La collana della Scuola di Alta Formazione Dottorale accoglie le migliori tesi di dottorato dell'Università degli Studi di Bergamo, insignite della dignità di stampa e sottoposte a procedura di *blind peer review*.

Stochastic optimization models have been extensively applied to financial portfolios and have proven their effectiveness in asset and asset-liability management; however, derivatives have not been studied much in detail within the context of dynamic stochastic programming. Derivatives such as options have non-linear payoff profile and their presence in the portfolio makes constraints in the optimization problems complex.
In this research, we focus on European options contracts (cash and physically-settled) within dynamic stochastic programming framework, starting from the single-stage models developed by other researchers, we develop multi-stage models where traditional asset classes are modeled with options available on them. We present a general model where not only long positions on derivatives are considered but also short-positions are modeled, we also verify that the general model we developed in particular cases reduces to the models developed by other researchers. We also developed models where we utilize physically-settled options contracts to manage the inventory dynamically. All the models developed are validated on the real data.

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Vivek Varun STOCHASTIC PROGRAMMING MODELS

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UNIVERSITÀ DEGLI STUDI DI BERGAMO

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This thesis is dedicated to my grandmother

Contents

Preface

Derivatives play a key role in the financial markets; they are used for speculation and hedging. In portfolio optimization theory, derivatives have been studied extensively. In this research, we explore the applications of European options in portfolio management. Several option strategies are developed in the literature to hedge or speculate under different market conditions. However, most of them focus on achieving single-period objectives. Coupling those static strategies with the multistage portfolio optimization could help in achieving investment objectives in the short, medium and long-run. This is the goal of the research we present in this book. We study European call and put options under different market conditions.

Most of the options traded in the market are cash-settled option contracts, nevertheless, there is also a high volume of physically settled contracts. These two types of contracts serve different investment objectives. Cash-settled options can be used for portfolio planning to offset losses in the underlying securities due to change in the market price or they can be used to improve the portfolio performance through speculation. Whereas, physically settled options contracts are aimed to increase or reduce the inventory of the underlying instrument, stock, or a commodity for instance.

In the literature, many successful applications of dynamic stochastic programming (DSP) can be found, DSP is a programming framework for modeling optimization problems with uncertainty. Real-world problems include unknown variables, whereas, deterministic optimization problems involve known parameters. Uncertainties are represented as a scenario tree and the goal is to find some policy that is feasible for all (or almost all) the data instances and maximizes the expectation of some objective function of the decisions and the random variables. DSP allows series of recourse actions taken at the stages (rebalancing periods) of the optimization program. Most of the applications of DSP are applied to asset-liability management in pension funds and in the insurance business and can model portfolio objectives for a planning horizon as long as 30 years. However, only a few applications of DSP can be found involving options under this framework, while most of them focusing on options expiring at the next rebalancing stage.

In this research, we focus on European options under the DSP framework. We introduce optimization models available in the literature, starting from a single-stage model we go on developing multistage models with European put and call options, modeling options for different expiry and also modeling them for short positions. Inclusion of short positions in the options would allow an investor to adopt trading strategies such as bull and bear spread. More complex strategies can be formulated using a combination of long and short positions in options across various strikes and maturities. We present models that buy and sell the options before expiry, option premiums are very volatile and could be rewarding in some situations. We validate the models we develop both analytically and numerically.

We also develop a model where physically-settled options contracts are used to increase or reduce the inventory effectively. This model is aimed to buy (or accumulate) stocks at a lower average price (Volume weighted average price) than the market price. This is obtained by taking positions in the options strategically which is achieved through the model. All the numerical results presented here are supported by generating realistic scenarios, simulation models forecasting realistic prices are validated.

The research developed answers many questions raised in the literature about the inclusion of options under the DSP framework. This book presents the relevant literature review of the stochastic programming and its applications, the option theory and the use of options under the DSP framework. We hope that it would give the reader an insight into the use and development of options strategies in a multi-stage environment.

Vivek Varun

Chapter 1

Derivatives and Stochastic Programming

Stochastic optimization models have been extensively applied to financial portfolios and have proven their effectiveness in asset and asset-liability management; however, hardly they are applied to the investment portfolios with derivatives such as options along with the underlying securities. The modeling of European type options in a multi-stage stochastic programming framework is a multi-utilitarian approach, for instance, a put option can be used to insure a portfolio against any unfavorable market outcome, a call option can be used maximize the profit in a bullish market situation, options can be bought or sold before their expiry to make profit from their volatile premiums, if options are physically-settled, then they can be used to increase or decrease the inventory strategically. We discuss other benefits and state-of-the-art of using options in multi-stage programming in this chapter.

1.1 Introduction

Financial derivatives have been used for mitigating downside risk and enhancing upside potential in portfolio management. Their different payoff structures give investors appropriate instruments to model their risk-reward profile. Different types of derivatives instruments can help in meeting different objectives. For instance, when buying/selling an asset a future/forward contract can be used to offset price fluctuation risk. Options can be used for multiple purposes, a put option can be used to provide a hedge against a downside movement in the underlying price, whereas a call option can be used to improve the upside potential of a long underlying position. Other common financial derivatives are swaps, collateralized debt obligations, and credit default swaps. However, in this research, we limit ourselves to options contracts only. Options contracts can either be cash-settled contracts or physically-settled contracts. Most of the options in the market are cash-settled contracts.

Physically-settled contracts are options contracts whereby settlement requires the actual physical delivery of the underlying asset. The most common physical settled options are stock options, since the delivery of underlying shares is easier due to their liquidity. For instance, if an investor buys a call option on Google with strike USD 500, at expiry of the option if the price of the Google share is above USD 500, say USD 510, then the investor has the right to buy Google shares at USD 500 instead of paying USD 510, saving USD 10 compared to the market price at expiry. On the other hand, if the call option expires out-of-the-money, i.e. price of the underlying falls below USD 500 at expiry, then the investment value of the option goes to zero. Similarly, a put option can be used to sell the underlying at a certain (strike) price if the put option expires in-the-money. These physical settled options contracts allow investors to bring down the average buying cost when the market is bullish and bring up the average selling cost when the market is bearish. In the latter part of this thesis, we develop a model that utilizes options contracts to optimally reduce or increase inventory when the market is bearish and bullish respectively.

The other type of option contracts is cash-settled contracts. These are contracts where settlement happens in cash. These types of options are required where the physical delivery of the underlying assets is not possible or inconvenient or costly. For example, index options on Standard and Poor's index (SPX) and the volatility index (VIX). These type of options do not exist physically, we cannot buy indices, therefore, options on these types of financial instruments are cash-settled.

The benefits of options are not limited to buying or selling the underlying at a certain price, options contracts can be used to develop hedging strategies where the loss can be minimized. Options contracts can also be used for speculation, out-of-the-money options cost much less as compared to the at-the-money options, a small movement in the underlying price would reflect big relative changes in the OTM option price. From hedging to speculation, options meet the demands of all types of investors and fit into their portfolios despite the complexities of the market.

In this research, we consider the use of both physical and cash-settled options contracts. Multi-fold benefits of options contracts make it a hot topic for researchers. Questions arise how to fit options into a portfolio for a desired shape of the portfolio returns; how to mitigate the overall risk of a portfolio using options; under what mathematical settings to model options in a portfolio; can dynamic approach be more efficient than the static ones; is it possible to buy/sell options before their expiry; can there be an optimal amount allocated for speculation using options and what are the implications of short-selling options contracts, etc. We answer these questions in our research. We develop models that give both mathematical and financial meaning to the use of options contracts in the multi-stage setting.

There have been many successful applications of dynamic stochastic programming applied to portfolio optimization answering asset-liability management challenges. However, this research considers only the use of broad asset classes, such as fixed income, equity, real estate, etc. Very limited research has been developed where options are treated as an asset class. At this stage, it is important to go through some studies to understand the benefits of including options in a portfolio, some successful applications of multi-stage stochastic programming and then finally options and dynamic stochastic programming together to study options in a multi-stage setting.

1.2 Literature Review of Options in Portfolio Optimization

Over the last four decades, many studies have been conducted on studying options in portfolio management, some researchers have focused on the hedging side of the options, some have focused on the speculative aspects and some have worked on combining different types of options to mimic certain investment policies. We present here some relevant contributions that motivate our research, we start from the earliest contributions by Merton all the way to the most recent studies.

Merton, Scholes, and Gladstein, 1978 studied on options investments in put options as term insurance to insure the portfolio against any possible loss, it was proved that no put strategy or call strategy can dominate any other strategy if options are priced correctly. It was found that investors can use uncovered put option writing and covered put option buying to produce patterns for returns on investments that cannot be proxied by any combination of equity and fixed income securities. Harrison and Pliska, 1981, talked about general hedging methods for options in complete market. Follmer and Sondermann, 1986, discussed hedging strategies in incomplete market.

Brennan and Cao, 1996, found that options/derivatives improve the Pareto efficiency as the trading gets continuous as with multiple trading sessions uninformed investors behave as rational trend followers. Aliprantis, Monteiro, and Tourky, 2004, presented minimum cost portfolio insurance investment strategy, when derivative markets are complete then holding a put option in conjunction with the reference portfolio provides minimum cost insurance at arbitrary arbitrage free security prices. It was analysed that if the asset span is a lattice-subspace, then the minimum-cost portfolio insurance can be easily calculated as a portfolio that replicates the targeted portfolio in a subset of states which is the same for every reference portfolio.

Haugh and Lo, 2001, showed that under certain conditions, a portfolio of a few numbers of options can be a good proxy for more complex dynamic investment policies due to the fact the derivative securities are equivalent to specific dynamic trading strategies in complete market and this is the motivation behind constructing buy-hold portfolios of options that mimic certain dynamic investment policies.

Liu and Pan, 2003, solved investment strategies in closed form given that investor has access to options along with stocks and bonds. It was shown that due to the volatile nature of derivatives they enable non-myopic investors to disentangle the simultaneous exposure to diffusive and jump risks in the stock market.

Similarly, Muck, 2010, analysed trading strategies with derivatives when an investor has full or partial access to the derivatives market, it's the case when options are not available on all the stocks in the portfolio. Potential benefits of adding derivatives to the market are studied, it was found that diffusion correlation and volatility or jump sizes may have a significant impact on the benefit of a new derivative product even if the market price of risk remains unchanged. Increasing or decreasing utility gains of the different types of options can be exploited for a more diversified portfolio.

Driessen and Maenhout, 2007, studied the economic benefits of giving investors access to index options in the standard portfolio problem, analysing both expected-utility and nonexpected-utility investors in order to understand who optimally buys and sells options.

These studies focused on hedging aspects of the options or on replication of some dynamic investment policy. Options have also been studied under the condition of risk-neutrality on the Greeks to obtain an arbitrage-free profit.

Gondzio, Kouwenberg, and Vorst, 2003,presented stochastic optimization hedging (SOH) model to consider transaction costs, stochastic volatility, and trading restrictions by introducing a dynamic trading strategy. The goal of the strategy is to minimize the hedging errors at the first few trading dates. Traditional hedging strategies like delta hedging or delta-vega hedging are not appropriate in this context. The main drawback of the model is that the number of constraints grows exponentially with the number of trading dates.

Papahristodoulou, 2004, formulated a linear programming model to select the optimal hedging strategies, unlike other approaches where hedging strategies are defined in advance. The main advantage of this approach is that it makes investment planning more rational and independent of market beliefs of the investor. Horasanlı, 2008, extended the work presented by Papahristodoulou, 2004 from a single asset to a multi-asset portfolio with options on them and considered all the hedging strategies using delta, gamma, theta, rho and vega. The model has many disadvantages in dealing with the options, linear constraints are forced on options while their payoff is nonlinear. Despite limitations, the model through linear programming provides many advantages to the investor.

Gao, 2009, presented a general linear programming model with bounds on each Greek letter and then performs a new post-optimality analysis of the model where risks are adjusted by the investor to suit the market dynamics. With the model and the method proposed, one can take the options strategies in terms of one's subjective personality, and meanwhile, adjust the risks to suit the needs of the market change. Sinha and Johar, 2010, further extended this work by introducing quadratic programming to tackle the non-linear payoff of the options, they formulated a quadratic programming model and then approximated that with a linear programming model, it was found that the risk of theportfolio can be hedged by reducing its delta gamma and vega and at the same time, it is possible to minimize the net premium to be paid for the creation of the hedged portfolio.

Liang, Zhang, and Li, 2008, studied a mean-variance formulation for the portfolio selection problem involving options. In particular, the portfolio in question contains a stock index and some European style options on the index. A refined mean-variance methodology is adopted in their approach to formulate this problem as multi-stage stochastic optimization. It turns out that there are two different solution techniques, both lead to explicit solutions to the problem: one is based on stochastic programming and optimality conditions, and the other one is based on stochastic control and dynamic programming.

Palma and Prigent, 2008, introduced a financial hedging model for global environmental risks using financial and environmental assets. It was found that options indeed provide a good hedge to the portfolio, however, there is a need to include new types of options that combine both equity and environmental assets contrary to current practice where two separate option markets are considered.

These types of models can solve the hedging issues in a very limited way since these are the static approaches so they are not able to handle the fast-changing market conditions. In addition, as the trading dates are increased the number of constraints on different hedging strategies grows exponentially. Dimson and Mussavian, 1999, presented a historical review of option pricing techniques.

Scheuenstuhl and Zagst, 2008, proposed a mean-variance portfolio management approach using stocks and options. In addition, they use additional investor preferences in terms of shortfall constraints to allow a more detailed portfolio specification. Also, they utilize an approximation of the return distribution and develop economically meaningful conditions to transform the complex optimization problem into a linear problem.

Zymler, Rustem, and Kuhn, 2011, proposed a novel robust optimization model for designing portfolios including European-style options. Their proposed model, which is on the basis of second-order cone programming, trades off weak and strong guarantees on the worst-case portfolio return. Fonseca and Rustem, 2012, proposed a robust formulation for the international portfolio management problem that maximizes the portfolio return for the worst possible outcome of returns. They further incorporate forward contracts and quanto options to mitigate currency and market risks. Since their proposed model is not linear, they reformulate both the uncertainty set and the objective function as a semi-definite problem. Fonseca, Wiesemann, and Rustem, 2012, used a similar robust approach to cope with international portfolio management. Instead of using currency forward contracts, they utilize equity options for domestic assets.

In the literature, there have been some studies available where disadvantages of including options in a portfolio are discussed, Neuberger and Hodges, 2002, have questioned the benefits of including options in a portfolio. They model an economy with a single risky asset, the model enables them to examine the benefits to an investor of using options to optimize their investments, it was found that including options makes minor improvements in portfolio performance only when it comes to hedge the volatility. The risk-reward in holding options is not limited to volatility risk, options may be a costly hedge to the volatility.

1.3 Literature Review of Stochastic Programming and its Applications

So far, we have discussed the advantages and successful applications of including options in the portfolio. The approaches we have discussed are the static approaches, hence, there is a need to integrate the use options in a multi-stage setting so that dynamic and robust hedging and trading policies can be formulated. For this, we use a multi-stage stochastic programming approach.

The approach is a mathematical framework for modeling optimization problem that involves uncertainty. Unlike deterministic optimization problems where all the parameters are known at the beginning of the optimization process. In stochastic programming, uncertainties are revealed with time and are not known at the beginning of the optimization problem. As an example, here the decision maker takes some action in the first stage, after which a random event occurs affecting the outcome of the first-stage decision. A recourse decision can then be made in the second stage that compensates for any bad effects that might have been experienced because of the first-stage decision. The optimal policy from such a model is a single first-stage policy and a collection of recourse decisions (a decision rule) defining which second-stage action should be taken in response to each random outcome.

