}\sum_{h} x_{i,h,\tau}(\omega) = \sum_{i\in\mathcal{A}}\sum_{h<\tau-} x_{i,h,\tau-}(\omega) + \sum_{i\in\mathcal{A}} x^{+}_{i,\tau}(\omega) - \sum_{i\in\mathcal{A}}\sum_{h<\tau} x^{-}_{i,h,\tau}(\omega).$$

The two quantities $W_{\tau-}(\omega)$ and $W_{\tau}(\omega)$ differ on the wealth composition: in the latter case inclusive of the real estate investment and associated liability and, consequently, they differ in terms of liquidity as further dicussed in section 2 and they differ by composition and portfolio weights. At τ dependending on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the wealth distribution will depend on the cumulative returns generated by the investment strategy so far, by the cash balance evolution and the market value of the real estate. After the real estate investment and before retirement, assuming a retirement income funded only through pension and insurance products, thus excluding a real estate disinvestment, a relevant mortgage in τ might jeopardise the achievement of the retirement goal $\tilde{W}^r_T$ at T. Notice that leaving liquidity aside, the wealth distributions in τ with and without the real estate should be the same: accordingly we introduce a stochastic dominance relationship by comparing $W_t(\omega)$ with the wealth distribution generated by a benchmark investment portfolio. What are then appropriate benchmark strategies B_k to consider when assessing the SD relationships? We consider $k = 1, 2, 3$ strategies of increasing complexity, actually adopted in practice.

1. B_1: the initial portfolio X_0 is invested in the financial market following an equally-weighted strategy $1/N$ in which the portfolio composition does not vary over time **(solitamente quando confronto con l'1/N cito DeMiguel et al. (2007))**.
2. B_2: the initial family wealth W_0 is invested in the money market and grows over time depending on the prevailing risk-free interest rates.
3. B_3 the family wealth W_t, $t \in \mathcal{T}$ is protected from inflation.

Under all cases, cash-flows determined by family income and living costs are considered exactly as in the goal-based ALM problem formulation. The following processes are considered to define the SD relationship for $t \in \mathcal{T}$: the family wealth $W_t(\omega)$ as defined in eq. (1), based on inflation-adjusted consumption flows, goals' revaluation paths, a portfolio value $X_t(\omega)$ and cash $z_t(\omega)$ and a benchmark wealth $W^k_t(\omega)$, $k = 1, 2, 3$ without the inclusion of any target defined for $k = 1$ as

$$W^1(t_j, \omega) = X^1_{t_j}(\omega) + z^1_{t_j}(\omega). \tag{3}$$

where $X^1_{t_j}(\omega)$ is a (non-optimal) portfolio with equally weighted assets also refered to as a $1/n$ **(io userei ovunque 1/N perche' e' coerente con DeMiguel e non si confonde con la notazione nodale)** portfolio. Here the cash account is determined by current consumptions and income flows together with the liquidity generated by buying and selling decisions induced by the benchmark strategy along scenario ω: $z^1_{t_j}(\omega) = z^1_{t_{j-1}}(1 + r_{z,t_{j-1}}) + I_{t_j}(\omega) - C_{t_j}(\omega) + X^{1,+}_{t_j}(\omega) - X^{1,-}_{t_j}(\omega)$. Under such strategy on a negative market phase, nothing will prevent the individual wealth from becoming negative, as we show below.

For $k = 2, 3$ we will instead consider the cases of an initial portfolio position entirely liquidated at $t = 0$ and invested either on a money market account earning a stochastic risk-free interest rate or, respectively, in a fully inflation protected account. Under either cases given the family income and current consumptions, the benchmark portfolio will surely yield a positive return over the planning horizon. In particular:

$$W^2(t_j, \omega) = X_0 \times R_{z,t_j}(\omega) + z^2_{t_j}(\omega) \tag{4}$$

$$W^3(t_j, \omega) = X_0 \times \Pi_{t_j}(\omega) + z^3_{t_j}(\omega). \tag{5}$$

In both equations (4) and (5) the cash account will evolve from an initial cash surplus z_{t_0} according to income inflows and consumption: neither equation implies portfolio rebalancing decisions. The terms $R_{z,t_j}(\omega)$ and $\Pi_{t_j}(\omega)$ are respectively a compounded risk-free amount generated by a sequence of short interest rates r_{z,t_j} for $j = 1, 2, ...$ along scenario ω and an inflation deflator **(perche' DE-flator?)** over the period $0, t_j$ again along scenario ω, a function of the inflation process $\pi_{t_j}(\omega)$. We see below that both processes will generate non-negative wealths and are defined as mean-reverting stochastic processes Consigli et al. (2011) **(spiegare referenza)**. In the optimzation problem we require the family wealth to stochastically dominate the above benchmark wealths to **(at invece di to)** the second order.

Following Kuosmanen (2004) and Kopa et al. (2018), first and second order SD in a discrete probability space can be characterised in terms of permutation matrices P, namely we have for $t \in \mathcal{T}$:

FSD : F_W *first-order stochastically dominates* (FSD) F_k in stage $t \in \mathcal{T}$ if and only if we can find a P_1 s.t. $W_t \geq P_1 W^k_t$ where W_t and W^k_t are vectors with dimension $\mathcal{N}_t$ the height of the scenario tree in stage t and P_1 is a permutation **(matrix)** whose rows and columns sum all to 1 and whose elements $p_{1,ij} = \{0, 1\}$.

SSD : Similarly F_W *second-order stochastically dominates* (SSD) F_k in stage $t \in \mathcal{T}$ if and only if we can find a P_2 s.t. $W_t \geq P_2 W^k_t$ where W_t and W^k_t are vectors with dimension $\mathcal{N}_t$ the height of the scenario tree in stage t and P_2 is a permutation **(matrix)** whose rows and columns sum all to 1 and whose generic elements $p_{2,ij} \in [0, 1]$. We have in this case a convex combination of possible states.

The above matrix permutations can enforce first and second order SD through appropriate constraints. Our interest is on the assessment of a SSD constraint on a complex long-term financial planning problem: under any random scenario ω the wealth distribution induced by the solution of the optimization problem is compared with that associated with $W^k_t(\omega)$, $k = 1, 2, 3$, at different stages over the decision horizon.

More specifically, we analyse in the case-study the dependence of the optimal asset-liability strategy on the introduction of SSD constraints at the horizon and then at the real-estate target time: under either cases we also evaluate **(ex-post the)** FSD under alternative assumptions on the decision maker risk aversion.

2 Households asset-liability management model

We introuce a discrete time households' ALM problem, in which, given a sequence of family goals and current living standard, an optimal investment policy must be determined: the asset universe includes mutual funds, life contracts, pension funds and a money account. Relevant practical and operational standards in the European market are considered as well as actual tax and pension conversion coefficients. Concerning the retirement income generation, the focus is primarily on alternative pension schemes offered now-a-days by private institutions and typically complementary to public Tier I welfare services. The optimal problem is formulated as an expected shortfall minimization with respect to the sequence of goals Rockafellar and Uryasev (2002) Consigli (2007) **(spiegare le referenze)**.