This ability of stochastic programming makes it useful for solving finance and economic related problems where the future prices are not known at the start of the optimization process. As the prices are revealed, new decisions are taken to achieve an objective. Another advantage of this approach is that it could be applied to any model problems with any time horizon, for instance, its application to pension fund planning, where a long-term horizon problem is solved with typical time horizon expanding to as long as 30 years, on the other hand, it can also be applied to a portfolio of few securities/assets that aim to achieve a target at the end of a year or six months period, stochastic optimization approach has proven to be helpful in many aspects.

It also has some assumptions, first, underlying stochastic processes are not influenced by the values of the decision variables, second, decisions adapt to available information at the time they are made, but do not depend (invariant wrt) on specific projected future outcomes (no hindsight).

Fundamental components of a multi-stage stochastic program are: (Vladimirou, SPXI tutorial 2007)

1. the description of the underlying (multivariate) discrete stochastic process for the uncertain parameters, dynamic information structure (Scenario-tree Generation).

2. discrete time dynamic stochastic optimization program capturing the structure of the decision process.

3. Mapping (1) & (2) in a logical conformable way.

4. Defining appropriate performance and risk measure for the problem under uncertainty.

Figure 1.1 shows a sample scenario tree with discrete time stages, $t = 0, 1, 2$ & 3 are the time steps where a decision is made based on information realized. Each atom in this scenario tree is called a node and each node is linked to a previous node or parent node, this defines the basis of sequential decision process with the information process capturing their connection. This makes easier to model cash flow equations, asset balance equations and wealth equations in each scenario along the planning horizon. Key advantages of stochastic programs are; they can handle multi-asset problem by determining optimal asset value at an individual level, modeling uncertainties irrespective of the type of distribution, can handle regulatory policies as constraints and can alternatively use flexible risk measures or performance objectives.

Use of multi-stage stochastic programs started in the early seventies, Ho and Manne, 1974. The first few applications were to optimize a portfolio of multiple fixed income securities, Bradley and Crane, 1975. Gradually, the stochastic optimization

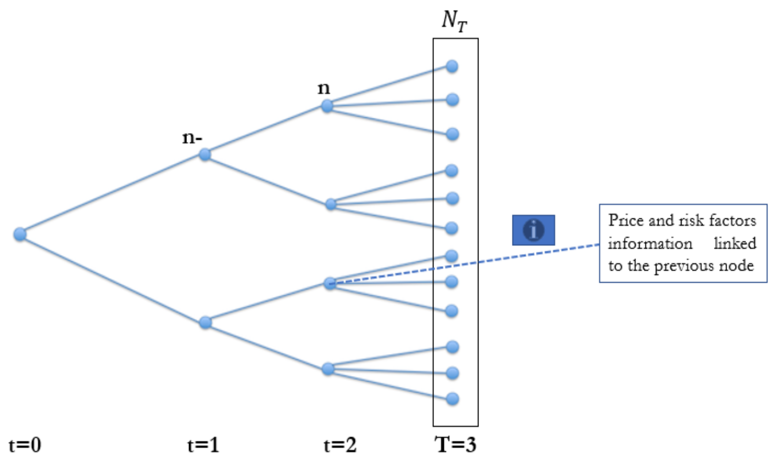

FIGURE 1.1: Scenario Tree Representation

technique was applied in many areas in finance and economics. We cite a few of them. The first successful application of the stochastic programming in finance was the famous Russell-Yasuda Kasai model, Carino et al., 1994 where an asset-liability management model was developed using multi-stage stochastic techniques. It determined an optimal investment strategy that incorporated a multi-period approach and allowed decision maker to define risk in tangible operational terms. Regulatory restrictions were implemented as constraints. The technique used yielded an extra 42 bps in the fiscal year 1991 and 1992.

The approach then was further explored by the academic researchers and many successful applications and advancements came. Nielsen and Zenios, 1996, developed a dynamic stochastic optimization model to tackle the uncertainties in liability in the insurance business. The model considered explicitly the uncertainties inherent in this problem due to both interest rate volatility and the behavior of individual investors. Dert, 1995 applied this technique to analyze the investment policy and funding policy of a pension fund and proved that the probability of underfunding can be reduced significantly.

Consigli and Dempster, 1998, developed a CALM model (Computer Aided Asset-Liability Management) to deal with uncertainties affecting both assets (in either portfolio or the market) and liabilities (in the form of scenario dependant payments or borrowing costs). This randomness in the assets/liabilities demanded a thorough investigation of the reliability of the scenario or event trees used to solve these multistage programs. Poor scenarios can lead to bad investment decisions. Kouwenberg, 2001, talked about the reliability of scenario trees in this context. He used both randomly sampled event trees and event trees fitting the mean and the covariance of the return distribution for generating the coefficients of the stochastic program. Hence, allowing to investigate the performance of the model and the scenario generations conducted on rolling horizons. It was found that the performance of the model can be improved drastically if the right model is adopted to generate scenario trees. Klaassen, 1998, talked about the use of arbitrage-free scenario trees in the optimization procedure.

Since policies are implemented as constraints under this programming framework, so it is possible to study the effectiveness of the regulations under which a company is working. Høyland and Wallace, 2001, developed a multi-stage stochastic optimization model for an insurance company where they showed how legal regulations put by the government are in the interest of insurance holders.

It was intuitive and evident that bad scenarios would lead to bad outcomes and poor investment decisions. Hence, there was a need of good scenarios and sufficient number of scenarios. Researchers then came up with arbitrage-free event trees and with sufficient number of scenario trees to reduce the computational complexity of the optimization problem. Mulvey, 1996 and Gaivoronski and De Lange, 2000 talked about the outcome of the optimization problem in context of economic projection model. Bertocchi, Moriggia, and Dupačová, 2000, Pflug, 2001, Dupačová, 2002, Römisch, 2009, Casey and Sen, 2005, Dupačová, Gröwe-Kuska, and Römisch, 2003, Heitsch and Römisch, 2007, Heitsch and Römisch, 2009, and Kuhn, 2008 have discussed scenario reduction techniques.

1.4 Derivatives in Stochastic Programming Framework

So far we have discussed the benefits of including options in a portfolio and stochastic programming technique that allows us to come up with dynamic investment policies. It's now time to review the combination of the two in the literature. Over the last two decades, researchers have applied stochastic programming techniques to options. Most of the applications are about hedging options since SP techniques can efficiently handle the inclusion of multiple risk factors avoiding myopic decisions, it becomes practical to use SP to apply to options hedging. Wu and Sen, 2000, presented a stochastic programming model to hedge currency options, American type options were studied where an importer wants to hedge currency risk on a fixed amount of US dollars at some time in future. The model includes some realistic features like sensitivity to delta and gamma, the objective function incorporates delta and gamma tracking error with some other risk factors to rebalance portfolio using different options contracts at the decision stages. It was shown that the SP based hedging model can have significant advantages over traditional approaches for currency hedging. It was reported that the modeling and solution approach proposed can be applied to a very broad spectrum of the hedging problems related to contingent claims, like mortgage-backed securities and some exotic fixed-income derivatives, where the returns or payoffs are path-dependent and the Monte Carlo simulation is widely adopted. King, 2002, analyzed the hedging of contingent claims in the discrete-time, discrete-state case from the perspective of modeling the hedging problem as a stochastic program. The model was extended to the analysis of options pricing when modeling risk management concerns and the impact of spreads and margin requirements for writers of contingent claims. It was found that arbitrage pricing in incomplete markets failed to model incentives to buy or sell options. An extension of the model to incorporate pre-existing liabilities and endowments revealed the reasons why buyers and sellers trade options. The model also indicated the importance of financial equilibrium analysis for the understanding of options prices in incomplete markets.

Villaverde, 2004, studied hedging European and Barrier options in a discrete-time and discrete space setting by using stochastic optimization to minimize the mean downside hedge error under transaction costs. Scenario trees were generated using a method that ensured the absence of arbitrage and which matched the mean and variance of the underlying asset price in the sampled scenarios to those of a given distribution. It was shown that SP based method produced a lower mean downside hedge error for both types of options for a range of transaction costs. The methodology was then implemented for the case where the underlying price was driven by a discretized Varriance-Gamma process in which case delta hedging methods were not readily available. The results were found to be similar to the case where the underlying asset follows a discretized Geometric-Brownian motion. Barkhagen and Blomvall, 2016, developed a more realistic model where they considered buying and selling at observed bid-ask prices. The SP model developed relied on realistic modeling of the important risk factors for the application, the price of the underlying security and the volatility surface. Volatility surface was estimated from a cross-section of observed option quotes that contain noise and possibly arbitrage. Non-parametric estimation approach was used to produce arbitrage-free volatility surfaces. By using a simple dynamic model of the squared LVS (local volatility surface) based on PCA (principal component analysis), they built an SP model that captures the most important joint dynamics of a collection of option prices. It was shown that the model presented is able to come up with a hedging strategy that performs better than both delta and delta-vega hedging in terms of producing lower realized risk and costs.

Most of these applications discuss hedging of different types of options. There are some studies that talk about profiting from including options in a portfolio in a multi-stage setting using SP techniques. Blomvall and Lindberg, 2003, presented options in a portfolio of stock index and risk-free asset. They use stochastic programming to analyze the performance of different portfolios, a portfolio with stock index, a portfolio with stock index and a risk-free asset and a portfolio of a stock index, a risk-free asset and call options on the stock index. It was found that portfolios with options contracts outperform the other portfolios in terms of mean and variance. They develop a two-stage model, where the second stage is the option expiry and therefore, the horizon of the planning problem is never longer than a month. Portfolios are rebalanced at a daily frequency and only those options are considered which are near the expiry.

The model presented is a very good starting point to study options in multi-stage settings, however, the model has some limitations. For pricing reasons Blomvall and Lindberg, 2003, did not consider extending the horizon of the problem, they used the Black-Scholes model to price the options. The formulation of a multi-stage model that tackles buying and selling of options requires a proper pricing approach on the scenario tree. The model considers only call options in the portfolio, it would be interesting to include also put option, given the protective features of put options which can serve as low cost insurance to hedge a portfolio Aliprantis, Monteiro, and Tourky, 2004. However, the return of a put option can be replicated using the underlying and call option. Hence, including put options from a hedging perspective is more relevant than including options for speculating high portfolio returns. Blomvall and Lindberg, 2003, ran the program for the period Feb 1990-June 1999 rebalancing the portfolio daily and running 80 scenarios each time. However, the in-sample stability of the model has not been discussed.

The second important contribution along these research lines was by Topaloglou, Vladimirou, and Zenios, 2011, which became the basis of developing mathematical models in a multi-stage setting where options are considered. However, they consider a single-stage model where options expiry is equal to the planning horizon of the problem. Unlike, Blomvall and Lindberg, 2003, where only call options are considered in the portfolio, Topaloglou, Vladimirou, and Zenios, 2011, considered both call and put options. As a result, it allows formulating different strategies on options contracts, such as long straddle, long strangle, strip and strap strategies. Options have non-linear payoffs, when we model them in a single stage framework (with the assumption that span of the planning problem is same as the maturity of the options in the portfolio) it becomes simpler to model their payoffs since the value of an option at the expiry depends solely on the value of the underlying. This fact was beautifully exploited by the Topaloglou, Vladimirou, and Zenios, 2011, so at the final stage either the option is in-the-money with a positive value or its value is zero if it's out-of-the-money. This actually simplifies the model equations.

Topaloglou, Vladimirou, and Zenios, 2011, developed this model in order to optimize an international portfolio with options and forwards. The main objective of the model is to control the overall risk exposure of the portfolio and to achieve a balance between risk and reward of the portfolio. CVaR is identified as the risk and expected return as a reward. Options are included in the portfolio to hedge market risk in the long underlying in the portfolio, whereas, forwards are included to hedge currency risk since the portfolio has securities in multiple currencies. The single-stage model developed was able to achieve the risk-reward balance. However, it opened many other questions about using options or other derivatives in a multi-stage setting and especially looking into the case when the maturity of the derivative is not the same as the span of the planning horizon. The successful implementation of the options hedging strategies motivated the researchers to look into the viability of such strategies to generate profit in the short-run while keeping positions in the underlying for a longer time period. Different options strategies are profitable in different market scenarios. For instance, when an investor is long on the underlying asset, a put option may be useful when the market is expected to be bearish. A straddle may be useful when the market is bullish. Each strategy implemented can be an answer to the market belief of the investor. A dynamic policy where different types of hedging strategies are used with changing market conditions could definitely yield better performance.

Topaloglou, Vladimirou, and Zenios, 2011, studied totally unhedged portfolios where no derivatives are considered, currency risk protection using forward contracts, control of market risk using options contracts, joint protection against market risk and currency risk using forwards and options and finally the use of quantos to protect the position in stock indices. It was reported that an unhedged portfolio exhibited the worst performance (lowest cumulative return and high volatility), whereas the introduction of derivatives improve the portfolio performance in terms of expected return and volatility. These results attracted other researchers to extend this work to a multi-stage setting. The work by Topaloglou, Vladimirou, and Zenios, 2011 was extended by the Yin and Han, 2013b, they developed a multi-stage stochastic model to include options in an integrated view. They moved from a single-stage model to a multi-stage model where options are available at each stage. This allows an investor to build options position at each stage along with the underlying, however, the model considers options that are expiring at the subsequent stages.

A multi-stage model can capture the evolution of underlying series in an effective way, availability of call and put options would allow to effectively adopt options as per the market scenario. Including options in the portfolio in multi-stage setting gives more freedom to the decision maker to dynamically rebalance his positions. The model developed has two main advantages over the model developed by Topaloglou, Vladimirou, and Zenios, 2011, first, multi-stage extension with time-varying investment opportunities and dynamic adjustment, second, they incorporate the overall risk management on five time-varying Greeks.

It was reported that the overall risk management scheme improved when all the five Greek letters are considered in a multi-stage setting, empirical analyses validated the effectiveness of the multi-stage model and the optimal solution of the model dominated traditional hedging strategy. Though the model developed by Yin and Han, 2013b, is a multi-stage model with options available for purchase at each decision stage, however, it considered options that are expiring at the subsequent stages, at each stage options are available that are expiring at the very next stage. Therefore, at any decision stage, the only options are available are the contracts expiring at the next stage. This simplifies the inventory equations for options. At each stage, we have fresh options expiring in the next stage and options that were bought in the previous stage that do not exist anymore. Still, the question of holding options at any decision stage is wide open.

Another work on the similar lines is presented by Davari-Ardakani, Aminnayeri, and Seifi, 2016, they developed a multi-stage model where options are not expiring at the subsequent stage. They exploit the fact that options can be traded before their expiry that indeed could be a very lucrative thing. They extended the work by the Yin and Han, 2013b, of a multi-stage stochastic model for a portfolio with options expiring at the subsequent stage to a generic model where an option can be traded before its expiry. Whilst, the authors have tried to answer a very interesting question through their model, we do not agree with the model they have presented. The equations presented in the model are confusing and are not correct according to us. The authors talked about including European type call and put options in the portfolio and then introduced decision variables on the number of positions on call and put options to be exercised at a node in the scenario tree. If the options are in-the-money then all the options positions should account for the profit, if not, then options expire worthlessly. There is no point in having a decision variable on how many positions of options to exercise (for European options). Also, the model presented could be redundant, they have talked about decision variables on the amount of call/put options and the number of positions on call/put options. This is confusing, it is not clear whether the authors are referring to a nominal amount model or a monetary model. They, however, do not consider short-selling options. Their research presented positive outcomes of including options in the portfolio.

Our work is the answer to the open questions in this context, how and why to include and model options in a portfolio, how they should be implemented in a multi-stage setting, what are the benefits of trading an option before its expiry, is short-selling of option beneficial and risky.

We present in this study multi-stage stochastic optimization models that answer all these questions. We start with a simple model where options are considered in a single-stage setting and then extend the model to take into account the other complexities of expiration at any stage along the planning horizon, we develop a model where options are not necessarily expiring at the decision stages. We generalize the work done by Topaloglou, Vladimirou, and Zenios, 2011 and Yin and Han, 2013b. We develop a model that considers a separate inventory for the options contracts and is parallel to the inventory of the underlying assets. We present a general model that consider short-selling of options contracts, we then finally, verify the model theoretically, how in special cases it reduces to the models presented by Topaloglou, Vladimirou, and Zenios, 2011, and Yin and Han, 2013b. We present strong empirical evidence of exploiting various options features.

We model optimization problems from the point of an investor who wishes to maximize the wealth at the end of the planning horizon, the investor is open to include options in the portfolio along with investments in an equity index, fixed income index and commodity index. Planning horizon of the problem is six months and frequency of portfolio rebalancing is monthly. We develop models to maximize wealth at the horizon by considering both cash-settled contracts and physically settled contracts. Figure 1.2 below summarizes how we extend the work in the literature and what contribution each model brings to the research area.

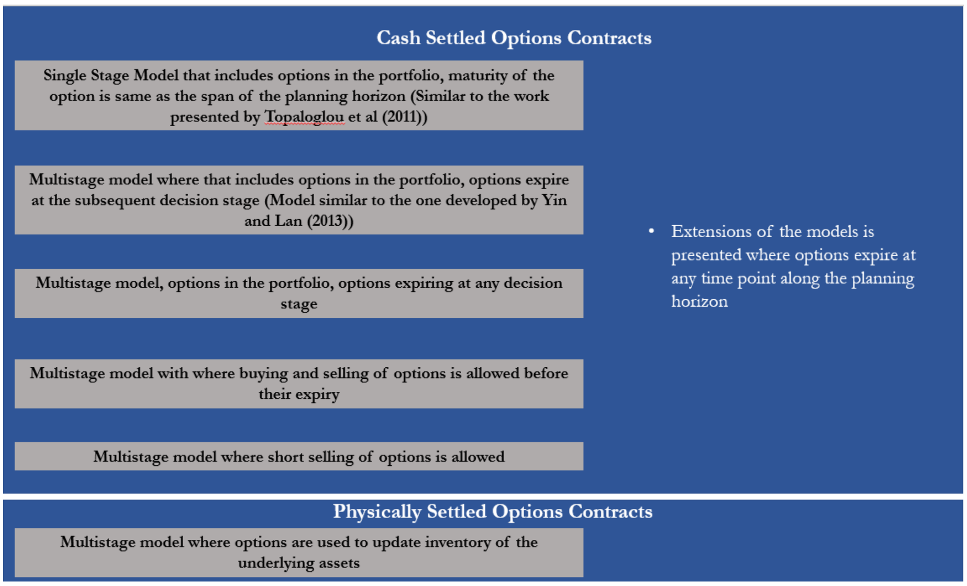

FIGURE 1.2: State-of-the-art and possible extensions

We now discuss the application of these models in brief:

- **Single stage model with options**: -We develop a single-stage stochastic model to optimize a portfolio that includes European style call and put options along with the underlying assets. The goal of the options is to protect the portfolio against market risk. A protective put option is used for insuring the underlying against any bearish movement in the market. Similarly, other strategies such as long straddle, long strangle, strip and strap are implemented and tested using the combination of call and put options in the portfolio. These strategies help in tackling different market conditions. Topaloglou, Vladimirou, and Zenios, 2011 have proved that options can effectively be used to contain market risk.

- **Multi-stage model with options in the model**: - Next, we develop a multi-stage model where options are available at each decision stage with the assumption that options expire at the subsequent stage. The model is similar to the model developed by Yin and Han, 2013b. The advantages of this model are; it gives more freedom to the investor at each decision stage as options are available for investment along with other underlying assets, secondly, it makes the investment policy more dynamic and robust in terms of changing market scenarios. Short-term hedging strategies could be profitable here, figure 1.3.

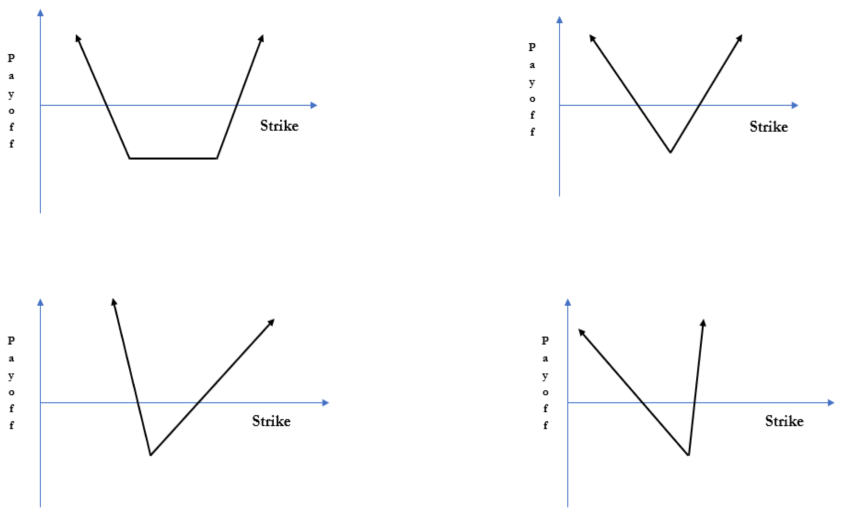

FIGURE 1.3: Options Strategies

- **Multi-stage model, options expiring at any stage**: -Next, we extend this model where we buy options that expire at any decision stage along the planning horizon. The advantage of this model is twofold: first, it allows the investor to hedge his position for different time stages without affecting the short-run profitability using options of shorter maturity. For example, at-the-money options with shorter maturity are cheaper than at-the-money options with a longer maturity, so they leave more space for speculation than options with higher premiums. Secondly, the inclusion of options with longer maturity insures the portfolio to achieve/maintain a specific wealth level at a distant horizon, since a long position in the options corresponds to limited losses. So, this model guarantees insurance against any bearish movements in the underlying. Mathematically, this model becomes more complicated than the model introduced by Yin and Han, 2013b, in that model options expire at the subsequent stage, hence, there is no need for option inventory, each decision stage has fresh options expiring at the next stage. While in our model it is required to have an inventory of the options, as all the options are not expiring on the subsequent stage, at any time it is possible that options bought in the previous stage are held to the next stages. Therefore, proper pricing of option prices is also required.

- **Multi-stage model, buy/sell of long options is allowed**: - In the previous model, we talked about hedging the underlying for a longer time. Now, let's consider the case where a put option is bought to hedge the underlying for six months. After the first month, it was found that the put option bought at time 0 is in-the-money, as a result, its premium would be higher. The model then sells the options and buys a new put option to hedge the underlying for the remaining period. Options premiums are very volatile and thus are risky and rewarding at the same time. We develop a model to exploit this lucrative nature of option premiums. Picture below shows how volatile option premiums are, we plot S&P500 Index against call and put options expiring in December 2017 with strike price equal to 2500 for the period October 2016 to October 2017, figure 1.4. Index values are plotted on the left y-axis and option prices are plotted on the right y-axis.

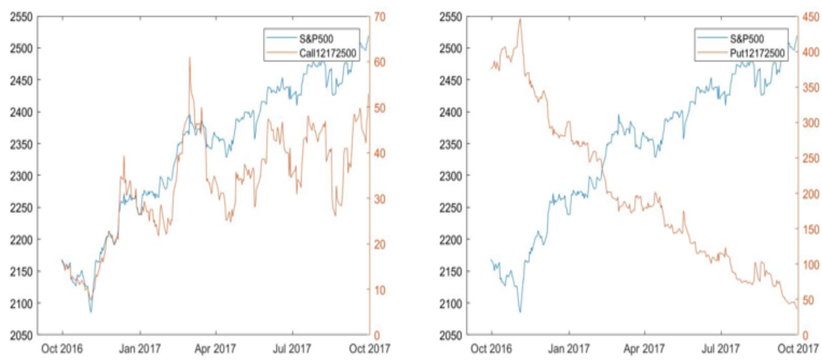

FIGURE 1.4: ATM Call and put options vs underlying index

- **Multi-stage model, buy/short-sell of options is allowed**: - We then develop a model where short selling of options is allowed. Short selling of options allows the investor to model multiple hedging strategies, like bull call spread, bear call spread, etc. It gives the investor a maximum degree of freedom to invest compared to all the models discussed above. This sort of model is actually a generalization of the previous models and we show how it improves the chances of achieving wealth targets for the investor. Each model discussed above is a special case of this model, this model is a summary and generalization of all the models in this line.

- **Multi-stage model, using options to update inventory**: - This model is different from the models discussed above and to the best of our knowledge, it is first of its kind. The multi-stage model uses a physical settled options contract to update the inventory of the underlying assets. The model gives flexibility to an investor who wishes to increase his inventory when the market is bullish while ensuring that the cost he pays is less than the market price. In the same way, if an investor holds a stock of underlying and anticipates that market is going to be bearish then the model generates a trading strategy that helps him reducing the inventory at a price higher than the average selling price in the market. This type of model helps to accumulate inventory or reduce inventory dynamically. This model has wide applications to commodity investors, stock holders or ETFs. This dynamic strategy can be applied to many areas.

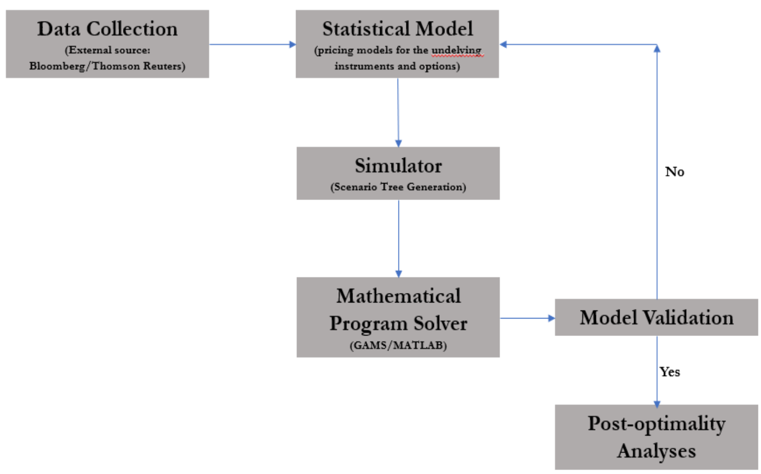

FIGURE 1.5: Methodology

The methodology that we adopt is summarized in the figure 1.5.

- **Data Collection**: - We collect data through external sources Bloomberg and Thomson Reuters Datastream.

- **Statistical Modeling**: - The data collected is then cleaned and used to develop a statistical model to fit for forecasting the underlying time series and the associated risk factors. The statistical model is validated before passing it to the scenario generation part. We forecast the data at discrete time steps as per the planning horizon. The reliability of this forecasting model is tested by rolling the model one period every time and seeing if the forecasted value at the first step is within the forecasted range or not.

- **Simulator**: - Once the statistical model is validated, it is then passed to the simulator to generate scenario/event trees for the optimization model.

- **MSP Solver**: - The generated scenarios are then passed to the mathematical standard program solver, which converts the scenario tree optimization to an equivalent deterministic linear program. CPLEX solver is used to solve the mathematical program in GAMS.

- **Model Validation**: - We then validate the models we have developed. We go by each constraint implemented in the models to check if we have achieved the desired outcome. If a constraint is supposed to put bound on investment in a particular asset then in the result that bound must be satisfied. If we implement a protective put strategy in the model on a certain underlying, then in the result its effectiveness must be reflected when the market is bearish. So, we go by all the variables and constraints to validate the model variables and model equations. Once the model is validated for the equations and constraints we then check the stability of the model. In-sample stability is checked by varying the number of scenarios passed to the optimization problem and observing how the value of the objective function is changing and how the distribution of the wealth is changing in the results. Once the model passes the stability test then we move to the post-optimality analyses.

- **Post-optimality Analyses**: - Once the results are validated we then analyze them on various schemes such as wealth distribution, portfolio composition at different time stages, portfolio performance in the mean and worst scenarios, optimal trading strategy, consistency in the portfolio performance, etc.

In the next chapter we are going to present the mathematical model to tackle options in a portfolio, we start with a single-stage model and gradually introduce extensions to make it a multi-stage model where option buying and selling is allowed before the expiry.

Chapter 2

Multi-stage Models for Portfolios with Derivatives

So far many successful applications of stochastic programming to asset-liability management (ALM) have been published in the literature. Some famous applications of stochastic programming to asset liability management have been reported e.g. in Nielsen and Zenios, 1996, Carino and Ziemba, 1998, Carino, Myers, and Ziemba, 1998, Høyland, 1998, Consigli and Dempster, 1998 and Kouwenberg, 2001. See also the collections Ziemba and Mulvey, 1998 and Zenios and Ziemba, 2004 and the references therein. For a general introduction to stochastic programming, we suggest the reader to the official (COSP) stochastic programming site: www.stoprog.org.

These studies, however, consider only traditional asset classes in the portfolio, rarely, there has been any study that talks about using derivatives in a multi-stage stochastic programming framework. Topaloglou, Vladimirou, and Zenios, 2011s have studied using options and forwards for managing international portfolios using stochastic programming techniques. The article introduces a single-stage model with options and forwards along with conventional asset classes. The study suggests that the use of derivatives improves the upside potential of the portfolio performance and it also helps in reducing downside risk. We extend the research carried by them to consider options in a multi-stage setting.

The motivation for including options in a portfolio can either be hedging or speculation. Different objectives of using options would require to develop different mathematical models. For instance, a long vanilla put option can be used to hedge a position in the underlying, similarly, a call option can be used to speculate on the price movements of the underlying. Both call and put options can be used together to make a straddle or strangle. If an option is bought to hedge a position and before the planning horizon if it is found to be in the money, then it can be sold and a new option can be bought to hedge the position in the underlying for the remaining investment horizon. Since the option premiums are very volatile, their price movements are risky but can be lucrative at the same time. Using options in a portfolio give rise to many possibilities, we develop a mathematical model for each of them. We start with the single-stage model presented by Topaloglou, Vladimirou, and Zenios, 2011.

2.1 Introduction to the Optimization Model with Options

The following model is similar to the model presented by Topaloglou, Vladimirou, and Zenios, 2011. The optimization problem is considered for the US investor having positions in some assets and options on them. The portfolio is composed of a stock index, bond index, and commodity index, then we have European call and put options on these indices. The portfolio thus is exposed to market risk. We develop a scenario-based approach to address the risks involved here. The deterministic inputs are initial position in each asset class and in options. European call and put options available at time 0 with exactly the same expiry as the horizon.

The scenario generated data for asset prices (and its related risk factors described later) defines the option payoffs at the horizon under each scenario. The model presented here is a nominal amount model and the decision variables (buy/sell) represent the quantity bought/sold in an asset class and option in a given node.

The model is a single-stage model, all decisions are taken at time t=0 of the planning horizon [0,T]. The objective is to maximize the expected wealth minus penalized risk measure which is expected shortfall in our case.

We go by introducing sets, parameters and decision variables to formulate the optimization model. The notations used are the following.