We rely on an event tree problem formulation Dempster (1988); Birge and Louveaux (2007); Consigli and Dempster (1998); Dupačová et al. (2000) **(spiegare le referenze)** with a planning horizon $\mathcal{T}$ specified as a discrete set of decision stages. A reference period θ, prior to time 0 is introduced in the model to define holding returns to be evaluated at time 0. In addition we define a time period $\tilde{\mathcal{T}}, \tilde{\tau} \in \tilde{\mathcal{T}}$ **(non chiara la nuova definizione di tau tilde)** beyond T for cash flows generated during the retirement period. Consistently with the stochastic programming formulation, random dynamics are modeled through *tree processes* with non-recombining sample paths in an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$ Dupačová et al. (2000); Consigli et al. (2012a) **(spiegare la referenza)**. Nodes along the tree, for $t \in \mathcal{T}$, are denoted by $n \in \mathcal{N}_t$ and for $t = 0$ the root node is labeled $n = 0$. The root node is associated with the partition $\mathcal{N}_0 = \{\Omega, \emptyset\}$ corresponding to the entire probability space. Leaf nodes $n \in \mathcal{N}_T$ correspond one-to-one to the atoms $\omega \in \Omega$. For $t > 0$ every $n \in \mathcal{N}_t$ has a unique ancestor $n-$ and for $t < T$ a non-empty set of children nodes $n+$. We denote with N_t the number of nodes or height of the tree in stage t and with t_n the time period associated with node n: $t_n - t_{n-}$ will then denote the time length between node $n-$ and node n. The set of all predecessors of node n: $n-, n--, .., 0$ is denoted by $\mathcal{P}_n$. We define the probability distribution $\mathbb{P}$ on the leaf nodes of the scenario tree so that $\sum_{n \in \mathcal{N}_T} p_n = 1$. A *scenario* is a path from the root to a leaf node and represents a joint realization of the random variables along the path to the planning horizon. We shall denote by $S = N_T$ the number of scenarios or sample paths from the root node to the leafs. Holding, buying and selling decisions on the investment portfolio and borrowing strategies define the control variables of the problem. The term *investment value* will be adopted when appropriate for investment net-asset-value or (NAV). We indicate with $\mathring{x}_i$ the value of the position in asset i at $t = 0$, before any rebalancing; $x_{i,h,n}$ denotes the value in node n of holdings in asset i purchased in node h; $x^+_{i,n}$, the value of asset i bought in node n; $x^-_{i,h,n}$ is the value of asset i sold in node n which was purchased in node h **(queste quantita' erano gia' state definite a meta' di pagina 6, c'e' un po' di ridondanza)**. Accordingly, sales at the root node on positions previously held are denoted by $x^-_{i,\tau,0}$: these are the values of asset i sold in the root node 0 which was purchased at time τ. Three asset classes $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ are considered for mutual funds (MF), pension funds (PF) and life annuities (LA), respectively.

Liabilities are denoted by y_n, the debt in node n, with y^+_n to denote a liability issued in node n and y^-_n a liability reduction in node n **(queste quantita' erano gia' state definite a meta' di pagina 6, c'e' un po' di ridondanza)**. The introduction of the investment nodes h for holding or selling decisions is motivated by the dependence of capital gains and penalties on holding periods. By summing over $h \in \mathcal{P}_n$, for $n \in \mathcal{N}_t$ we derive the value $x_{i,n}$ of the investment in asset i in node n: $x_{i,n} := \sum_{h \in \mathcal{P}_n} x_{i,h,n}$. The overall portfolio value in node n is $X_n = \sum_{i \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3} x_{i,n}$. Surplus or deficit balances on the cash account are denoted respectively by z^+_n and z^-_n.

An extended set of random coefficients is required to model the assets payoff. NAV returns of asset i in node n are denoted by $r_{i,n}$. We consider a short interest rate b_n and a 10-year long rate l_n in each node n of the scenario tree, while π_n will denote the inflation rate in node n. $\zeta^{+/-}_n$ define the interest rates on positive (+) and negative (-) cash positions in node n. Specific asset-class coefficients are: $a_{i,h,n}$: **(il doppio due-punti sta un po' male)** the unit value of life annuity i in node n which was bought in node h; $\varphi_{i,h,n}$: the unit capital gain in node n on asset i which was bought in node h net of any penalty. We indicate explicitly with $p_{i,h,n}$ the percentage penalty on asset i in node n that was purchased in h **(non mi e' chiaro cosa rappresenti questa penalty)**.

Each household is characterized by a financial wealth process W_n in node n, $n \in \mathcal{N}_t$ and $t \in \mathcal{T}$, which from an initial state W_0 will evolve according to cumulative investment returns, the individual's income, inflation-adjusted living costs, intermediate consumption and investment targets Consigli et al. (2011) **(spiegare la referenza)**. We denote by C_n the family living costs, by I_n the family income in nominal terms and by $\widetilde{W}^{f,r}_n$ the investment and retirement goals, respectively, in node n **(queste quantita' erano gia' state definite a inizio di pagina 6, c'e' un po' di ridondanza)**.

Variable life annuities, $i \in \mathcal{A}_2$ **(le life annuities sono state definite a inizio pagina come** A_3**)** carry a complex payoff function which is determined by a constant annuity rate and a random annuity evolution Consigli et al. (2012b) **(spiegare la referenza)**. In the ALM model, we denote with $A_{i,h,n}$ the value paid by the annuity in node n after a holding period $t_n - t_h$. The annuity payment is determined by the lump-sum investment $x^+_{i,n}$ in node n, an annuity conversion coefficient and the variable unit annuity coefficient $a_{i,h,n}$

in each descending node $n \in n+$ Consigli et al. (2012a); Konicz et al. (2015) **(spiegare la referenza)**.

2.1 Model instance

We formulate a stochastic linear program based on a convex combination between a maximum expected wealth and a shortfall minimization problem with respect to the targets under an extended set of constraints. A problem instance is specified by introducing alternative benchmark policies for $k = 1, 2, 3$ with associated SSD relationships and cash balances. As λ varies in the objective function, we derive different convex combinations between the expected wealth and the drawdown **(Da' errore, come si scrive?)** measure. We may assume that a relatively high λ, close to 1, would characterise an investor with a low tolerance for targets under-achievement.

$$\max_{x \in X, p_{2,n,\bar{N}} \in P_2} (1-\lambda)\mathbb{E}_{\mathcal{F}_T}\left[W_n\right] - \lambda \sum_j \mathbb{E}_{\mathcal{F}_j}\left[\widetilde{W}_n^j - W_n | W_n < \widetilde{W}_n^j; n \in \mathcal{N}_{t_j}\right] \quad (6)$$

s.t for all $n \in \mathcal{N}_t$, $t \in \mathcal{T}$ almost surely:

$$W_n = \sum_{i \in \mathcal{A}} x_{i,n-}(1 + r_{i,n}) + E_n \mathbb{I}_{n|n \in \mathcal{N}_T} + z_n + \sum_{h \in \mathcal{P}_n} \tilde{W}^f_{h,n} - y_n \quad (7)$$

$$W_n \geq \sum_{\bar{n} \in \mathcal{N}_t} p_{2,n,\bar{n}} W^k_{\bar{n}} \quad (8)$$

$$1 = \sum_{\bar{n} \in \mathcal{N}_t} p_{2,n,\bar{n}} \quad (9)$$

$$x_{i,0} = \mathring{x}_i + x^+_{i,0} - x^-_{i,\tau,0}(1 + \varphi_{i,\tau,0}) \quad t = 0, \quad \forall i \in \mathcal{A} \quad (10)$$

$$x_{i,h,n} = x_{i,h,n-}\left(1 + r_{i,n}\right) - A_{i,h,n} - x^-_{i,h,n} \quad \forall i \in \mathcal{A}, \quad h \in \mathcal{P}_n \quad (11)$$

$$x_{i,n} = x^+_{i,n} + \sum_{h \in \mathcal{P}_n} x_{i,h,n} \quad \forall i \in \mathcal{A} \quad (12)$$

$$z_0 = \mathring{z}^+ + \sum_{i \in \mathcal{A}} \sum_{\tau < 0} x^-_{i,\tau,0} - \sum_{i \in \mathcal{A}} x^+_{i,0} \quad t = 0 \quad (13)$$

$$z_n = I_n - C_n + z^+_{n-}\left(1 + r_{z,n-}\right) - z^-_{n-}\left(1 + r_{z,n-}\right) + \sum_{i \in \mathcal{A}_3} \sum_{h \in \mathcal{P}_n} A_{i,h,n} \quad (14)$$
$$+ \sum_{i \in \mathcal{A}} \sum_{h \in \mathcal{P}_n} x^-_{i,h,n} - \sum_{i \in \mathcal{A}} x^+_{i,n} + y^+_n - y_{n-} b_n - y^-_n - \widetilde{W}^f_n \mathbb{I}^f_{n|n \in \mathcal{N}_{t_j}}$$