2.1.1 Sets, Parameters and Variables

Sets

- $\mathcal{T}$, set of discrete time space indexed by t, $\mathcal{T} : t = \{0, 1, 2..., T\}$
- $\mathcal{N}$, Set of nodes in the scenario tree indexed by n, $\mathcal{N}_t$ is the set of nodes at time stage t
  (Every $n \in \mathcal{N}_t$ has a unique ancestor $n- \in \mathcal{N}_{t-1}$ and for $t \leq T-1$ there exists a non-empty set of nodes $n+ \in \mathcal{N}_{t+1}$)
- $\mathcal{I}$, set of financial assets, indexed by i
- $\mathcal{O}$, set of vanilla options
  - $\mathcal{O}^c$ & $\mathcal{O}^p$ are set of call and put options respectively
  - $\mathcal{J}_i$, set of expiries of the options $\mathcal{O}$, indexed by j, $\mathcal{O}_{ij}$ represents the set of options on asset i expiring at maturity j, $\mathcal{J}_i : j = \{J_{i1}, J_{i2}...\}$
  - $\mathcal{K}_i^j$, set of strikes of the options in $\mathcal{O}$, indexed by k, $\mathcal{O}_k$ K_i^j represents the vector of strikes at maturity j on asset i, $K_i^j : k = \{K_{i1}^j, K_{i2}^j, K_{i3}^j....\}$

Input Parameters

- $\bar{x}_i$, initial position in asset $i \in \mathcal{I}$
- C_0, initial available cash
- $\bar{C}$, is the initial available cash
- T, length of planning horizon

- χ^+ and chi^-, are the proportional transaction cost for purchase and sale in underlying
- χ_o^+, χ_o^- and χ_o, are proportional transaction costs on buying, selling and exercising option respectively.
- $\bar{\mu}$, user defined target
- v_{i0}, current price of the asset i per unit face value
- $O_{i0}^c(j,k)$, is the current price of the European call option on asset i with expiry j and strike price k
- $O_{i0}^p(j,k)$, is the current price of the European put option on the asset i with expiry j and strike price k

Now, we introduce parameters that would model the flow of information along the scenario tree.

Scenario Dependent Parameters

- $p(n)$, probability of node $n \in \mathcal{N}$ such that $\sum_{n \in \mathcal{N}_T} p(n) = 1$ and for every non-terminal node $p(n) = \sum_{m \in n+} p(m)$, $\forall n \in \mathcal{N}_t$, $t \leq T-1$.
- r_n, is the annual risk-free rate in node n.
- v_{in}, price of asset i, in node n.
- $t(n)$, $t(n-)$ and $t(n+)$, represent the time stages of node n, its predecessor node $n-$ and its successor node $n+$.
- $O_{in}^c(j,k)$, is the price of the European call option on the asset i in node n, with strike price equal to $K(j,k), \forall K(j,k) \in \mathcal{K}$ that expires at $t_j, j \in \mathcal{J}$.
- $O_{in}^p(j,k)$, is the price of the European put option on the asset i in node n, with strike price equal to $K_k, \forall K_k \in \mathcal{K}$ that expires at $t_j = J, j \in \mathcal{J}$.

Computed Parameters

Value of the initial portfolio is the sum of position in each asset.

$$W\_0 = \bar{C} + \sum_i \bar{x}_i v_{i0}, \tag{2.1}$$

Assumption: there is no initial position in the options

Decision Variables

- x_{in}^+, nominal amount of asset i purchased in node n, *buying decision*
- x_{in}^-, nominal amount of asset i sold in node n, *selling decision*
- x_{in}, nominal amount of asset i held in node n in the revised portfolio, *hold decision*
- $c_{in}(j,k)$, units purchased of a European call option on asset i with expiry j and strike price k, *buying decision in call option*

- $p_{in}(j,k)$, units purchased of a European put option on asset i with expiry j and strike price k, *buying decision in put option*

Auxiliary Variables

- W_n, value of portfolio in node n

2.1.2 The Objective Function

The objective function of a mathematical program is the key to choosing good decisions over the bad ones. It depends on the type of objective function that drives optimal solution, if the objective is to maximize wealth at the planning horizon then the procedure (sequence of decisions over stages) would try to move in the direction that maximizes wealth. If the objective is to find an optimal trade-off between risk and reward then the objective function would take the optimal solution that gives a balance between risk and reward. We formulate the optimal decision problem as multi-stage optimization problem with recourse(Birge and Louveaux, 1997, Dempster and Ireland, 1988, Dupačová and Bertocchi, 2001).

We consider the following objective function which is the convex combination of two parts *(risk and reward)*:

$$\max(1-\lambda)E[W^T] - \lambda R_\zeta \tag{2.2}$$

$$A_0 X_0 = D_0 \tag{2.3}$$

$$A_n X_{n-} + G_n X_n = D_n, \forall n \in \mathcal{N} \tag{2.4}$$

The first part is the expected wealth ($E[W^T]$) (*reward*) at the planning horizon T and the second part is the risk measure R_ζ (*risk*), λ is the risk aversion coefficient that defines how risk-averse the investor is. Expected wealth is defined as $\sum p_n W_n, \forall n \in \mathcal{N}_T$. We discuss the risk measure in the latter part of this section. A, G & D are the constraint matrices and define inventory balance equations, cash balance equations and other constraints optimization model is subjected to. X_n are the control variables, this vector decides buy, sell and hold decisions for each asset in each node of the tree. The idea behind choosing this type of objective function is the trade-off between risk and reward; as the future price of financial instruments is an uncertain phenomenon and so the expected value of portfolio is not well defined, this convex combination of risk and reward allows the investor to maximize their expected wealth while keeping a check on the risk measure.

It is important to identify the correct risk measure for every financial planning problem. Traditionally used tools for assessing and optimizing market risk assume that the portfolio return is normally distributed. In this way, the two statistical measures, mean and standard deviation, can be used to balance return and risk. However, in certain cases distribution of portfolio returns is far from the normal distribution. Since we are trying to achieve minimum return at the planning horizon, we consider *Expected Average Shortfall* (EAS) as a risk measure. It was introduced by Dempster et al., 2007, where minimum guaranteed return fund problem was solved. The advantage of such a risk measure is twofold; firstly, to manage the strategies of the fund and secondly to take into account guarantees given to the investors of the fund. Dempster et al., 2007, averaged the shortfall at each decision stage, however, in our research we consider expected average shortfall only at the horizon. The mathematical formulation of the risk measure considered is the following:

$$R_\zeta = \sum [\bar{\mu} - W_n]^+ p_n, \forall n \in \mathcal{N}_T \tag{2.5}$$

Where, $\bar{\mu}$ is user-defined target, any scenario that yields wealth (W_n) lower than this target would be reflected in the expected shortfall, shortfalls are then weighted by their probability (p_n) to calculate the expected average shortfall (EAS).

In this research, we formulate a nominal amount model (where we deal with the number of assets) to calculate the optimal investments in options and other securities. The reason for choosing this particular approach is that it would help us in tracking the number of options contracts in the inventory. For instance, in a protective put strategy, where we are going to use put options to hedge the position in the underlying and we need exactly the same number of put options as the number of underlying contracts. Consider in other cases, where we need to buy straddle or a strip or strap, we need to have proportional number of call options and put options, nominal amount model makes calculation easier. In the case of monetary amount model where inventory equations are modeled through investment in the assets, it would bring additional complexity to the model to calculate the number of options contracts to hedge the underlying position. Let us also consider the case where we buy/sell options to update underlying inventory, a nominal amount model would make it trivial to handle the inventory equations, we show this in Chapter 3.

2.1.3 Options Payoff Modelling Approach

Unlike stocks, the option payoff is non-linear. The payoff is expressed as $\max$ function. This $\max$ function if implemented in constraints in the optimization program (GAMS) would make the program non-linear. To avoid this, we define a variable $moneyness$ to track the intrinsic value of the option contract. This would help us in modeling problems where options are expiring on or before a decision stage.

We define *moneyness* (δ) of the call option on asset i with maturity j and strike k in node n as:

$$\delta^c_{in}(j,k) = max(v_{in} - K^j_{ik}, 0) \tag{2.6}$$

Similarly, *moneyness* for put option would be:

$$\delta^p_{in}(j,k) = max(K^j_{ik} - v_{in}, 0) \tag{2.7}$$

2.2 Problem Formulation: Single Stage Models

As described earlier, options can be used for different purposes depending on the requirements of the investor/trader, it can be either hedging or speculation. We now present a nominal amount model to optimize a portfolio of stock, bond and commodity indices and European call and put options on them. The maturity of the option is the same as the planning horizon for the single-stage model, later we extend the model to take into account additional complexities. Buy/sell decisions are made at time $t = 0$ and the option payoff depends on the price of the underlying assets evolved in the leaf (terminal) nodes of the scenario tree. We first present a case where options on indices are cash-settled contracts. So, the payoff of options (positive cash flow) from option expiry adds to the cash available. All the computation here is done in the context of a scenario tree optimization framework and therefore all the variables are scenario/node dependent to channel the information flow from the root node to the leaf nodes.

The aim is to optimize a portfolio with index instruments and to buy options on them to maximize profitability at the end of the planning horizon. Options expire at the planning horizon and are cash-settled. Next, we present here the set of decision variables and constraints implemented to characterize the random constraint matrices $A_n, G_n, D_n, \forall n \in \mathcal{N}$ and solve the problem.

We consider three types of constraints to be satisfied: the *inventory balance equations* define the portfolio evolution over time; the *cash balance constraints* include in each node all cash inflows and outflows generated by the current strategy; the *upper and lower bounds* on the decision vector which define policy constraints on the adoptable strategy.

For each node n of the scenario tree and the asset/derivative i, the optimal strategy is defined through the following possible decisions, x_{in} is the nominal amount *held* in asset i in node n; $c_{in}(j,k)$ and $p_{in}(j,k)$ is the nominal amount *bought* in call and put options contract on asset i in node n with strike k and maturity j respectively; x_{in^+} refers to a *buying decision* in asset i in node n; while x_{in}^- refers to a *selling decision* in asset i in node n. All the decision variables are constrained to be nonnegative.

At root node:

We first introduce the inventory equation at root node $n = 0$, the quantity (number of contracts) in root node is equal to the sum of the number of contracts held initially and the number of contracts bought in the root node less what is sold in the node n.

$$x_{i0} = \bar{x}_i + x_{i0}^+ - x_{i0}^-, \quad \forall\, i \in \mathcal{I} \tag{2.8}$$

Option Inventory:

$$c_{i0}(j,k) = c_{i0}^+(j,k), \forall (j,k) \tag{2.9}$$

$$p_{i0}(j,k) = p_{i0}^+(j,k), \forall (j,k) \tag{2.10}$$

Option inventory at time 0 is due to any purchase (buy) decisions made at the root node.

We impose a cash constraint in the first stage decision (which is the only decision stage in a single-stage problem)

$$C_0 = \bar{C} + \sum_{i\in\mathcal{I}} x^-_{i0} v_{i0}(1-\chi^-) - \sum_{i\in\mathcal{I}} x^+_{i0} v_{i0}(1+\chi^+)$$
$$- \sum_{j=T,k} [c^+_{i0}(j,k)O^c_{i0}(j,k) + p^+_{i0}(j,k)O^p_{i0}(j,k)](1+\chi^+_o) \tag{2.11}$$

Cash in the root node is the sum of cash that is held initially and the value of the assets sold in the root node less the amount invested (buying) in assets and options.

$$W_0 = C_0 + \sum_i x_{i0}v_{i0} + \sum_{i,j=T,k} [c_{i0}(j,k)O^c_{i0}(j,k) + p_{i0}(j,k)O^p_{i0}(j,k)] \tag{2.12}$$

Wealth in root node is the sum of the cash C_0 (cash in root node after rebalancing), the value of assets and the value of options in the root node.

At node: $n \in \mathcal{N}_T$

In the node n, the inventory would be equal to what is held in the previous node and what is bought in the current node less the contracts sold in the current node. Since no buying or selling decision is allowed in the leaf node the amount held in the parent node would be the same as the amount held in the current node n.

$$x_{in} = x_{i0}, \quad \forall\, i \in \mathcal{I} \tag{2.13}$$

$$x^+_{in} \;=\; x^-_{in} \;=\; 0 \tag{2.14}$$

Option Inventory:

$$c_{in}(j,k) = 0, \forall (i,j,k) \tag{2.15}$$

$$p_{in}(j,k) = 0, \forall (i,j,k) \tag{2.16}$$

At node n the position in options will always be 0, as it is the final stage and options either expire in the money or expire worthless, there are no options in the portfolio in the leaf nodes.

$$C_n = C_0 e^{r_n \Delta t} + \sum_{j,k} [c_{i0}(j,k) max(0, v_{in} - K^j_k) + p_{i0}(j,k) max(0, K^j_k - v_{in})] \tag{2.17}$$

Cash at node n is the sum of cash carried from the previous stage,(it is compounded at the annual risk-free rate r_n over the time period $\Delta t = t(n) - t(n-)$, expressed in years), and any cash inflows from options expiring in-the-money. Wealth in node n is the sum of cash available in that node and the value of assets held.

$$W_n = C_n + \sum_{i \in \mathcal{I}} x_{i0} v_{in}, \ \forall n \in \mathcal{N} - \{0\} \tag{2.18}$$

We now introduce policy constraints to consider bounds in the investment in various asset classes and options on them. These policy constraints also help in formulating strategies such as straddle and other options strategies, we show this later in this chapter. Let ϕ_L and ϕ_U be the set of lower and upper bounds on the securities, ϕ_{icL}, ϕ_{icU}, ϕ_{ipL} & ϕ_{ipU} be the upper and lower bounds on investment in call and put options on asset i respectively. All these ϕs are $card(\mathcal{I})$ X 1 vectors of values between 0 (no investment) and 1 (100% investment), this corresponds to upper or lower investment bound in the corresponding assets or derivatives, equation 2.19. These constraints can help limit investments in assets. Similarly, Equations (2.20 and 2.21) define the upper and lower bounds on investment in derivatives. Then we have non-negativity constraints (equations 2.22 & 2.23) to ensure that the decision variables are positive on all the nodes of the event tree. Equation 2.24, is the cash constraint, γ_C is the fractional wealth that is allowed to be kept in the cash account. It is important to note the index 0 in all the constraints below, it corresponds to the root node.

Policy Constraints:

$$\begin{aligned} &\phi_{iL} W_0 \leq x_{i0} v_{i0} \leq \phi_{iU} W_0, \ \phi_L = \{\phi_{iL}\}', \ \phi_{iL} \in [0,1] \\ &\phi_U = \{\phi_{iU}\}', \phi_{iU} \in [0,1], \ \forall \ i \end{aligned} \tag{2.19}$$

$$\phi_{icL} W_0 \leq c_{i0} O^c_{i0}(j,k) \leq \phi_{icU} W_0, \quad \phi_{icL} \in [0,1], \ \phi_{icU} \in [0,1], \forall \ i,j,k \tag{2.20}$$

$$\phi_{ipL} W_0 \leq p_{i0} O^p_{i0} \leq \phi_{ipU} W_0, \quad \phi_{ipL} \in [0,1], \ \phi_{ipU} \in [0,1], \forall \ i,j,k \tag{2.21}$$

Non-negativity constraints:

$$x^+_{i0} \geq 0, \ x^-_{i0} \geq 0, \ x_{i0} \geq 0, \forall \ i \ \in \mathcal{I} \tag{2.22}$$

$$c_{i0}(j,k) \geq 0, \ p_{i0}(j,k) \geq 0, \ \forall \ j \ \& \ k \tag{2.23}$$

Cash Constraint:

$$0 \leq C_0 \leq \gamma_C W_0, \ \gamma_C \in [0,1] \tag{2.24}$$

This completes the single-stage model for optimizing a portfolio of multiple asset classes with options on them, the model is similar to Topaloglou, Vladimirou, and Zenios, 2011 single-stage model.

Possible Extension: Single stage model with options in the portfolio that are expiring on or before the Horizon

Next, we extend the above model for a different possibility. It is possible that options expire before the decision stage, in that case, we need to take the profit (if options expire in-the-money) to the next decision stage compunded at the risk-free rate. We need to know all the expiries between two rebalancing stages and the price of the underlying instrument at those expiries. We need to define some variables to track the moneyness of the options.