$$L_i X_n \leq \sum_{h \in \mathcal{P}_n} x_{i,h,n} \leq U_i X_n \quad \forall i \in \mathcal{A} \quad (15)$$

$$0 \leq y_n \leq \gamma \widetilde{W}^f_n \quad (16)$$

The above problem statement (6) to (16) follows the model rationale already introduced in Consigli et al. (2011): what is new in the present formulation comes from the specification of the SSD constraints and the benchmark wealths in (8) with respect to one among (3), (4), and (5). According to (9) and (9) **(c'e' due volte la stessa referenza)**, P_2 is a double stochastic matrix whose dimension depends on the stage t and the height of the scenario tree at that stage. We denote with $p_{2,n,\bar{n}}$ the generic element of the permutation matrix P_2 associating W_n with $W^k_{\bar{n}}$, where $0 \leq p_{2,n,\bar{n}} \leq 1$ and $\sum_{n\in\mathcal{N}_t} p_{2,n,\bar{n}} = \sum_{\bar{n}\in\mathcal{N}_t} p_{2,n,\bar{n}} = 1$ **(io sostituirei direttamente queste nel modello al posto della (9))**.

Depending on the stage t, $n \in \mathcal{N}_t$ may be very large: constraint (8) requires indeed the definition of a square matrix P_2 whose generic elements $p_{2,n,\bar{n}}$ are decision variables having cardinality equal to the square of the number of nodes at the SSD stage t, i.e. N_t^2, while the number of additional linear constraints is $3N_t + N_t^2$. For instance, in a case study with 512 scenarios, to impose the SSD condition on the last stage implies the definitions of 262 144 variables and of 263 680 linear constraints. SSD feasibility on several stages and for increasing number of scenarios, thus, has relevant computational implications due to the dimension of the resulting stochastic program.

The conditional expectation in (6) is defined as $\frac{\sum_{n\in\mathcal{N}_t:W_n<\widetilde{W}_n} p_n(\widetilde{W}_n - W_n)}{\sum_{n\in\mathcal{N}_t:W_n<\widetilde{W}_n} p_n}$, with p_n to denote the probability of the scenario passing through $n \in \mathcal{N}_t$. The terminal wealth W_n with $n \in \mathcal{N}_T$ will include the terminal portfolio, the end-effect E_n and cash surpluses minus the liability y_n. For $t < T, n \in \mathcal{N}_t$ the same equation for the financial wealth applies with however99 $E_n = 0$ **(non e' chiaro il "with however99")**.

At the horizon both investments in variable annuities and pension funds will generate cash-flows beyond the decision horizon. A discrepancy may then arise between the terminal NAV of pension funds' and annuities' and the value of their discounted cash-flows. As proposed already in Consigli et al. (2012b) such difference is captured by the end effect E_n **(ma non abbiamo detto a termine della sezione precedente che E_n = 0?)**. At each leaf node n, for given 10 year interest rates l_n, we compute the discount factors $e^{-l_n(\tilde{\tau}-T)}$ for payments to be received over $\tilde{\mathcal{T}}$. To take such adjustment into account we consider the following *end effect for cash-flows beyond the horizon*:

$$\begin{aligned} E_n &= \sum_{i\in\mathcal{A}_1\cup\mathcal{A}_3} x_{i,n}\left(\sum_{\tilde{\tau}\in\tilde{\mathcal{T}}} c_{i,z}\psi_{i,\tilde{\tau},z}e^{-l_n(\tilde{\tau}-T)}\right) - \sum_{i\in\mathcal{A}_1\cup\mathcal{A}_3} x_{i,n} \\ &= \sum_{i\in\mathcal{A}_1\cup\mathcal{A}_3} x_{i,n} f_{i,n,z} \end{aligned} \tag{17}$$

(perche' A_2 non c'e'? attenzione che la life annuity e' A_3) In equation (17) for given terminal portfolio $\sum_{h\in\mathcal{P}_n} x_{i,h,n}$ **(questa sommatoria su h non c'e' nella (17))** we define a constant pension payment over the retirement period by multiplying such value for the conversion coefficients $c_{i,z}$:

these are the conversion rates appropriate for positions held in asset i by an individual in pension class z. The $\sum_{i\in\mathcal{A}_1\cup\mathcal{A}_3} x_{i,n}c_{i,z}$ **(perche' A_2 non c'e'? attenzione che la life annuity e' A_3)** represents the constant annuity associated with the retirement vehicle i held by individual z on the pension portfolio in the leaf node n. The end-effect coefficients $f_{i,n,z}$ are defined as $\left(\sum_{\tilde{\tau}\in\tilde{\mathcal{T}}} c_{i,z}\psi_{i,\tilde{\tau},z}e^{-l_n(\tilde{\tau}-T)} - 1\right)$: they express the difference at the horizon (here coinciding with the age of retirement) between the discounted value of all future pension payments and the current portfolio NAV.

The set of constraints (10) and (12) define the inventory balance equations. The first stage decision, or root-node decision also referred to as the *implementable* decision of the MSP problem, is the only one under complete uncertainty regarding the markets' future evolution. We consider an initial investment $\mathring{x}_i$ **(gia' detto a fine pag 9)** in each asset $i \in \mathcal{A}_k, k = 1, 2, 3$ prior to rebalancing. Depending on the holding period τ selling decisions on such portfolio may generate a capital gain $\varphi_{i,\tau,0}$. $x_{i,0}$ defines the NAV of portfolio holdings in asset i after rebalancing.

For $i \in \mathcal{A}_2$ **(attenzione che la life annuity e' A_3 e questa cosa della complex payoff era gia' stata detta a meta' di pag 10)** annuities carry a complex payoff function determined by a constant annuity rate and a random annuity evolution. The annuity payment is determined by a lump-sum investment $x^+_{i,n}$ in node n, by the conversion coefficient and by the annuity coefficient $a_{i,h,n}$ in each descending node. The value $A_{i,h,n}$ paid by a VLA **(acronimo non definito)** bought in node h and held in node n, as explained in appendix, will depend on the initial investment $x^+_{i,h}$ and the revaluation path over the $t_n - t_h$ period.

The cash balance constraints (15) allow the tracking of all cash inflows and outflows at each stage. At time 0 we assume a cash surplus $\mathring{x}^+_1$ before rebalancing which is then affected by investment and selling decisions from the input portfolio at $n = 0$. On subsequent stages for $n \in \mathcal{N}_t, t > 0$ we consider cash flows generated in each node by the individual income I_n, the consumption C_n, interest inflows on positive cash positions (rate ζ^+_n net of the tax on interest payments $\vartheta_1 = 27\%$), negative interests (rate ζ^-_n) and annuities $A_{i,h,n}$. Rebalancing decisions $x^-_{i,h,n}$ and $x^+_{i,n}$ also generate cash flows as well as liability increases and reductions. The real estate target is denoted by $\widetilde{W}^f_{h,n}$ to express the nodal value in n of a target acquired in node h.

Finally, policy and maximum borrowing constraints (15) and (16) are typically problem dependent and include lower and upper bounds on investment decisions, respectively L_i and U_i, and a maximum liability, here defined in relative terms as function of the target wealth. At the horizon, for $n \in \mathcal{N}_T$, to avoid anticipative strategies, no decisions are allowed: $x^+_{i,n} = x^-_{i,h,n} = 0$ and $y^+_n = y^-_n = 0$.

The minimization of the objective value in equation (6) under the constraints (10) to (15) **(mancano vincoli (7) (8) (9) e (16))** is the output of the solution algorithm. The above mathematical instance is first expressed in algebraic form (GAMS 23.2), then the inclusion of the scenario coefficients for each equation will lead to the definition of a large scale stochastic linear program in MPS format solved with a commercial solver (CPLEX 12.1) Consigli and Dempster (1998); Consigli et al. (2011) (e.g. Cplex LP) **(spiegare referenze)**. We summarise next the elements of the adopted stochastic model for portfolio returns and interest rates and the associated tree generation procedure.