Let $\tau(j) = [\tau_1, \tau_2, \tau_3..]$ be the maturities between $[0, T]$, a scenario based notation would be $\tau_n(j)$, i.e. the maturities between $[0, T]$ in the nth scenario. Let $S_{\tau_n(j)}$ be the price of the underlying on the intermediate maturity. Cash inflow from such expiries Θ_n would be:

$$\Theta_n = \sum_{i,j \in \tau(j) \leq T, k} [c_{i0}(j,k)\delta^c_{in}(j,k) + p_{i0}(j,k)\delta^p_{in}(j,k)]e^{r_n(t-\tau_n(j))} \tag{2.25}$$

The new model that allows options expiry before horizon would have a modified cash equation 2.17, all other features including asset balance equations and policy constraints will remain the same. So, we just write the modified cash equation for this particular case and not repeating the entire model formulation.

$$C_n = C_0 e^{r_n \Delta t} + \Theta_n, \forall n \in \mathcal{N}_T \tag{2.26}$$

We can also rewrite this equation in terms of indicator variable δ and moneyness λ defined earlier in this chapter.

$$C_n = C_0 e^{r_n \Delta t} + \sum_{i,j \in \tau(j) \leq T, k} [c_{i0}(j,k)\delta^c_{in}(j,k) + p_{i0}(j,k)\delta^p_{in}(j,k)]e^{r_n(t-\tau_n(j))} \tag{2.27}$$

This completes the single-stage model in this study. The single-stage model presented here is a basic model that sets up the base for the multi-stage models.

2.3 Multi-stage Model: Long Position in European Options

In this section, we are going to talk about European put and call options in a multi-stage setting. There could be many possibilities with options when discussing them in a multi-stage setting, selling before expiry or short selling are such possibilities to be modelled.

2.3.1 Multi-stage Model (Buying options expiring at the next decision stage)

We aim to optimize a portfolio where the investor along with the assets has European call and put options available on the underlying assets, however, to avoid any volatility risk the investor wants only those options that are expiring at the next stage, so that at any rebalancing stage no options are held in the portfolio and only buying (no selling) decisions in options contracts are made. We are going to extend the single-stage model to a multi-stage model. At each node, new options are available that expire at subsequent nodes (children nodes), Yin and Han, 2013b. The equations developed in the previous model would change because of the increased number of decision stages, equations for the multi-stage model are below:

The asset inventory equation at the root node would remain the same as in the single-stage model.

Asset Inventory Balance Equation

$$x_{i0} = \bar{x}_i + x_{i0}^+ - x_{i0}^-, \; \forall \, i \in \mathcal{I} \tag{2.28}$$

We keep the option inventory separate from the underlying assets' inventory, making it easier to track the position in options contracts.

Option Inventory Balance Equation

$$c_{i0}(j,k) = c_{i0}^+(j,k), \forall (j,k) \tag{2.29}$$

$$p_{i0}(j,k) = p_{i0}^+(j,k), \forall (j,k) \tag{2.30}$$

The cash balance constraint is imposed in the first stage takes into account the number of options contracts bought in the root node n.

$$\begin{aligned} C_0 = \bar{c} + \sum_{i \in \mathcal{I}} x_{i0}^- v_{i0}(1-\chi^-) - \sum_{i \in \mathcal{I}} x_{i0}^+ v_{i0}(1+\chi^+) \\ - \sum_{i,j=t+1,k} [c_{i0}^+(j,k) O_{i0}^c(j,k) + p_{i0}^+(j,k) O_{i0}^p(j,k)](1+\chi_o^+) \end{aligned} \tag{2.31}$$

We now write the equations and above the constraints at node n (after the first decision stage). The inventory balance constraints reflect the decision problem Markovian structure: as time evolves, along each scenario, the portfolio evolution will be fully specified in nominal value through holding, buying and selling decisions. Each such decision generates, jointly with other commitments, cash flows in each node resulting in cash surpluses or deficits to be compounded to the following stage.

Asset Inventory Balance Equation: At node $n \in \mathcal{N} - \{0\}]$

$$x_{in} = x_{in-} + x_{in}^+ - x_{in}^-, \; \forall \, i \in \mathcal{I}, \tag{2.32}$$

Contracts held in the current node is equal to contracts held in the previous node and the contracts bought less the amount sold in the current node.

Option Inventory:

$$c_{in}(j,k) = c_{in}^+(j,k), \forall (i,j,k) \tag{2.33}$$

$$p_{in}(j,k) = p^{+}_{in}(j,k), \forall (i,j,k) \tag{2.34}$$

The number of options held in a node are exactly same as the number of options bought in that node. Any options held previously would be expiring on the current decision stage, since the model assumes option expiry same as the time length between two consecutive decision stages which is constant for the entire planning horizon.

Cash Balance Constraint in node n

$$\begin{aligned} C_n = C_{n-}e^{r_n \Delta t} &+ \sum_{i\in\mathcal{I}} x^{-}_{in} v_{in}(1-\chi^{-}) - \sum_{i\in\mathcal{I}} x^{+}_{in} v_{in}(1+\chi^{+}) \\ &- \sum_{i,j=t(n^{+}),k} [c^{+}_{in}(j,k)O^{c}_{in}(j,k) + p^{+}_{in}(j,k)O^{p}_{in}(j,k)](1+\chi^{+}_{o}) \\ &+ \sum_{i,j=t(n),k} [c_{in-}(j,k)max(0, v_{in} - K^{j}_{k}) + p_{in-}(j,k)max(0, K^{j}_{k} - v_{in})](1-\chi_o) \end{aligned} \tag{2.35}$$

Cash at node n would be the sum of cash carried from the previous stage,(it is compounded by the annual risk free rate r_n over the time period $\Delta t = t(n) - t(n-)$, expressed in years), and cash inflows from options expiring in-the-money. The above equation can be re-written in terms of δ.

$$\begin{aligned} C_n = C_{n-}e^{r_n \Delta t} &+ \sum_{i\in\mathcal{I}} x^{-}_{in} v_{in}(1-\chi^{-}) - \sum_{i\in\mathcal{I}} x^{+}_{in} v_{in}(1+\chi^{+}) \\ &- \sum_{i,j=t(n^{+}),k} [c_{in}(j,k)O^{c}_{in}(j,k) + p_{in}(j,k)O^{p}_{in}(j,k)](1+\chi^{+}_{o}) \\ &+ \sum_{i,j=t(n),k} [c_{in-}(j,k)\delta^{c}_{in}(j,k) + p_{in-}(j,k)\delta^{p}_{in}(j,k)](1-\chi_o) \end{aligned} \tag{2.36}$$

Wealth at node n is the sum of the cash available in that node, value of asset investments and value of options held in that node.

$$W_n = C_n + \sum_{i\in\mathcal{I}} x_{in}v_{in} + \sum_{i,j=t(n^{+}),k} [c_{in}(j,k)O^{c}_{in}(j,k) + p_{in}(j,k)O^{p}_{in}(j,k)],\ \forall\, n \in \mathcal{N} \tag{2.37}$$

The policy and non-negativity constraints remain same as in the previous model with the exception that constraints now must hold for all the nodes on which a decision is taken i.e. $n \in \mathcal{N}_{0,T-1}$. We add one more constraint 2.46 to make sure that no decision is taken on the leaf nodes, $n \in \mathcal{N}_T$.

Policy Constraints:

$$\begin{aligned} &\phi_L W_n \le x_{in}v_{in} \le \phi_U W_n,\ \phi_L = \{\phi_{iL}\}',\ \phi_{iL} \in [0,1], \\ &\phi_U = \{\phi_{iU}\}',\ \phi_{iU} \in [0,1],\ \forall\, i \in \mathcal{I} \end{aligned} \tag{2.38}$$

$$\phi_{icL} W_n \leq c_{in} O^c_{in}(j,k) \leq \phi_{icU} W_n, \quad \phi_{icL} \in [0,1],\ \phi_{icU} \in [0,1], \forall\, i,j,k \tag{2.39}$$

$$\phi_{ipL} W_n \leq p_{in} O^p_{in}(j,k) \leq \phi_{ipU} W_n, \quad \phi_{ipL} \in [0,1],\ \phi_{ipU} \in [0,1], \forall\, i,j,k \tag{2.40}$$

Non-negativity constraints:

$$x^+_{in} \geq 0,\ x^-_{in} \geq 0,\ x_{in} \geq 0,\ \forall\, i \in \mathcal{I},\ \forall\, n \in \mathcal{N}_{0,T-1} \tag{2.41}$$

$$c_{in}(j,k) \geq 0,\ p_{in}(j,k) \geq 0,\ \forall\, n \in \mathcal{N}_{0,T-1} \tag{2.42}$$

$$c^+_{in}(j,k) \geq 0,\ p^+_{in}(j,k) \geq 0,\ \forall\, n \in \mathcal{N}_{0,T-1} \tag{2.43}$$

Cash Constraint:

$$0 \leq C_n \leq \gamma_C W_n,\ \forall n \in \mathcal{N},\ \gamma_C \in [0,1] \tag{2.44}$$

Cash Constraint:

$$0 \leq C_n \leq \gamma_C W_n,\ \forall n \in \mathcal{N},\ \gamma_C \in [0,1] \tag{2.45}$$

No decision is made on the leaf nodes:

$$x^+_{in} = 0;\ x^-_{in} = 0;\ c^+_{in} = 0,\ p^+_{in} = 0,\ \forall\, n \in \mathcal{N}_T,\ \forall\, i \in \mathcal{I} \tag{2.46}$$

This completes the multi-stage model when options are expiring at the subsequent decision stage.

Model Validation: We present some observations that validate the model we have presented. Let us consider a case where put option contracts are used to hedge the long position in the underlying. Above model then would have call option decision variable fixed at 0 and allowing purchase only in put options. We set $c_{in} = 0$ and $p_{in} \leq x_{in}$ to ensure that number of put options contracts are always less than the number of assets.

To validate the model, we consider a scenario tree having 64 branches with branching structure [1 2 2 2 2 2 2], every time step is equal to one month, monthly revision of the portfolio is allowed. Equity pricing model is used to generate prices for equity index described in Chapter 4, we use Black-Scholes model to calculate option prices, historical volatility of the underlying asset is used and risk free rate is assumed to be 0. We consider only one asset in the portfolio and ATM options on it. We analyse the worst case scenario (2.1), model buys put option in stage 6 to hedge equity positions, price does fall 1.775% and due to put options in the portfolio loss realized is only 0.5%. The constraints are implemented such that it is not obliged to have options in the portfolio in each node, depending on how the market evolves

TABLE 2.1: Model Validation: Protective Put Case

Month	1	2	3	4	5	6	7
Equity Index Value ($)	1870.85	1866.167	1891.48	1899.937	1958.478	1945.264	1910.727
Wealth ($)	100000	99980	101336.1	101789.1	104352.6	103648.4	103081.1
Units of Equity Index in the portfolio	0	53.57505	53.57505	53.28249	53.28249	52.99087	52.99087
Cash	100000	0	0	0	0	0	1830.01
Number of Call Options	0	0	0	0	0	0	0
Number of Put Options	0	0	0	53.28	0	52.99	0
% change in Equity	na	-0.0025	0.013564	0.004471	0.030812	-0.00675	-0.01775
% change in Wealth	na	-0.0002	0.013564	0.00447	0.025185	-0.00675	-0.00547
Cumulative Equity performance	0	-0.0025	0.011027	0.015548	0.046838	0.039776	0.021315
Cumulative Wealth performance	0	-0.0002	0.013361	0.017891	0.043526	0.036484	0.030811

algorithm or the optimization program decides when to buy and how many to buy. The objective here is to see if the put option purchased actually serve the hedging purpose or not. In our findings, it is validated that put options provide protection against a downside movement.

Let us now consider a case where the investor is willing to adopt a straddle strategy, strip or strap strategy. This would require to modify the constraints. We know that number of call and put options in these strategies are proportional, $c_{in} = [1, 2, 1/2]p_{in}$ for straddle, strap and strip strategies respectively. We add this constraint to the model and we see if the desired outcome is achieved or not.

TABLE 2.2: Model Validation: Straddle Strategy

Month	1	2	3	4	5	6	7
Equity Index Value ($)	1870.85	1862.765	1894.558	1877.791	1889.181	1876.274	1859.588
Wealth ($)	100000	99980.12	102215.2	101030.9	101030.9	101010.7	100112.4
Units of Equity Index in the Portfolio	0	53.06179	53.33754	0	0	53.83582	53.83582
Cash	100000	0	0	101031	101031	0	0
Number of Call Options	0	53.06	53.34	0	0	0	0
Number of Put Options	0	53.06	53.34	0	0	0	0
% change in Equity	na	-0.00432	0.017068	-0.00885	0.006065	-0.00683	-0.00889
% change in Wealth	na	-0.0002	0.022356	-0.01159	0	-0.0002	-0.00889
Cumulative Equity performance	0	-0.00432	0.012672	0.00371	0.009798	0.002899	-0.00602
Cumulative Wealth performance	0	-0.0002	0.022152	0.010309	0.010309	0.010107	0.001124

The model buys (2.2)a long straddle at the second month, equity price goes up by 1.7%, because of call options in the straddle the realised return in wealth is about 2.2%. At the very next stage it again buys a long straddle, the price of equity goes down by 0.8% and because of put options in the straddle strategy the loss stays around 1.1% (it is due to the fact that a significant amount of wealth was invested in buying the options).

The model buys a strip when the asset prices are going down (2.3). Percentage change in equity is -0.88% while in the portfolio with strip the percentage change is -0.86%. Strip strategy is more effective when market is highly volatile.

The model buys a strap in month 2 (2.4), when the asset prices are increasing, portfolio beats benchmark (equity index) by 1.6% due to call options in the portfolio. At the month 3, price decreases and portfolio realised 0.8% more loss than the

TABLE 2.3: Model Validation: Strip Strategy

Month	1	2	3	4	5	6	7
Equity Index Value ($)	1870.85	1862.765	1894.558	1877.791	1889.181	1876.274	1859.588
Wealth ($)	100000	99980	101686.2	100807.6	100807.6	100787.5	99891.13
Units of Equity Index in the Portfolio	0	53.6729	53.21964	0	0	53.71681	53.71681
Cash	100000	0	0	100807.6	100807.6	0	0
Number of Call Options	0	0	26.61	0	0	0	0
Number of Put Options	0	0	53.22	0	0	0	0
% change in Equity	na	-0.00432	0.017068	-0.00885	0.006065	-0.00683	-0.00889
% change in Wealth	na	-0.0002	0.017065	-0.00864	0	-0.0002	-0.00889
Cumulative Equity performance	0	-0.00432	0.012672	0.00371	0.009798	0.002899	-0.00602
Cumulative Wealth performance	0	-0.0002	0.016862	0.008076	0.008076	0.007875	-0.00109

TABLE 2.4: Model Validation: Strap Strategy

Month	1	2	3	4	5	6	7
Equity Index Value ($)	1870.85	1862.765	1894.558	1877.791	1889.181	1876.274	1859.588
Wealth ($)	100000	99980.18	103294.7	101494.6	101494.6	101474.3	100571.9
Units of Equity Index in the Portfolio	0	52.75175	53.58231	0	0	54.08288	54.08288
Cash	100000	0	0	101494.6	101494.6	0	0
Number of Call Options	0	105.5	107.16	0	0	0	0
Number of Put Options	0	52.75	53.58	0	0	0	0
% change in Equity	na	-0.00432	0.017068	-0.00885	0.006065	-0.00683	-0.00889
% change in Wealth	na	-0.0002	0.033151	-0.01743	0	-0.0002	-0.00889
Cumulative Equity performance	0	-0.00432	0.012672	0.00371	0.009798	0.002899	-0.00602
Cumulative Wealth performance	0	-0.0002	0.032947	0.014946	0.014946	0.014743	0.005719

benchmark, as all the call options bought expired worthless. Similarly, other strategies can be modified using different constraints.