2.2 Scenario generation and stochastic dominance

(nel titolo c'e' stochastic dominance ma poi non se ne parla, si potrebbe invece mettere benchmark definitions)

Problem (6) to (15) requires the specification of the tree process of asset returns, liability costs, money market account and benchmark wealths. We follow the stochastic model presented in Consigli et al. (2011): specifically the short- and long-term interest rates and the inflation process are modeled as mean-reverting correlated Cox-Ingersol-Ross (CIR) processes Cox et al. (1985).

In (18) for $j = 1, 2, 3$ we assume for the short rate $r_{z,n} = \omega_n^1$, the long rate $l_n = \omega_n^2$ and inflation $\pi_n = \omega_n^3$. The coefficients $\alpha^j, \omega^{j,*}$ and σ^j denote respectively the mean reversion coefficient, the long term equilibrium values and the standard deviations for each process, whereas $t_n - t_{n-}$ defines the time increment between the nodes $n-$ and n. Correlation is introduced directly on the realizations e_n^r of three standard normal variables via the Choleski elements $c_{j,r}$ of the correlation matrix. Given the initial states $\omega^j(0) = \omega_0^j$ for $t \in \mathcal{T}$, $n \in \mathcal{N}_t$ we have:

$$\omega_n^j = \omega_{n-}^j + \alpha^j(\omega^{j,*} - \omega_{n-}^j)(t_n - t_{n-}) + \sigma^j \sqrt{\omega_{n-}^j}\sqrt{t_n - t_{n-}} \sum_{r=1,2,3} c_{j,r} e_n^r \quad (18)$$

The coefficients of the CIR processes are estimated through the method of moments. In Table 1 we report the estimated model coefficients adopted in the case study.

	mean reversion	long term rate	CIR volatility	long term Choleski matrix		
short rate	0,103	1,90%	4,57%	1		
long rate	0,158	1,33%	4,28%	0,888776	0,458355	
inflation	0,110	1,70%	4,33%	0,667990	0,058399	0,741794

Table 1: Interest rate model and inflation rate coefficients, monthly data 1/1/2000-31/12/2016

These processes will determine the asset returns according to a hierarchical structure in which fixed income and money market returns are derived according to a duration-convexity approximation and equity returns, require in addition the specification of a risk premium process. For the detailed model instance, we refer to Consigli et al. (2011).

For given initial conditions and input tree structure, the return coefficients $r_{i,n}$ along the tree are derived through Monte Carlo (MC) simulations linking each node to the descending nodes and spanning all the event tree. Scenario generation will here below be extended to infer the benchmark wealth distributions, an input to the optimization problem. The procedure can be sketched as follows: **(i λ nell'algoritmo NON sono definiti. Che differenza c'e' tra DERIVE e COMPUTE? Ho eliminato W_t che NON e' calcolato dall'algoritmo e sostituito con T per coerenza con la Fig. 1)**

Return scenario generation algorithm

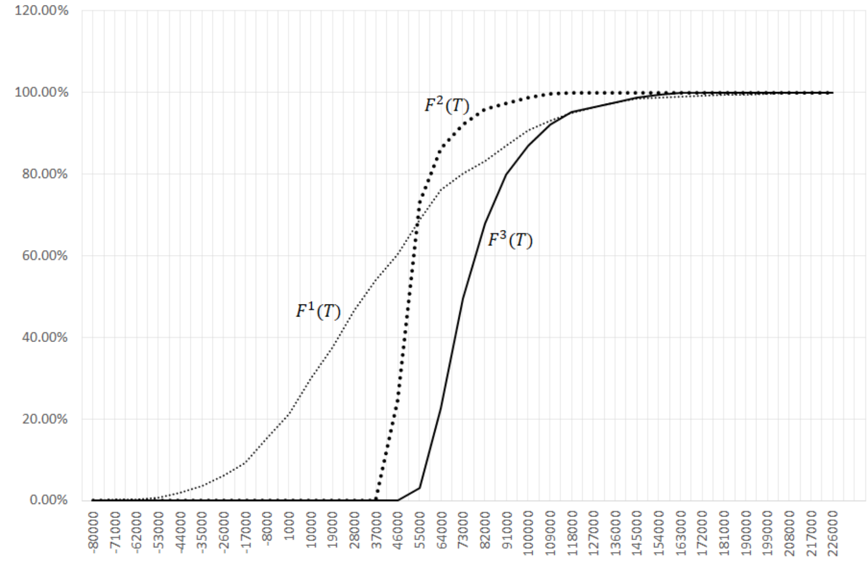
```
given all starting conditions and coefficients:
r_{z,0}, l_0, π_0, X_{0−}, D^j, C^j and σ^j, λ^j_0, β^j_0, β^j_1, β^j_2, β^j_3, w_{ij}
  for t ∈ T
    for n ∈ N_t
        generate CIR scenarios r_{z,n}, l_n, π_n
        for j = 1, 2
            generate the bond indices B^j_n
        for j = 3
            derive the equity premium λ_{j,n}
            compute the random drift μ_{j,n} = r_{z,n} + σ^j λ^j_n
            generate the equity value B^j_n
        end for j
        compute r_{i,n} = Σ_j w_{ij} r(B^j_n)
        for k = 1, 2, 3
            determine the benchmark processes: W^k_n
        end for k
    end for n
  end for t
  for k = 1, 2, 3
    derive the benchmarks' distribution: F_k(T)
  end for k
```

For every investment opportunity $i \in \mathcal{A}_k, k = 1, 2, 3$ the algorithm will generate the set of nodal returns needed by the ALM model specification. **(non ho trovato dove si spiega il passaggio da B_n a $\mathcal{A}_k$)** The short rate $r_{z,n}$ provides also a basis for the mortgage rate and the interest rates on the cash account $z_n^{+/-}$. Additional random coefficients in the model are defined by the unit annuity $a_{i,h,n}$ in node n of asset i that was purchased in node $h \in \mathcal{P}_n$ and by the end-effect coefficients $f_{i,n,z}$ introduced in eq. (17).

Once the tree processes for the asset returns, the risk-free money market account and the inflation have been generated, it is possible to derive the tree processes for the benchmark returns for $k = 1, 2, 3$. Fig. 1 shows the associated wealth distributions at the end of the planning horizon.

A $1/n$ **(modificherei con 1/N)** strategy, by enforcing a stage-by-stage rebalancing to preserve equal weights across all assets, will surely limit the portfolio value dispersion around the mean, but it won't necessarily avoid negative returns as shown in Fig. 1. The other benchmarks will generate only positive returns: the inflation-adjusted wealth dynamics is the one that over the 20 years is expected to yield the higher returns. For a given input scenario

Fig. 1: Benchmark wealths distributions at the end of the planning horizon. k=1 for 1/N pflio, k=2 for risk-free pflio, k=3 for inflation-adjusted pflio.



tree, we will consider in the case study an extended set of evidences related to different benchmark strategies and optimization problems' specifications.

3 Case-study: goal-based investing and stochastic dominance

We consider a 512 scenario tree spanning a 20 year horizon with the following branching structure $4^2 - 2^5$. The results consider a first stage decision based on a scenario tree spanning the period from January 1, 2017 to 31-Dec-2036.

We refer to Consigli et al. (2015) and references therein for a summary on the steps from the stochastic program formulation to its solution and the associated output analysis. In this project we rely on a Matlab 7.4 and GAMS 23.2 interface for scenario generation, alegbraic model formulation and MPS file generation for a large scale deterministic equivalent problem Birge and Louveaux (2007) **(spiegare referenza)**.