Extension: Multi-stage model (Buying the options that are expiring on or before the next rebalancing stage)

Now, we add one more complexity to the model presented. It is possible that options are not expiring on the decision stages. So, we generalize cash equation for any node n to account for cash flows coming in from any option expiry before the next rebalancing stage.

$$\Theta_n = \sum_{i,j \in t(n) < \tau(j) \leq t(n^+),k} [c_{in}(j,k)\delta^c_{in}(j,k) + p_{in}(j,k)\delta^p_{in}(j,k)]e^{r_n(t(n)-\tau_n(j))} \tag{2.47}$$

The new model in this case will have some equations different from the previous model. The cash constraint at the root node would be different in a way that options are available for purchase on expiry set $[t, t+1]$, that is from $t = 0$ to $t = 1$, while in the previous model options are available for purchase at expiry time $t+1$.

$$\begin{aligned} C_0 = \bar{c} + \sum_{i\in\mathcal{I}} x^-_{i0} v_{i0}(1-\chi^-) - \sum_{i\in\mathcal{I}} x^+_{i0} v_{i0}(1+\chi^+) \\ - \sum_{i,t(n)\leq j \leq t(n+),k} [c_{i0}(j,k)O^c_{i0}(j,k) + p_{i0}(j,k)O^p_{i0}(j,k)](1+\chi^+_o), n = \{0\} \end{aligned} \tag{2.48}$$

Cash constraint in node n would be different from the cash constraint presented in the previous model, since the options are now available for purchase on the set $[t(n), t(n+)]$ and options are exercised on the interval $(t(n-), t(n)]$. The equation in terms of δ can be written as:

$$\begin{aligned} C_n = C_{n-}e^{r_n \Delta t} &+ \sum_{i \in \mathcal{I}} x_{in}^{-} v_{in}(1-\chi^{-}) - \sum_{i \in \mathcal{I}} x_{in}^{+} v_{in}(1+\chi^{+}) \\ &- \sum_{i, t(n) \leq j \leq t(n^{+}), k} [c_{in}^{+}(j,k) O_{in}^{c}(j,k) + p_{in}^{+}(j,k) O_{in}^{p}(j,k)](1+\chi_o^{+}) \\ &+ \sum_{i, t_{n-} < j \leq t(n), k} [c_{in-}(j,k) \delta_{in}^{c}(j,k) + p_{in-}(j,k) \delta_{in}^{p}(j,k)] e^{r_n(t(n)-\tau_n(j))}(1-\chi_o) \end{aligned} \tag{2.49}$$

The third and the last change from the previous model would be in the wealth equation. Again, the time interval of existing options would be $[t(n), t(n+)]$. Wealth then would be the sum of the cash held in the current node, investment in the assets and investment in the options.

$$W_n = C_n + \sum_{i \in \mathcal{I}} x_{in} v_{in} + \sum_{i, t(n) \leq j \leq t(n^{+}), k} [c_n(j,k) O_n^{c}(j,k) + p_n(j,k) O_n^{p}(j,k)], \ \forall\, n \in \mathcal{N} \tag{2.50}$$

This completes the multi-stage model where options are expiring on or before the next rebalancing stage.

Extension: Multi-stage model options expiring at any decision stage along the planning horizon)

We consider options in the portfolio that expire at any decision stage along the planning horizon. The motivation behind having such a portfolio is that the investor may be looking to hedge his portfolio over a long time period. This brings another complexity to the model, if an option contract has not expired then it must be reflected in the inventory equation in the children nodes. Comparing to the previous models, now options inventory is going to have a different structure. We assume that purchasing of options contracts is allowed only at time 0.

We are now extending the model presented by Yin and Han, 2013b. This would require us to modify cash constraints at root node as well as for the nth node. At root node, time span for option purchase would now be between $t = 0$ and $t = T$.

$$\begin{aligned} C_0 = \bar{C} &+ \sum_{i \in \mathcal{I}} x_{i0}^{-} v_{i0}(1-\chi^{-}) - \sum_{i \in \mathcal{I}} x_{i0}^{+} v_{i0}(1+\chi^{+}) \\ &- \sum_{i, j \leq T, k} [c_{i0}(j,k) O_{i0}^{c}(j,k) + p_{i0}(j,k) O_{i0}^{p}(j,k)](1+\chi_o^{+}) \end{aligned} \tag{2.51}$$

The option inventory at time 0 would remain the same as in the previous model.

$$c_{i0}(j,k) = c_{i0}^{+}(j,k), \forall (i,j,k) \tag{2.52}$$

$$p_{i0}(j,k) = p_{i0}^{+}(j,k), \forall (i,j,k) \tag{2.53}$$

However, at node n the inventory would be different. Since no option purchase is allowed at the decision stages and only options that are bought in root node are allowed to carry to the next nodes, therefore, the position in the option in the current node would simply be the option position in the previous node, provided that option contract has not expired. If option contracts have expired then $c_{in}(j,k) = 0$ & $p_{in}(j,k) = 0$, otherwise

$$c_{in}(j,k) = c_{in-}(j,k), \forall (j,k) \tag{2.54}$$

$$p_{in}(j,k) = p_{in-}(j,k), \forall (j,k) \tag{2.55}$$

This change in inventory equations must be reflected in cash constraint at node n. So, the cash in node n would be the sum of cash carried from the previous node, cash inflow and outflow due to selling or buying the underlying assets and cash coming in from the expiry of the options in that node.

$$\begin{aligned} C_n = C_{n-}e^{r_n \Delta t} + \sum_{i \in \mathcal{I}} x_{in}^{-} v_{in}(1-\chi^{-}) - \sum_{i \in \mathcal{I}} x_{in}^{+} v_{in}(1+\chi^{+}) \\ + \sum_{i,j=t(n),k} [c_{in-}(j,k)\delta_{in}^{c}(j,k) + p_{in-}(j,k)\delta_{in}^{p}(j,k)](1-\chi_o) \end{aligned} \tag{2.56}$$

Wealth in node n would be the sum of cash held and the investment in assets and options in that node.

$$W_n = C_n + \sum_{i \in \mathcal{I}} x_{in} v_{in} + \sum_{i,t(n)<j\leq T,k} [c_{in}(j,k) O_{in}^{c}(j,k) + p_{in}(j,k) O_{in}^{p}(j,k)], \ \forall\, n \in \mathcal{N} \tag{2.57}$$

This completes the multi-stage model where we have options expiring at any decision stage along the planning horizon.

Extension: Multi-stage model for options expiring at any time point along the planning horizon)

We add a little complexity to the previous model that options are expiring at any time point along the horizon (not necessarily on the decision stages). We would now extend the previous model with the equations developed in the beginning to consider options expiring before decision stages. The only equations that would change are the cash balance equation and wealth equation in node n.

$$C_n = C_{n-}e^{r_n \Delta t} + \sum_{i \in \mathcal{I}} x_{in}^{-} v_{in}(1-\chi^{-}) - \sum_{i \in \mathcal{I}} x_{in}^{+} v_{in}(1+\chi^{+})$$
$$+ \sum_{i, t(n-)<j \leq t(n), k} [c_{in-}(j,k)\delta_{in}^{c}(j,k) + p_{in-}(j,k)\delta_{in}^{p}(j,k)] e^{r_n(t(n)-\tau_n(j))}(1+\chi_o) \tag{2.58}$$

The difference here is the time interval on which the options are exercised, $(t(n-), t(n)]$ unlike at $t(n)$ in the previous model. This completes a multi-stage model where options are expiring at any time point along the horizon.

2.3.2 Multi-stage model: Buying, selling (no short selling) and exercising options at any time point along the horizon

We introduce another complexity to the previous models. This time we allow selling of the options that have not expired. This allows us to transfer risk from the buyer of the option to writer of the option, for instance, we buy an option at time 0 to hedge our position before expiry at any decision stage the option was found to be in the money, so we can sell this option and can profit from its premium and buy another option to hedge our position. Option premiums are very lucrative, these are risky but at the same time they can be rewarding as well. This is what we try to capture in the model discussed below, Davari-Ardakani, Aminnayeri, and Seifi, 2016 also attempted to develop a model like this, however, we do not agree with some technicalities of the model.

Asset Inventory Equation at root node

$$x_{i0} = \bar{x}_i + x_{i0}^{+} - x_{i0}^{-}, \; \forall \, i \in \mathcal{I} \tag{2.59}$$

Option Inventory Equation:

$$c_{i0}(j,k) = c_{i0}^{+}(j,k) - c_{i0}^{-}(j,k); \tag{2.60}$$

$$p_{i0}(j,k) = p_{i0}^{+}(j,k) - p_{i0}^{-}(j,k); \tag{2.61}$$

The options in the node n are equal to options bought (+ superscript) in that node minus options sold (- superscript) in that node. Since no short selling is allowed constraints are introduced in the later part of the model.

Cash Balance Constraint at root node:

$$\begin{aligned} C_0 = \bar{C} &+ \sum_{i \in \mathcal{I}} x_{i0}^{-} v_{i0}(1-\chi^{-}) - \sum_{i \in \mathcal{I}} x_{i0}^{+} v_{i0}(1+\chi^{+}) \\ &- \sum_{i, j \leq T, k} [c_{in}^{+}(j,k) O_{in}^{c}(j,k) + p_{i0}^{+}(j,k) O_{i0}^{p}(j,k)](1+\chi_o^{+}) \end{aligned} \tag{2.62}$$

While imposing cash balance constraint at the root node, we make sure that no selling of options is allowed, as it is our assumption that there are no initial position in options, selling at time 0 would refer to short selling which is not in the scope of this model.

Asset Inventory Equation: At node $n \in \mathcal{N} - \{0\}]$

$$x_{in} = x_{in-} + x^{+}_{in} - x^{-}_{in},\ \forall\, i \in \mathcal{I}, \tag{2.63}$$

Asset inventory would have the same Markovian structure as in the previous multistage models.

Option Inventory Equations in node $n \in \mathcal{N} - \{0\}]$

$$c_{in}(j,k) = \begin{cases} c_{in-}(j,k) & \text{if } j \le t_n, \forall\, k, \\ c_{in-}(j,k) + c^{+}_{in}(j,k) - c^{-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

$$p_{in}(j,k) = \begin{cases} p_{in-}(j,k) & \text{if } j \le t_n, \forall\, k, \\ p_{in-}(j,k) + p^{+}_{in}(j,k) - p^{-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

We replicate the inventory equations to formulate the option inventory equations. The only difference comes from the maturity of the options. If an option is at maturity then the position in the option in the previous node is moved to the current node as it is and it settled in the cash equation. If not, then buying and selling in that option contract is allowed. The net position would be the sum of what is held previously and what is bought in the current node less what is sold in the current node.

Cash Balance Constraint in: $n \in \mathcal{N} - \{0\}]$

$$\begin{aligned} C_n = {} & C_{n-}e^{r_n \Delta t} + \sum_{i\in\mathcal{I}} x^{-}_{in} v_{in}(1-\chi^{-}) - \sum_{i\in\mathcal{I}} x^{+}_{in} v_{in}(1+\chi^{+}) \\ & - \sum_{i,t(n)\le j\le T,k} [c^{+}_{in}(j,k)O^{c}_{in}(j,k) + p^{+}_{in}(j,k)O^{p}_{in}(j,k)](1+\chi^{+}_{o}) \\ & + \sum_{i,t(n)\le j\le T,k} [c^{-}_{in}(j,k)O^{c}_{in}(j,k) + p^{-}_{in}(j,k)O^{p}_{in}(j,k)](1-\chi^{-}_{o}) \\ & + \sum_{i,t(n^{-})\le j\le t(n),k} [c_{in}(j,k)\delta^{c}_{in}(j,k) + p_{in}(j,k)\delta^{p}_{in}(j,k)]e^{r_n(t(n)-\tau_n(j)})(1-\chi_o) \end{aligned} \tag{2.64}$$

Cash constraint clearly identifies cash outflow and inflow due to trade in underlying assets, any cash outflow due to options purchase, cash inflow due to options selling and expiries. Wealth in node n then would simply be the sum of cash available in that node, value of the underlying assets and options contracts.

$$W_n = C_n + \sum_{i\in\mathcal{I}} x_{in}v_{in} + \sum_{i,t(n)<j\le T,k} [c_{in}(j,k)O^{c}_{in}(j,k) + p_{in}(j,k)O^{p}_{in}(j,k)],\ \forall\, n \in \mathcal{N} \tag{2.65}$$

Policy constraints remain the same as in the previous model, non-negativity constraints are now extended to $c^{+}_{in}, p^{+}_{in}, c^{-}_{in} \& p^{-}_{in}$.

Policy Constraints:

$$\phi_L W_n \leq x_{in} v_{in} \leq \phi_U W_n,\ \phi_L = \{\phi_{iL}\}',\ \phi_{iL} \in [0,1], \tag{2.66}$$
$$\phi_U = \{\phi_{iU}\}',\ \phi_{iU} \in [0,1],\ \forall\, i \in \mathcal{I} \tag{2.67}$$

$$\phi_{icL} W_n \leq c_{in} O^c_{in}(j,k) \leq \phi_{icU} W_n, \quad \phi_{icL} \in [0,1],\ \phi_{icU} \in [0,1], \forall\, i,j,k \tag{2.68}$$

$$\phi_{ipL} W_n \leq p_{in} O^p_{in}(j,k) \leq \phi_{ipU} W_n, \quad \phi_{ipL} \in [0,1],\ \phi_{ipU} \in [0,1], \forall\, i,j,k \tag{2.69}$$

Non-negativity constraints:

$$x^+_{in} \geq 0,\ x^-_{in} \geq 0,\ x_{in} \geq 0,\ \forall\, i \in \mathcal{I},\ \forall\, n \in \mathcal{N}_{0,T-1} \tag{2.70}$$

$$\forall\, n \in \mathcal{N}_{0,T-1} \quad \begin{array}{c} c_{in}(j,k) \geq 0,\ p_{in}(j,k) \geq 0, \\ c^+_{in}(j,k) \geq 0, p^+_{in}(j,k) \geq 0, \\ c^-_{in}(j,k) \geq 0, p^-_{in}(j,k) \geq 0 \end{array} \tag{2.71}$$

$$x^-_{i0} \leq \bar{x}_i,\ \forall\, i \in \mathcal{I} \tag{2.72}$$

Cash Constraint:

$$0 \leq C_n \leq \gamma_C W_n,\ \forall n \in \mathcal{N},\ \gamma_C \in [0,1] \tag{2.73}$$

No decision is made on the leaf nodes.

$$x^+_{in} = 0;\ x^-_{in} = 0;\ c_{in} = 0,\ p_{in} = 0,\ \forall\, n \in \mathcal{N}_T,\ \forall\, i \in \mathcal{I} \tag{2.74}$$

This completes the model where buying and selling of option is allowed before expiry.

Model Validation

We now validate this model where buying and selling of options take place before their expiry. Table 2.5 shows how the price evolves in the scenario corresponding to the 75th percentile of the price distribution at the horizon. We have the prices for equity, bond and commodity indices and prices for ATM call and put options expiring at 1 month, 3 months and 6 months time. It can be seen clearly that after the maturity of options their price in the scenario is 0. Table 2.6, 2.7 and 2.8 show hold, buy and selling decisions made in the model. Option payoff at expiry is either 0 or some positive value, depending on if it expires ITM or OTM. So, in the GAMS code option payoff can be treated like a payoff of an asset that goes to 0 at a certain point, forcing the algorithm to not buy when the price in the next node is 0 and sell all the quantity if the price in the next node is 0.

Table 2.6, 2.7 and 2.8 show that options were bought and sold before the expiry. Call option on Equity with 6 month expiry was bought in the 3rd stage and then sold in the 4th stage, and was again bought in the 5th stage to drive the optimal investment strategy.