The MPS file input to CPLEX dual solver due to the high number of scenarios is very large-scale with 155 549 rows and 461 601 columns and 1 369 455 non-zero coefficients: the generation of the MPS file and the technology matrix reduction before solution requires on a Toshiba labtop with Intel(Core) processor of 2.60GHz and 12 GB of RAM roughly 65 minutes of CPU time while the solution algorithm took roughly 3 minutes of CPU time to solve one problem instance. In this case study we analyse the impact of the SSD constraints on the optimal asset-liability strategy for different $\lambda = \{0.25, 0.5, 0.75\}$ as the benchmark portfolio changes from $k = 1, 2, 3$. In what follows one problem instance, for brevity, will be referred to as $\mathcal{L}(\lambda, k, \tau)$ to span: (i) different values of λ in the objective function (6), (ii) the benchmark policies adopted for SSD feasibility and (iii) the decision stage τ in which the SSD constraint is imposed: we consider below two cases $\tau = 10, 20$ years where the real estate and retirement targets are set. We denote with $F_k(W^k_{\lambda,\tau})$ and $F_W(W_{\lambda,\tau})$ **(la notazione non mi sembra coerente. Per la ricchezza cumulata benchmark io userei solo $F_k(t)$, per la ricchezza cumulata della ricchezza ottima userei $F_W(\lambda, k, t)$ come quella che usiamo nelle figure 5 e 6)** the associated benchmark and optimal wealth cumulative distributions. On a given support their difference is denoted by $\delta^k_{\lambda,\tau}(w) = F_W(W_{\lambda,\tau}(w)) - F_k(W^k_{\lambda,\tau}(w)) = \mathbb{P}(W_{\lambda,\tau} \leq w) - \mathbb{P}(W^k_{\lambda,\tau} \leq w)$.

We consider the following evidences in this case study:

- The overall impact of SSD constraints on the problem solution and the individuals wealth dynamics;
- The sensitivity of the optimal asset-liability strategy to SSD constraints;
- The implication on first order SD **(modificherei con FSD)** of SSD constraints;
- The trade-off between intermediate and terminal goals under SSD constraints;

The following goals will determine the optimal financial plan:

- *Investment goals*: a 150 000 Eur real estate investment in 10 years time. This will imply a leverage and the selling of financial investments, according to a share to be determined by the optimiser and it will lead to an increase of the family wealth. In (16) we allow with $\gamma = 0.7$ a maximum liability worth 70% the current real estate value.
- *Retirement goal*: the family wishes to preserve the living standard during retirement and to this aim a projected final goal at the horizon is set in nominal terms to 750 000 Eur: such value will generate a stochastic inflation-adjusted pension target at the horizon, from which an annual pension of $750\,000 \times 0.04 = 30\,000$ would be generated in nominal terms at current pension conversion coefficient: such annuity will however worth more when received under a no-inflation very low interest rate scenario, while it would be discouraged under a high interest rate scenario. To take the issue into account as mentioned above we include at the horizon an adjustment.

Our representative decision maker and family member faces annual costs currently around 30 000 Euros and relies on an income of 50 000 Euros: for precautionary reasons a lower bound of 15 000 Euros is set by the decision maker to be held in cash. Finally at $t = 0$ the family may rely on an initial financial portfolio worth 100 000 Euros. Over time, consumption costs, goals and income will grow according to the prevailing inflation rates.

Table 2: Stochastic program solutions $\mathcal{L}(\lambda, k, \tau)$: scenario proportions, min, average and max positive exceedances of the wealth distributions relative to the benchmark distributions, scenario-wise, in Euros.

	$\tau = 20$			$\tau = 10$		
w.r.t.	W^1	W^2	W^3	W^1	W^2	W^3
W(0,25;1)	100.0%	99.6%	91.4%	100.0%	100.0%	94.5%
min	0.00	-1.66	-234.74	5157.976	0	-107.838
mean	89233.28	72248.93	46085.02	55760.18	42568.87	33645.857
max	545066.26	459566.40	369354.67	134508.6	156095	140231.18
W(0,50;1)	100.0%	99.2%	87.3%	100.0%	100.0%	96.1%
min	0	-3.42	-371.937	12636.21	0	-42.836
mean	65930.75	50457.97	26662.5645	54688.01	41649.85	32379.107
max	279779.2	361226.6	307328.419	134601.6	155357	140288.12
W(0,75;1)	100.0%	99.0%	85.5%	100.0%	100.0%	96.1%
min	0	-8.882	-352.168	12636.21	0	-69.588
mean	65330.35	47396.75	24857.1969	54557.83	41645.8	32373.407
max	279979.4	361226.6	307750.535	138055.5	157297.9	140231.84

3.1 Benchmark strategies, SSD feasibility and optimal wealth

Consider the two cases of SSD constraints set at $\tau = 20$ and $\tau = 10$ years and the corresponding wealth distributions, those associated with: (i) the benchmark strategies and the optimal AL **(abbiamo definito AL o solo ALM?)** strategies (ii) under SSD constraints and (iii) without SSD constraints. In Tables 2 and 3 we present a set of evidences associated with the above three distributions as $\lambda = 0.25, 0.5, 0.75$. We check the impact of the SSD constraints and the adopted benchmark strategies on the terminal and intermediate wealth distributions as the family objectives penalise increasingly the shortfall with respect to the terminal retirement target.

Consider first Table 2: we report evidences on the wealth distribution generated at $t = 20$ and $t = 10$ by the optimal policies as λ increases, when the SSD constraints are associated with the benchmark wealths W^k, $k = 1, 2, 3$: the percentage values represent the proportion of scenarios in which $W(\lambda)$ exceeds W^k, immediately below the minimal, average and maximal differences between the two quantities scenario-wise. At the end of the planning horizons the values $W(\lambda)$ **(qui manca l'1 nella parentesi)** are computed before retirement and can thus be compared with the retirement goals. The flag 1 in $W(\lambda, 1)$ refers to active SSD constraints **(perche' invece di usare l'1 non inseriamo direttamente il benchmark di riferimento con cui stata calcolata e mettiamo il "noSSD" se invece non c'e' vincolo attivo?)**, needed in Table 3. Every pair $W(\lambda, 1) - W^k$ identifies a problem solution with active SSD constraints on the associated benchmark **(non mi e' chiaro che benchmark e' considerato quando l'SSD e' attivo. Cioe', $W(\lambda, 1)$ specifica il λ e che l'SSD e' attivo ma non rispetto a quale benchmark. Dato che nei confronti dei benchmark W^2 e W^3 ci sono valori negativi significa che stata utilizzata la ricchezza ottima ottenuta avendo come benchmark W^1...)**.

The table can be read columnwise taking into account the first benchmark distribution for W^1 generated by the $1/n$ **(modificherei con 1/N)** strategy and then the distributions of W^2 and W^3 generated by the money market and the inflation adjusted strategies, first at the end of the investment horizon and then at the intermediate stage. The benchmark policy $k = 1$ is liable to generate at the horizon negative wealth values, depending on the introduced market scenarios. On the other hand, for $k = 2, 3$ all wealth scenarios are bounded to be positive and above the initial wealth level of the family.

For $k = 1, 2$ we see that the associated SSD-feasible strategy do actually lead, scenario-wise, to an almost sure positive excess of the family wealth relative to the benchmark wealths at both the 20 and 10 year horizons. As λ increases from 0.25 to 0.75 we also see that, according to the *min*, *mean* and *max* statistics, the associated wealth distributions shift slightly to the left: this is true in particular for the terminal wealth distributions while the intermediate wealth distributions are relatively insensitive to the λ's. According to the fourth and seventh columns, with SSD constraints on W^3 at $\tau = 20$ and 10, SSD-feasibility still leads on average to positive $\delta^3_W(\lambda, \tau)$ and with very low negative exceedances when they occur. The key evidence is that $W^3(\omega)$ generates the most demanding benchmark policy and when $\lambda = 0,75$ at the horizon the family is left on average with a low income and in the best case scenario with less than half the retirement target. Similar evidences can be taken at $t = 10$ when the SSD constraints are set at that stage.

To further evaluate the impact of the SSD constraints on the wealth distributions we consider in Table 3 the evidences collected when solving the stochastic program (6)-(16) with and without SSD constraints: in the first case we took $k = 3$.

We denote with $W(\lambda, 1)$ and $W(\lambda, 0)$ wealth values associated with optimal solutions with and without SSD constraints at the given stages, respectively. The table can be read again column-wise first on the terminal horizon, $\tau = 20$, and then at the 10 year horizon. Every pair $W(\lambda, 1)$-$W(\lambda, 0)$ is generated by the solution of two stochastic programs: we present the share of scenarios in which $W(\lambda, 1) \geq W(\lambda, 0)$ and immediately below the minimum, average and maximum excesses.

When $\tau = 20$, we see that as λ increases, with or without stochastic dominance constraints the wealth distributions shift to the left. When $\lambda = 0.25$ and 0.50 the introduction of SSD constraints does not penalise the wealth distribution while when $\lambda = 0.75$ the SSD constraints lead to a left shift which generates mostly negative differences. At the intermediate stage, however, when removing the SSD constraints, independently of λ, the intermediate wealth distributions shift significantly to the right with negative mean values on all problems comparisons. Notice that of specific rilevance are on Table 3 the figures on the sub-tables diagonals, for same τ and λ with and without SSD. The other figures are left for further evidence. According to the figures, we can summarise that the presence of SSD constraints generates a significant penalty on the family wealth when $\lambda = 0.75$ and a less relevant one when

Table 3: Comparison of wealth scenarios SSD constraints, for $k = 3$, and without. Percentages of positive exceedances of $W(\lambda, 1)$ over $W(\lambda, 0)$, at $\tau = 20$ and 10 and min, mean and max differences in Euros.

	$\tau = 20$			$\tau = 10$		
	W(0,25;0)	W(0,5;0)	W(0,75;0)	W(0,25;0)	W(0,5;0)	W(0,75;0)
W(0,25;1)	100.0%	97.7%	25.8%	16.4%	16.4%	16.4%
min	56823.02	-25552.43	-232268.54	-33830.02	-29766.33	-33830.02
mean	87538.00	21558.44	-1695.28	-6482.392	-5410.222	-6482.257
max	312797.72	156439.83	127429.88	49612.69	51600.91	49612.691
W(0,50;1)	100.0%	37.9%	25.8%	16.4%	16.4%	16.4%
min	43483.38	-92650.87	-294294.795	-44875.3	-29880.39	-44875.3
mean	68115.55	2135.986	-21117.7368	-7749.141	-6676.971	-7749.006
max	259156.8	156439.8	127429.878	49612.69	51600.91	49612.691
W(0,75;1)	100.0%	37.9%	25.8%	16.4%	17.2%	16.4%
min	39925.53	-89825.64	-293872.679	-43892.77	-34937.41	-43892.77
mean	66310.18	330.6182	-22923.1044	-7754.842	-6682.672	-7754.707
max	259578.7	156439.8	127429.878	49612.69	51600.91	49612.691

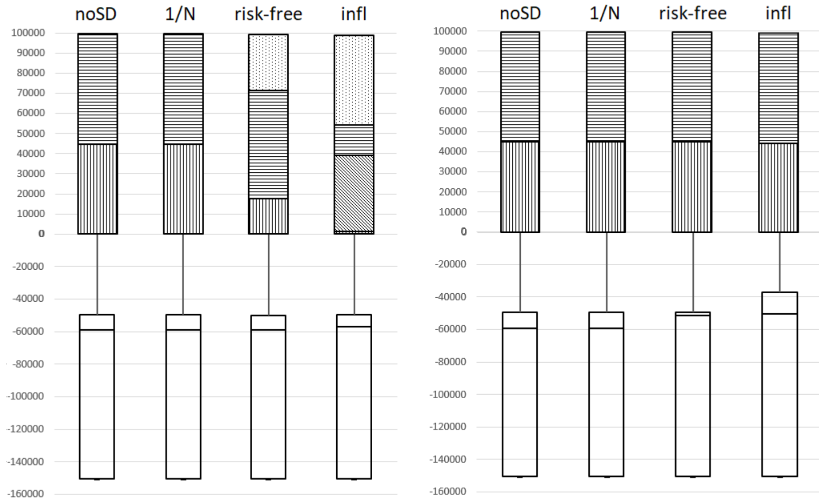
$\lambda = 0.50$ leading to a positive shortfall with respect to the terminal retirement goal.

3.2 Optimal asset-liability strategy under SSD constraints

We report next the collected evidences in terms of optimal asset-liability strategy associated with the solution of programs $\mathcal{L}(\lambda, k, \tau)$. We study the impact on the root node decision and the liability decision at the intermediate stage of the SSD constraints initially at $\tau = 20$ years and then receding to $\tau = 10$. The evidences are again compared also with the optimal strategy that would have been generated without SSD constraints. Notice that the liability is limited to the mortgage that would be generated by the real estate target. Under the different benchmark strategies we check whether the root node decision is affected or not by SSD constraints far down the planning horizon and specifically in correspondence with the targets' stages, **(qui c'e' una virgola, non so se volevi aggiungere altro)**

Fig. 2 shows on the left that SSD constraints set at stage 6, after 10 years, do actually influence the root node investment allocation but negligibly the mortgage decision, why it is not so when set at the end of the planning horizon: the optimal root node portfolio remains the same under any benchmark while in this case a moderate impact is on the liability. Interestingly the total removal of SSD constraints does generate the same optimal allocation as the one with SSD on benchmark strategy 1. We see below in Fig. 3 and Fig. 4 that as λ increases when $\tau = 10$ the very same sensitivity of the here and now decision to SSD constraints on the benchmark policies is recorded while the borrowing decision changes quite dramatically and it reaches its upper bound on all scenarios. Interestingly however when $\tau = 20$ in both figures below the here and now asset allocation is sensitive to the SSD constraints on benchmark 3 and the stage 6 borrowing decision is in this case scenario dependent.

Fig. 2: $\mathcal{L}(0, 25, k, \tau)$: H&N allocations and liability strategy. Top graphs are the H&N allocations where vertical-line represents the bank account (i1), horizontal-line the unit-linked monetary (i8), dots unit-linked (i9) and obliques-line the unit link bond government (i6). Bottom plots show the stage 6-year 10 borrowing strategy in a debt boxplot: the minimum value, the first, second and third quartiles and the maximum value. Left the optimal decisions when SSD constraints are at $t = 10$, right at $t = 20$.

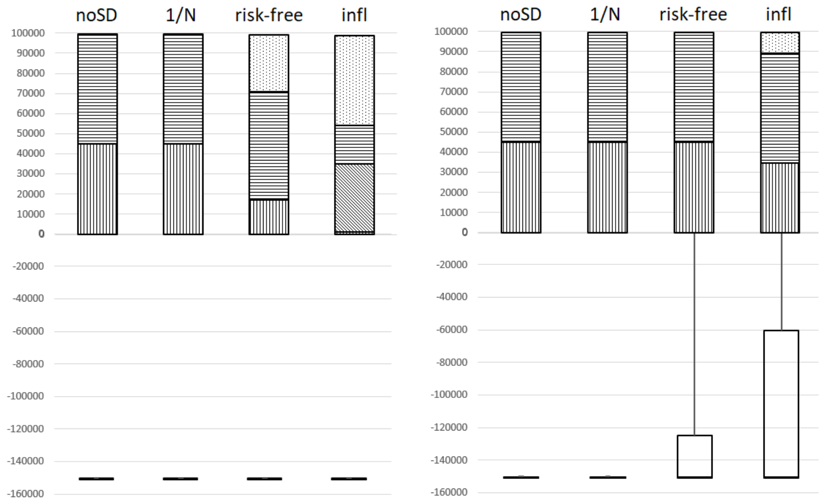


When $\lambda = 0.75$ similar remarks on the optimal root node asset allocation and borrowing strategy apply.