TABLE 2.5: Mode Validation: Buy/Sell Options *Price Evolution*

Month	0	1	2	3	4	5	6
Equity	1870.8	1904.4	1845.5	2061.6	2060.3	2055.3	2103
Bond	1931.2	1931	1950.6	1944.1	1925.7	1928.6	1921.7
Commodity	5018.1	4997.7	4916.2	4830.4	4708.4	4573.3	4409.2
Eq_Call_1M	16.552	33.594	0	0	0	0	0
Eq_Call_3M	24.198	39.398	2.6274	190.75	0	0	0
Eq_Call_6M	33.178	48.527	12.228	193.43	191.66	185.46	232.18
Eq_Put_1M	16.552	0	0	0	0	0	0
Eq_Put_3M	24.198	3.6751	27.118	0	0	0	0
Eq_Put_6M	33.178	9.6151	34.013	0.0004548	6.70E-06	1.17E-10	0

TABLE 2.6: Model Validation: Buy/Sell options before expiry (*holding decisions*)

Month	0	1	2	3	4	5	6
Equity	26.439	0	27.528	24.981	0	24.384	24.384
Bond	25.839	52.268	26.082	28.976	58.074	29.631	29.631
Commodity	0	0	0	0	0	0	0
Eq_Call_1M	26.439	0	0	0	0	0	0
Eq_Call_3M	0	0	27.528	0	0	0	0
Eq_Call_6M	0	0	0	24.981	0	24.384	24.384
Eq_Put_1M	0	0	0	0	0	0	0
Eq_Put_3M	0	0	0	0	0	0	0
Eq_Put_6M	0	0	0	0	0	0	0

The intrinsic value of the option is driven by the price of its underlying security and the time to maturity. When the price was declining the option was sold and when it was increasing, it was held in the portfolio. This completes validation of the model developed for selling options before their expiry.

2.4 A Generic Multi-Stage Model: Long/Short Positions in Options

So, far we have talked about long positions in options. Now, we are going to introduce a generic model that considers short selling of options. Short-selling of options would allow us to make profit the price movements in the underlying and it would help us in implementing hedged strategies like butterfly spread.

TABLE 2.7: Model Validation: Buy/Sell Options (*buying decisions*)

Month	0	1	2	3	4	5	6
Equity	26.439	0	27.528	0	0	24.384	0
Bond	25.839	26.429	0	2.8944	29.098	0	0
Commodity	0	0	0	0	0	0	0
Eq_Call_1M	26.439	0	0	0	0	0	0
Eq_Call_3M	0	0	27.528	0	0	0	0
Eq_Call_6M	0	0	0	24.981	0	24.384	0
Eq_Put_1M	0	0	0	0	0	0	0
Eq_Put_3M	0	0	0	0	0	0	0
Eq_Put_6M	0	0	0	0	0	0	0

TABLE 2.8: Model Validation: Buy/Sell Options (*selling decisions*)

Month	0	1	2	3	4	5	6
Equity	0	26.439	0	2.5467	24.981	0	0
Bond	0	0	26.186	0	0	28.443	0
Commodity	0	0	0	0	0	0	0
Eq_Call_1M	0	26.439	0	0	0	0	0
Eq_Call_3M	0	0	0	27.528	0	0	0
Eq_Call_6M	0	0	0	0	24.981	0	0
Eq_Put_1M	0	0	0	0	0	0	0
Eq_Put_3M	0	0	0	0	0	0	0
Eq_Put_6M	0	0	0	0	0	0	0

We are going to replicate the option inventory equations introduced in the above model. However, we would separate inventory for long and short positions. We introduce $c^l_{in}, c^{l+}_{in}, c^{l-}_{in}$ for *hold*, *buy* and *sell* decisions in long call options and $c^s_{in}, c^{s+}_{in}, c^{s-}_{in}$ for *hold*, *buy* and *sell* decisions in short call options. Similarly, we have variables for put options, $p^l_{in}, p^{l+}_{in}, p^{l-}_{in}, p^s_{in}, p^{s+}_{in}, p^{s-}_{in}$ for *hold*, *buy* and *sell* decisions in long and short put options. The asset inventory equations remain the same. This model in special cases should be equivalent to the models discussed by, Topaloglou, Vladimirou, and Zenios, 2011 and Yin and Han, 2013b. The model goes as follows:

Asset Inventory Equation: At root node

$$x_{i0} = \bar{x}_i + x^+_{i0} - x^-_{i0}, \; \forall \, i \in \mathcal{I} \tag{2.75}$$

The options in the node n are equal to options bought (+ superscript) in that node minus options sold (- superscript) in that node.

Options Inventory Equation:

For long position:

$$c^l_{i0}(j,k) = c^{l+}_{i0}(j,k) - c^{l-}_{i0}(j,k); \tag{2.76}$$

$$p^l_{i0}(j,k) = p^{l+}_{i0}(j,k) - p^{l-}_{i0}(j,k); \tag{2.77}$$

For short position:

$$c^s_{i0}(j,k) = c^{s+}_{i0}(j,k) - c^{s-}_{i0}(j,k); \tag{2.78}$$

$$p^s_{i0}(j,k) = p^{s+}_{i0}(j,k) - p^{s-}_{i0}(j,k); \tag{2.79}$$

The cash constraint in node 0 tackles cash inflow and outflow due to sale and purchase in the underlying assets, any long position in call and put options is modeled as cash outflow and short position in option is modeled as cash inflow.

Cash Constraint at node 0:

$$\begin{aligned} C_0 = \bar{C} &+ \sum_{i\in\mathcal{I}} x^-_{i0} v_{i0}(1-\chi^-) - \sum_{i\in\mathcal{I}} x^+_{i0} v_{i0}(1+\chi^+) \\ &- \sum_{i,j\le T,k} [c^{l+}_{in}(j,k) O^c_{in}(j,k) + p^{l+}_{i0}(j,k) O^p_{i0}(j,k)](1+\chi^+_o) \\ &+ \sum_{i,j\le T,k} [c^{s+}_{in}(j,k) O^c_{in}(j,k) + p^{s+}_{i0}(j,k) O^p_{i0}(j,k)](1-\chi^-_o) \end{aligned} \tag{2.80}$$

Asset Inventory Equation: At node $n \in \mathcal{N} - \{0\}]$
It remains the same as there are no modifications done on this part of the model.

$$x_{in} = x_{in-} + x^+_{in} - x^-_{in},\ \forall\, i \in \mathcal{I}, \tag{2.81}$$

updating option inventory

For long Position:

$$c^l_{in}(j,k) = \begin{cases} c^l_{in-}(j,k) & \text{if } j \le t_n, \forall\, k, \\ c^l_{in-}(j,k) + c^{l+}_{in}(j,k) - c^{l-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

$$p^l_{in}(j,k) = \begin{cases} p^l_{in-}(j,k) & \text{if } j \le t_n, \forall\, k, \\ p^l_{in-}(j,k) + p^{l+}_{in}(j,k) - p^{l-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

For Short Position:

$$c^s_{in}(j,k) = \begin{cases} c^s_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ c^s_{in-}(j,k) + c^{s+}_{in}(j,k) - c^{s-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

$$p^s_{in}(j,k) = \begin{cases} p^s_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ p^s_{in-}(j,k) + p^{s+}_{in}(j,k) - p^{s-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

We replicate the inventory equations to formulate the long/short option inventory equations. The only difference comes from the maturity of the options. If an option is at maturity then the position in the option in the previous node is moved to the current node as it is and it settled in the cash equation. If not, then buying and selling in that option contract is allowed. The net position would be the sum of what is held previously and what is bought in the current node less what is sold in the current node. However, the short position would need to take into account margin adjustments. This goes as follows.

$$m^c_{in}(j,k) = c^s_{in}(j,k)max(0, v_{in} - K^j_i)$$

$$m^p_{in}(j,k) = p^s_{in}(j,k)max(0, K^j_i - v_{in})$$

Cash Constraint in node n:

$$\begin{aligned} C_n = {} & C_{n-}e^{r_n \Delta t} + \sum_{i \in \mathcal{I}} x^-_{in} v_{in}(1-\chi^-) - \sum_{i \in \mathcal{I}} x^+_{in} v_{in}(1+\chi^+) \\ & - \sum_{t_n \leq j \leq T, k} [c^{l+}_{in}(j,k)O^c_{in}(j,k) + p^{l+}_{in}(j,k)O^p_{in}(j,k) - c^{s+}_{in}(j,k)O^c_{in}(j,k) - p^{s+}_{in}(j,k)O^p_{in}(j,k)] \\ & + \sum_{t_n \leq j \leq T, k} [c^{l-}_{in}(j,k)O^c_{in}(j,k) + p^{l-}_{in}(j,k)O^p_{in}(j,k) - c^{s-}_{in}(j,k)O^c_{in}(j,k) + p^{s-}_{in}(j,k)O^p_{in}(j,k)] \\ & + \sum_{t_{n^-} \leq j \leq t_n, k} [c^l_{in}(j,k)max(0, v_{in} - K^j_i) + p^l_{in}(j,k)max(0, K^j_i - v_{in}) \\ & - c^s_{in}(j,k)max(0, v_{in} - K^j_i) + p^s_{in}(j,k)max(0, K^j_i - v_{in})] \\ & - \sum_{k, j > t_n} (m^c_{in}(j,k) + m^p_{in}(j,k) \end{aligned} \tag{2.82}$$

The cash constraint here is the most complex of all the cash constraints discussed so far. It is a combination of cash outflows and inflows due to sale or purchase in underlying assets and long or short position in options, cash inflows from the options expiry and in addition, any margin outflow.

Wealth in node n would be the sum of cash available and value of investments in long assets and derivatives less the position in short derivatives contracts.

$$
\begin{aligned}
W_n = C_n + \sum_{i \in \mathcal{I}} x_{in} v_{in} + \sum_{t_n < j \leq T, k} &[c^l_{in}(j,k) O^c_{in}(j,k) \\
+ p^l_{in}(j,k) O^p_{in}(j,k) - c^s_{in}(j,k) O^c_{in}(j,k) &- p^s_{in}(j,k) O^p_{in}(j,k)], \; \forall \, n \in \mathcal{N}
\end{aligned} \tag{2.83}
$$

Policy constraints remain the same as in the previous model, non-negativity constraints are extended to the new variables introduced.

Policy Constraints:

$$\phi_L W_n \leq x_{in} v_{in} \leq \phi_U W_n, \; \phi_L = \{\phi_{iL}\}', \; \phi_{iL} \in [0,1], \tag{2.84}$$

$$\phi_U = \{\phi_{iU}\}', \; \phi_{iU} \in [0,1], \; \forall \, i \in \mathcal{I} \tag{2.85}$$

$$\phi_{icL} W_n \leq c_{in} O^c_{in}(j,k) \leq \phi_{icU} W_n, \quad \phi_{icL} \in [0,1], \; \phi_{icU} \in [0,1], \forall \, i,j,k \tag{2.86}$$

$$\phi_{ipL} W_n \leq p_{in} O^p_{in}(j,k) \leq \phi_{ipU} W_n, \quad \phi_{ipL} \in [0,1], \; \phi_{ipU} \in [0,1], \forall \, i,j,k \tag{2.87}$$

Non-negativity constraints:

$$x^+_{in} \geq 0, \; x^-_{in} \geq 0, \; x_{in} \geq 0, \; \forall \, i \in \mathcal{I}, \; \forall \, n \in \mathcal{N}_{0,T-1} \tag{2.88}$$

$$
\begin{aligned}
& c^l_{in}(j,k) \geq 0, \; p^l_{in}(j,k) \geq 0, \\
& c^s_{in}(j,k) \geq 0, \; p^s_{in}(j,k) \geq 0, \\
\forall \, n \in \mathcal{N}_{0,T-1} \quad & c^{l+}_{in}(j,k) \geq 0, p^{l+}_{in}(j,k) \geq 0, \\
& c^{l-}_{in}(j,k) \geq 0, p^{l-}_{in}(j,k) \geq 0 \\
\forall \, n \in \mathcal{N}_{0,T-1} \quad & c^{s+}_{in}(j,k) \geq 0, p^{s+}_{in}(j,k) \geq 0, \\
& c^{s-}_{in}(j,k) \geq 0, p^{s-}_{in}(j,k) \geq 0
\end{aligned} \tag{2.89}
$$

$$x^-_{i0} \leq \bar{x}_i, \; \forall \, i \in \mathcal{I} \tag{2.90}$$

Cash Constraint:

$$0 \leq C_n \leq \gamma_C W_n, \; \forall n \in \mathcal{N}, \; \gamma_C \in [0,1] \tag{2.91}$$

No decision is made on the leaf nodes.

$$x^+_{in} = 0; \; x^-_{in} = 0; \; ncl^+_{in} = 0, \; npl^+_{in} = 0, \; \forall \, n \in \mathcal{N}_T, \; \forall \, i \in \mathcal{I} \tag{2.92}$$

This completes the model where buying and short selling of option is allowed before expiry.

2.4.1 Theoretical Validation of the Generic Model

The theoretical model developed above can be reduced to the single-stage model we started with, or to the other models that we have developed in this chapter. As the asset inventory equations remain the same throughout all these models and the only difference comes from the options intricacies. We see the option inventory equation of the generic model first and then start making reductions to it:

For long Position:

$$c^l_{in}(j,k) = \begin{cases} c^l_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ c^l_{in-}(j,k) + c^{l+}_{in}(j,k) - c^{l-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

$$p^l_{in}(j,k) = \begin{cases} p^l_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ p^l_{in-}(j,k) + p^{l+}_{in}(j,k) - p^{l-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

For Short Position:

$$c^s_{in}(j,k) = \begin{cases} c^s_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ c^s_{in-}(j,k) + c^{s+}_{in}(j,k) - c^{s-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

$$p^s_{in}(j,k) = \begin{cases} p^s_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ p^s_{in-}(j,k) + p^{s+}_{in}(j,k) - p^{s-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

We first remove the short-selling feature from this model, then we get rid of c^s, p^s variables. The model reduces to the buy/sell (long) model with the following inventory equations. (c^l, p^l are replace by c, p respectively)

$$c_{in}(j,k) = \begin{cases} c_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ c_{in-}(j,k) + c^{+}_{in}(j,k) - c^{-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

$$p_{in}(j,k) = \begin{cases} p_{in-}(j,k) & \text{if } j \leq t_n, \forall\, k, \\ p_{in-}(j,k) + p^{+}_{in}(j,k) - p^{-}_{in}(j,k) & \text{if } j > t_n, \forall\, k \end{cases}$$

Next we remove the selling feature from the model to reduce it to a multi-stage model where options are expiring at any stage (bought only at time 0)

$$c_{in}(j,k) = c_{in-}(j,k), \forall (j,k), j > t(n) \tag{2.93}$$

$$p_{in}(j,k) = p_{in-}(j,k), \forall (j,k), j > t(n) \tag{2.94}$$

Next we reduce it to a model where options are expiring at the subsequent stages, Yin and Han, 2013b. The option inventory is due to the option bought in that node.

$$c_{in}(j,k) = c^{+}_{in}(j,k), \forall (j,k), j = t(n+) \tag{2.95}$$

$$p_{in}(j,k) = p^{+}_{in}(j,k), \forall (j,k), j = t(n+) \tag{2.96}$$

Finally, to a single stage model, where in node n, nc,np are equal to 0, Topaloglou, Vladimirou, and Zenios, 2011.

$$c_{0n}(j,k) = c^{+}_{0i-}(j,k), \forall (j,k), j = T \tag{2.97}$$

$$p_{0n}(j,k) = p^{+}_{0i}(j,k), \forall (j,k), j = T \tag{2.98}$$

Model Validation: Generic Model

We are now going to validate the model where short positions on options are considered. To validate the model, we are going to put some constraints in the model that would restrict the naked short selling of options. We are interested in short-selling but in a hedged environment, the short position in options could expose the portfolio to unlimited losses. Therefore, we implement a vertical spread (bull call spread) strategy to check its implications in the optimization process.