We may summarise the collected evidences on the root node asset allocation and mortage decisions by saying that their sensitivity to changing λ is limited while they are affected by SSD constraints even when set at the end of the planning horizon as the benchmark policy becomes more demanding, as in the case of an inflation-adjusted wealth.

(Qui secondo me il commento da fare e' che i vincoli SSD risultano chiaramente pi stringenti quando il vincolo e' allo stadio 6, ma talvolta anche quando e' allo stadio 8. In ogni caso, quando la dominanza si fa' sentire, questo si traduce in una maggiore diversificazione e questa conseguenza e' stata evidenziata in tutti i lavori che hanno a che fare con la SD: dominanza stringente $\Rightarrow$ maggiore diversificazione. Inoltre, e' per me molto interessante che la dominanza induca di fatto una posizione diversificata ma anche pi conservativa (per tenere sotto controllo la coda di sx) e questo fa si che l'evoluzione della ricchezza sia pi bilanciata e induca spesso una minore esposizione alle liabilities: perdo meno $\Rightarrow$ ho piu' ricchezza $\Rightarrow$ chiedo meno mutuo.)

Fig. 3: $\mathcal{L}(0, 50, k, \tau)$: H&N allocations and liability strategy. Top graphs are the H&N allocations where vertical-line represents the bank account (i1), horizontal-line the unit-linked monetary (i8), dots unit-linked (i9) and obliques-line the unit link bond government (i6). Bottom plots show the stage 6-year 10 borrowing strategy in a debt boxplot: the minimum value, the first, second and third quartiles and the maximum value. Left the optimal decisions when SSD constraints are at $t = 10$, right at $t = 20$.



3.3 Second and first order stochastic dominance

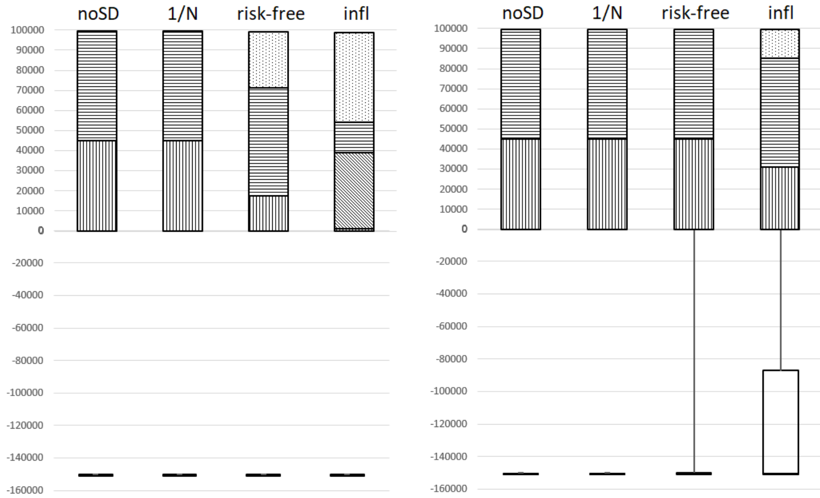
Following up from Tables 2 and 3, we consider next more explicitly the implications of SSD constraints on first- and second-order SD relationships at different stages: we focus again on the target stages, $\tau = 10, 20$. We address the following questions:

- Depending on the benchmark policy, $k = 1, 2, 3$ and the risk coefficient λ, when imposing an SSD constraint on stage t would that influence the SD relationships in prior or subsequent stages?
- Furthermore: under which conditions would that also imply FSD with respect to the benchmark policies?

From above, the answers are to a large extent benchmark-policy-dependent and we see that in the given context, very often SSD carries also the FSD property, particularly when the benchmark wealth distribution is not particularly challenging, as for $k = 1$. We consider on the left the evidences collected when setting the SSD constrains on stage 6, at the 10 year horizon. On the right the stage 8, year 20 case study. We see from both tables that SSD constraints influence if any SD relations at current and previous stages and that, on the same scenario set, depending on the benchmark policy very often SSD also implies FSD.

We summarise the evidences by considering each benchmark policy at the time:

Fig. 4: $\mathcal{L}(0,75,k,\tau)$: H&N allocations and liability strategy. Top graphs are the H&N allocations where vertical-line represents the bank account (i1), horizontal-line the unit-linked monetary (i8), dots unit-linked (i9) and obliques-line the unit link bond government (i6). Bottom plots show the stage 6-year 10 borrowing strategy in a debt boxplot: the minimum value, the first, second and third quartiles and the maximum value. Left the optimal decisions when SSD constraints are at $t = 10$, right at $t = 20$.



$k = 1$ For any λ SSD with respect to the $1/N$ strategy implies FSD at the same stage when $\tau = 10$ and jointly SSD and FSD in previous stages when $\tau = 20$;

$k = 2$ Under any λ SSD implies FSD on the same stage when $\tau = 10$ while it is not necessarily so **(sostituirei con "it's not the case")** when $\tau = 20$ even if very limited FSD infeasibilities. On the other hand when set at the terminal stage SSD feasibility holds only at that stage and not before, nor does FSD.

$k = 3$ Finally in this case SSD feasibility implies only in one case FSD feasibility at the same stage, which is when $\lambda = 0,75$ while it doesn't have any implication neither on SSD nor on FSD conditions in previous or subsequent stages when $\lambda \neq 0,75$ or $\tau = 20$ **(attenzione! io non vedo questo caso, a me sembra che con k=3 non ci sia mai FSD)**.

To summarise, when the benchmark policy is challenging then SSD feasibility is problem specific and likely to affect the optimal asset-liability strategy since the first stage and throughout the planning horizon when set either at stage 6 or 8. It is novel and interesting the impact of SSD constraints on the optimal asset-liability strategy as partially witnessed by the Fig. 3 and 3 in particular. We complete this case-study by analysing the relationship between the SSD constraints and the real estate and retirement goals: *a-priori* we may expect the constraint when particularly strict, by reducing the feasibility region to yield a sufficient wealth to achieve the real estate target with minimal

Table 4: SD relations when imposing the SSD constraint on stage 6

Lambda	Benchmark	6 stage	8 stage
0.25	1/N	SSD	noSSD
		FSD	noFSD
	Risk-free	SSD	noSSD
		FSD	noFSD
	Inflation	SSD	noSSD
		noFSD	noFSD
0.50	1/N	SSD	noSSD
		FSD	noFSD
	Risk-free	SSD	noSSD
		FSD	noFSD
	Inflation	SSD	noSSD
		noFSD	noFSD
0.75	1/N	SSD	noSSD
		FSD	noFSD
	Risk-free	SSD	noSSD
		FSD	noFSD
	Inflation	SSD	noSSD
		noFSD	noFSD

Table 5: SD relations when imposing the SSD constraint on stage 8

Lambda	Benchmark	6 stage	8 stage
0.25	1/N	SSD	SSD
		FSD	FSD
	Risk-free	noSSD	SSD
		noFSD	noFSD
	Inflation	noSSD	SSD
		noFSD	noFSD
0.50	1/N	SSD	SSD
		FSD	FSD
	Risk-free	noSSD	SSD
		noFSD	noFSD
	Inflation	noSSD	SSD
		noFSD	noFSD
0.75	1/N	SSD	SSD
		FSD	FSD
	Risk-free	noSSD	SSD
		noFSD	noFSD
	Inflation	noSSD	SSD
		noFSD	noFSD

borrowing and in so doing facilitate the achievemant of a sufficient retirement income.

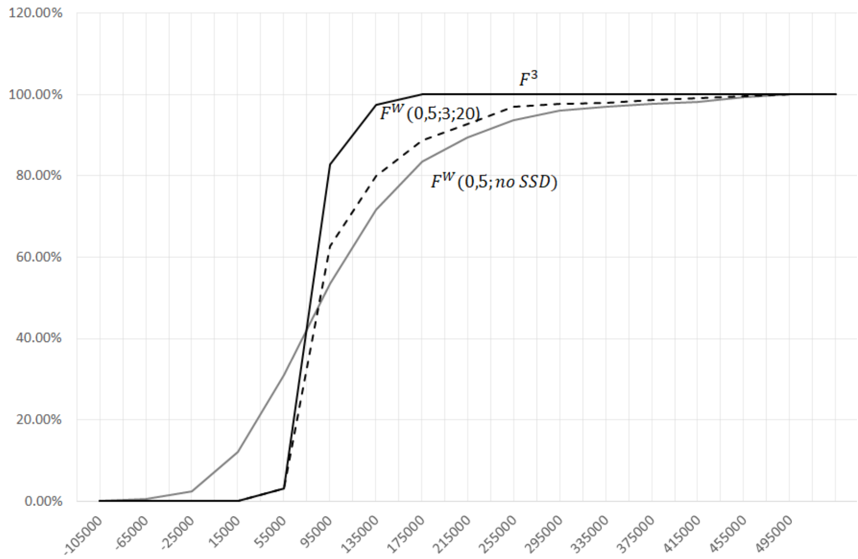
(bene il commento. Io sottolinerei che l'achievement del FSD non e' in generale garantito se non nei casi in cui anche la stessa SSD e' relativamente da ottenere. Inoltre sottolinerei che quando la SSD e' imposta sullo stadio 6 non si ottiene MAI sullo stadio 8, a sottolineare che la funzione obiettivo tende in una direzione che e' in qualche modo contrastante con la SD e quindi non appena il portafoglio e' "libero" dalla SD si sposta dal benchmark per tendere all'ottimizzazione dell'obj.)

3.4 Goals trade-off analysis

Individual ALM problems have been proposed by several authors Mulvey and Ziemba (1998); Medova et al. (2008); Consigli (2007) with relevant real-world applications already in the past. In this article we have extended a previous modeling effort Consigli et al. (2011) to incorporate stochastic dominance principles in a goal-based model. In this final set of evidences we analyse the implication of SSD constraints at the final stage specifically on the intermediate leverage and the shorfall with respect to the retirement target for the specific case of $\mathcal{L}(0.5;3;\tau)$: the choice is justified by the SD relationships between the benchmark strategies as reported in Fig. **??** and the evidence on the optimal problems solutions as λ varies.

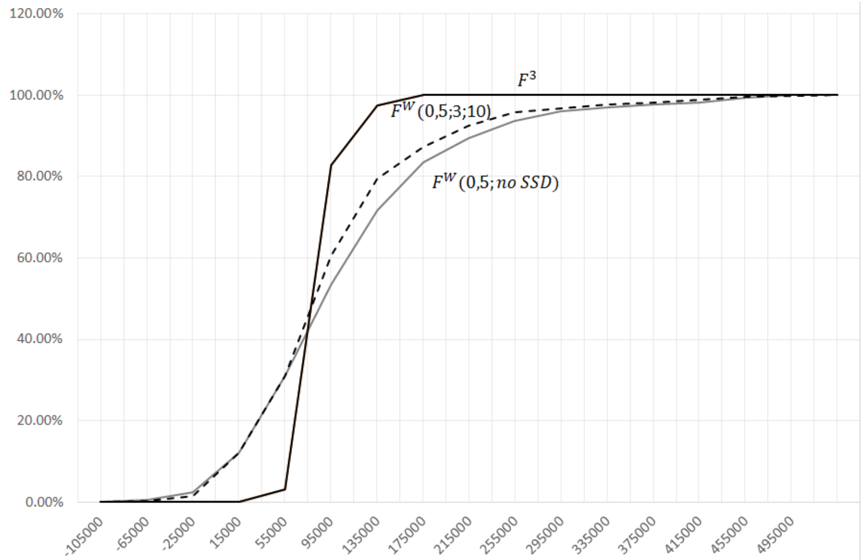
The real estate and the retirement targets are both revalued in real terms and thus their scenario evolution is determined by the same scenarios of the initial portfolio: given the different at $t = 0$ between the initial portfolio value and the real estate target, at $t = 10$, depending on the optimal portfolio returns the acquisition of the (ambitious) retirement goal may be jeopardised by the intermedate leverage decision: in Fig. 3 we see that the borrowing decision varies from a minimum of roughly 90 000 to a maximum of 150 000 Euros. We may interpret these amounts as shortfall with respect to the intermediate target and consider how the introduction of SSD constraints with respect to benchmark 3 affected the wealth distribution at the end of the planning horizon, where the retirement goal was set. We display in Fig. 5 the terminal wealth distributions associated with the optimal solution without SSD constraints $F_W(0.50)$, with SSD constraints $F_W(0.50, 3, 20)$ and the benchmark wealth distribution for the case $k = 3$: $F^3(20)$ **(questa notazione e' ancora diversa, da omogenizzare)**. Notice that when removing the SSD constraints the two goals at the intermediate and terminal stages are not affected and based on $\lambda = 0.50$ the decision maker is assumed to seek a maximum expect wealth together with a penalty on the terminal wealth shortfall with respect to the target.

Fig. 5: Terminal wealth distributions: benchmark (solid black line) and optimal wealths with (dashed black line) and without (solid gray line) SSD constraints at $T = 20$



Seba vorrei confrontare la distribuzione della ricchezza ottima con SSD di cui sopra in T=20 con quella che avremmo sempre in T=20 qualora l'SSD sia settato sempre su k=infl ma allo stadio 6. Tra i risultati non sono riuscito a trovarli

Fig. 6: Terminal wealth distributions: benchmark (solid black line) and optimal wealths with (dashed black line) and without (solid gray line) SSD constraints at $T = 10$



(Aggiunta! referenza "fig:comparisonT10". Bisogna aggiungere il relativo commento)

The CDF's plotted in Fig. 5 show that by imposing the SSD constraint the lower tail of the wealth distribution shifts to the left but when there is the possibility to achieve higher terminal wealth the constraints will no longer be binding and the upper tails of F^W's will be close to each other. Accordingly the shortfall with respect to the retirement goal across all scenarios will be reduced by requiring SSD feasibility. In the *best case scenario* the family will buy the house and move into retirement with a minimal wealth of around 28 000 euros per year. In the *worst case* with around 2 900 euros per year, far below the retirement goal but at least positive.

In presence of an SSD constraint at the intermdiate stage, under any scenario the borrowing decision would be maximal (see Fig.3) and the shortfall with respect to the retirement goal will increase. In a goal-based investment problem the inclusion of SSD constraints defined with respect to a challenging distribution helps achieving the final target or in any case limit the dispersion around that target. Furthermore the SSD constraint preserves its mathematical rationale even for high values of λ in the objective function when increasing weight is set on targets' shortfall minimization objective due to the impact on the lower tail of the generated wealth distribution.

(anche qui riporterei e sottolinerei lo stesso commento fatto prima, cioe' che la dominanza sullo stadio 6 non influisce sulla dominanza allo stadio 8 e quindi il portafoglio e' libero. Bene il tuo commento che sistemare la coda di sinistra non incide eccessivamente sul minor guadagno nella coda di destra.)

Conclusions

Main focus of this article is an extensive study of the financial and methodological implications a simple extension of a canonical goal-based individual ALM problem to accommodate SSD principles through a set of linear constraints Kuosmanen (2004); Kopa et al. (2018): given a set of investment and retirement goals we have first evaluated the impact of SSD constraints on the family terminal wealth distribution and then their implications on first stage investment decisions and asset-liability strategy: interestingly we have reported that SSD-constrained portfolios will typically lead to higher performances with an effective hedge of high losses without in general jeopadising the portfolio upside. Furthermore even far away SSD constraints may have a signficant impact on first stage investment allocation and liability policies.

In a practical context that SSD feasibility in presence of a relatively coarse benchmark policy, such as in the $1/n$ portfolio case will easily imply FSD, while as the benchmark return distribution shifts to the right this will no longer be the case. Such shift will lead however to a reduced shortfall with respect to end-of-the-horizon retirement goals.

gc: complete after last read! before submission

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