A bull call spread is formed by buying a call option in-the-money and going short on a call option which is out-of-the-money, with the same expiry. The constraint for bull call spread would be:

$$\sum_{k \in K^*} c^l_n(j,k) + \sum_{k \in K^{**}} c^s_n(j,k) \leq \phi_s W_n, \forall j = t(n), n \in \mathcal{N} \tag{2.99}$$

$$c^l_n(j,k) = c^s_n(j,k), \forall (j,k),\ n \in \mathcal{N}, \tag{2.100}$$

where K^* and K^{**} are sets of in-the-money (ITM) and out-of-the-money (OTM) options. We implemented this constraint in the model and validated it. In the figure 2.1, we show the payoff of a call bull spread formed using 5% ITM and OTM call options expiring in six months time. In the figure, x-axis and y-axis represent monthly time steps and strategy payoff respectively. Both profit and loss are capped.

TABLE 2.9: Model Validation: Short Position in options (Hold decisions)

Month	0	1	2	3	4	5	6
Equity	21.366	20.769	19.815	0	0	30.897	30.897
Bond	25.839	26.599	27.94	0	28.629	28.629	28.629
Commodity	0	0	0	0	0	0	0
call option (ITM)	106.83	103.85	99.076	0	0	0	0
CALL Option (OTM)	-106.83	-103.85	-99.076	0	0	0	0

It can be seen from the table 2.9 & 2.10 that short and long positions in options have changed as the price of the underlying was changing while the number of long and short options are exactly the same at each time step, the position always remained hedged. The buy/sell decisions in long/short positions in options drive the

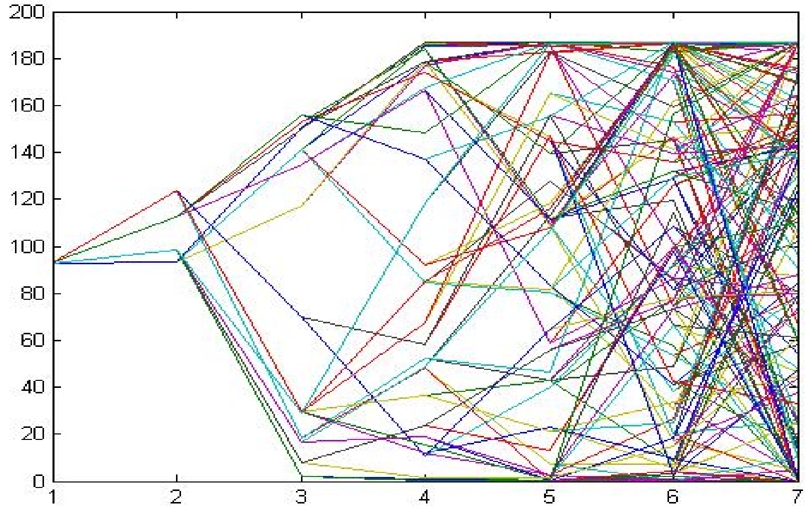

FIGURE 2.1: Call Bull Spread Payoff

TABLE 2.10: Model Validation: Short Position in Options (Price)

Month	0	1	2	3	4	5	6
Equity	1870.8	1893.1	1944.2	1979.5	1927.4	1760.3	1759.9
Bond	1931.2	1917.1	1921.5	1929.3	1940.1	1930.9	1927.5
Commodity	5018.1	4956	4925.3	4902.1	4791.3	4743.6	4611.5
Call Option 5% ITM	95.65	116.34	166.86	202.18	150.05	3.86	0
Call Option 5% OTM	2.59	3.9	13.82	28.12	3.75	0	0

optimal investment policy.

So far, we have discussed cash-settled European options in the optimization models. In the next Chapter (3), we are going to discuss an optimization model to update inventory using physical options.

Chapter 3

Model Extension Based on Derivatives' Inventory

In the previous chapter, we have discussed models and approaches where options contracts are cash-settled. In this chapter we explore physically settled options contracts in a multi-stage programming framework, this could have wide applications across various markets. For instance, a production house that needs copper to manufacture its products can buy copper form a commodity exchange. Options contracts can be used to buy the physical copper commodity at the settlement of the option (at the strike price). If options purchased expire in the money at the maturity then the buyer would have the right to buy the underlying at the strike price which would be lower than the market price, this way a firm can accumulate more copper metal (underlying asset) from the exchange at a better price than the price in the spot market. On the other hand, for a firm that has copper in its inventory fearing a decline in the prices of copper in the future could use put options contracts to sell underlying at a price higher than the market price.

This approach is not limited to a commodity trader, let us consider a broker who wants to purchase a number of shares of a company XYZ over the next few months for his client at a price lower than the market price. He can use options contracts to accumulate stocks, the objective function in this optimization program gives flexibility to the trader/broker to achieve an inventory with a certain buying cost, for instance, *VWAP* (Volume Weighted Average Price).

Barbaros and Bagajewicz (2004) introduce a model to use inventory and options to hedge financial risk, they analyze how the shape of the risk curves change when options are introduced in capacity expansion problem. The study showed that the usual assumption that with options contracts will by itself reduce the risk exposure at small profits is not always true and that proper risk management tools are needed for this purpose as well. Some authors have also considered futures contracts to manage inventory. This has seen wide applications where energy derivatives are involved, (Bertochhi et al (2011)).

In this chapter, we limit our research to the use of options contracts. The rest of the chapter is organized as follows. Section 1 of the chapter introduces sets and parameters that are needed to formulate a stochastic programming model. Section 2 introduces a multistage model. In section 3 we discuss possible extensions of the model. Section 4 is dedicated to model validation.

3.1 Multi-stage model: Using options contracts to update the inventory

3.1.1 Sets, Parameters and Variables

We describe here parameters and sets needed to setup multistage model.

Sets

- $\mathcal{T}$, set of discrete time space indexed by t, $\mathcal{T} : t = \{0, 1, 2..., T\}$
- $\mathcal{N}$, Set of nodes in the scenario tree indexed by n, $\mathcal{N}_t$ is the set of nodes at time stage t
  (Every $n \in \mathcal{N}_t$ has a unique ancestor $n- \in \mathcal{N}_{t-1}$ and for $t \leq T - 1$ there exists a non-empty set of nodes $n+ \in \mathcal{N}_{t+1}$)
- $\mathcal{I}$, set of financial assets, indexed by i
- $\mathcal{O}$, set of vanilla options
  - $\mathcal{O}^c$ & $\mathcal{O}^p$ are set of call and put options respectively
  - $\mathcal{J}$, set of expiries of the options $\mathcal{O}$, indexed by j, $\mathcal{O}_j$ represents the set of options expiring at maturity j, $\mathcal{J} : j = \{J_1, J_2...\}$
  - $\mathcal{K}$, set of strikes of the options in $\mathcal{O}$, indexed by k, $\mathcal{O}_k$ represents the set of options with strike price K, $\mathcal{K} : k = \{K_1, K_2...\}$, K^j represents the vector of strikes at maturity j, $K^j = \{K_1^j, K_2^j, K_3^j....\}$ indexed by k^j

Input Parameters

- $\bar{x}_i$, the initial position in asset $i \in \mathcal{I}$
- $\bar{c}$, is the initial available cash
- T, length of planning horizon
- χ^+ and chi^-, are the proportional transaction cost for purchase and sale in underlying
- χ_o^+, χ_o^- and χ_o, are proportional transaction costs on buying, selling and exercising option respectively.
- μ, user defined target
- v_{i0}, current price of the asset i per unit face value
- $O_{i0}^c(j, k)$, is the price of the European call option on asset i with expiry j and strike price k
- $O_{i0}^p(j, k)$, is the price of the European put option on the asset i with expiry j and strike price k

Now, we introduce parameters that would model the flow of information along the scenario tree.

Scenario Dependent Parameters

- $p(n)$, probability of node $n \in \mathcal{N}$ such that $\sum_{n \in \mathcal{N}_T} p(n) = 1$ and for every non-terminal node $p(n) = \sum_{m \in n+} p(m)$, $\forall n \in \mathcal{N}_t$, $t \leq T-1$
- v_{in}, price of asset i, in node n
- $O^c_{in}(j,k)$, is the price of the European call option on the asset i in node n, with strike price equal to $K_k, \forall K_k \in \mathcal{K}$ that expires at $t_j = J, j \in \mathcal{J}$
- $O^p_{in}(j,k)$, is the price of the European put option on the asset i in node n, with strike price equal to $K_k, \forall K_k \in \mathcal{K}$ that expires at $t_j = J, j \in \mathcal{J}$

Computed Parameters

Value of the initial portfolio

$$W\_0 = \bar{c} + \sum_i \bar{x}_i v_{i0}, \tag{3.1}$$

Decision Variables

- x^+_{in}, nominal amount of asset i purchased in node n
- x^-_{in}, nominal amount of asset i sold in node n
- x_{in}, nominal amount of asset i held in node n in the revised portfolio
- $c_{in}(j,k)$, units purchased of a European call option on asset i with expiry j and strike price k
- $p_{in}(j,k)$, units purchased of a European put option on asset i with expiry j and strike price k

We assume that there is no initial position in the options contracts.

$$\bar{c_{i0}} = 0, \forall i \in I, \bar{p_{i0}} = 0, \forall i \in I \tag{3.2}$$

Auxiliary Variables

- W_n, value of portfolio in node n

We consider an optimization problem to update the inventory of the underlying assets over a six-month horizon with monthly rebalancing stages. We assume that options that are available at a decision stage are expiring at the subsequent stage. This gives us the flexibility to look into choices available to update inventory at the nearest maturity of the derivatives contracts. The optimal strategy is determined for the given parameter λ, by adopting the risk-reward function mentioned in Chapter 2.

$$\max(1-\alpha)E[W^T] - \alpha R_\zeta \tag{3.3}$$

$$A_0 X_0 = D_0 \tag{3.4}$$

$$A_n X_{n-} + G_n X_n = D_n, \forall n \in \mathcal{N} \tag{3.5}$$

The first part is the expected wealth ($E[W^T]$) (*reward*) at the planning horizon T and the second part is a risk measure R_ζ (*risk*), λ is the risk aversion coefficient that defines how risk-averse the investor is. Expected wealth is defined as $\sum p_n W_n, \forall n \in \mathcal{N}_T$. We discuss risk measures in later part in this section. $A, G\&D$ are the constraint matrices and define inventory balance equations, cash balance equations and other constraints model is subjected to. X_n are the control variables, this vector decides buy, sell and hold decisions for each asset in each node of the tree. The risk measure R_ζ adopted is expected shortfall and is defined as:

$$R_\zeta = \sum [\bar{\mu} - W_n]^+ p_n, \forall n \in \mathcal{N}_T \tag{3.6}$$

Where, $\bar{\mu}$ is a user-defined target, any scenario that yields wealth (W_n) lower than this target would be reflected in the expected shortfall, shortfalls are then weighted by their probability (p_n) to calculate the expected shortfall. (10). Portfolio revisions imply, for $t = 0, 1, 2...T-1$ a transition along the tree from the portfolio allocation at the ancestor node to a new allocation through holding, buying and selling decisions on individual securities. The last possible revision is at stage $T-1$ with one period to go.

Since the feasibility region and the optimal strategy are scenario dependent so their derivation requires the specification of the return coefficients and scenario probabilities along the tree. We present here the set of decision variables and constraints actually implemented to characterize the random constraint matrices $A_n, G_n, D_n, \forall n \in \mathcal{N}$ and solve the problem.

We consider three types of constraints, to be satisfied: the *inventory balance equations* define the portfolio evolution over time; the *cash balance constraints* include in each node all cash inflows and outflows generated by the current strategy; the *upper and lower bounds* on the decision vector which define policy constraints on the adoptable strategy.

For each node n of the scenario tree and the asset/derivative i, the optimal strategy is defined through the following possible decisions, x_{in} is the nominal amount *held* in asset i in node n; $c_{in}(j,k)$ and $p_{in}(j,k)$ is the nominal amount *bought* in call and put options contract on asset i in node n with strike k and maturity j respectively; x_{in^+} refers to a *buying decision* in asset i in node n; while x^-_{in} refers to a *selling decision* in asset i in node n. All the decision variables are constrained to be nonnegative.

Let $\bar{x}_i$ be the initial holding in asset $\mathcal{I}$ then the inventory balance equation at root node can be written as:

Inventory Balance Equation at root node

$$x_{i0} = \bar{x}_i + x^+_{i0} - x^-_{i0}, \ \forall \ i \in \mathcal{I} \tag{3.7}$$

The inventory balance constraints reflect the decision problem Markovian structure: as time evolves, along each scenario, the portfolio evolution will be fully specified in nominal value through holding, buying and selling decisions. Each such decision generates, jointly with other commitments, cash flows in each node resulting in cash surpluses or deficits to be compounded to the following stage.

The cash balance constraint is imposed for the first stage in the following way; it takes into account the number of options contracts bought in the root node n.

$$\begin{aligned} C_0 =& \bar{c} + \sum_{i \in \mathcal{I}} x_{i0}^{-} v_{i0}(1-\chi^{-}) - \sum_{i \in \mathcal{I}} x_{i0}^{+} v_{i0}(1+\chi^{+}) \\ &- \sum_{j=t+1,k} [c_0(j,k)O_0^c(j,k) + p_0(j,k)O_0^p(j,k)](1+\chi_o^{+}) \end{aligned} \tag{3.8}$$

3.1.2 Options Payoff Modeling Approach

Option payoff is a non-linear function unlike other assets, we need to break this non-linearity somehow in order to make the problem simpler. We need to introduce some variables that can tell us whether the option at expiry is in the money or not. The advantage of such variables is two-fold, first, it would get rid of us of the $\max$ function in the equations, so that would translate the problem to a linear system making it easier for computations, etc. Secondly, we are considering nominal amount model and these variables would help us in tracking the number of options, that is going to be a great help in inventory update problem using physically settled options, it is shown later in this chapter in more detail.

We define *moneyness* (δ) of the call option on asset i with maturity j and strike k in node n as:

$$\delta_{in}^c(j,k) = max(v_{in} - K_k^j, 0) \tag{3.9}$$

Similarly, *moneyness* for put option would be:

$$\delta_{in}^p(j,k) = max(K_k^j - v_{in}, 0) \tag{3.10}$$

We then define an indicator variable λ to check if the options are in-the-money or not, $\lambda = 1$, if $\delta \geq 0$, otherwise 0. $\lambda_{in}^c(j,k), \lambda_{in}^p(k,j)$ are the indicator variables for call and put option respectively having strike $k \in \mathcal{K}$ and expiring at $j \in \mathcal{J}$. The product of $lambda_{in}^c(j,k)$ & $\delta_{in}^c(j,k)$ is the option payoff at maturity.

We now write the equations and above constraints at node n (except the root node). The options purchased in the previous stage are going to update the inventory (in case of in-the-money expiry of options). So, first, we need to know if the options have expired in the money or not. This we can track by the indicator variables defined in chapter 2. If the option expires in the money then λ takes value equal to 1 otherwise 0. Buy decisions in options are define as c_n and p_n for call and put options. Since we have adopted a nominal amount model, these c_n and p_n are actually the number of options, therefore, if multiplied by λ we would know the quantities to added or subtracted from the inventory.

Assumption

To make the problem simple, we assume that there exists only stock index in the portfolio and options are available on stock index, $\mathcal{I} = \{1\}$, now, we can rewrite

$c_{in}(j,k)$ as $c_n(j,k)$.

Inventory Balance Equation at node n

$$x_{in} = x_{in-} + x^+_{in} - x^-_{in} + c_{n-}(j,k)\lambda^c_n(j,k) - p_{n-}(j,k)\lambda^p_n(j,k),\ \forall\, i \in \mathcal{I} = \{1\}, n \in \mathcal{N} - \{0\} \tag{3.11}$$

It is clear from the above equation that if a call option expires in-the-money then it would increase inventory, on the other hand, if a put option expires in-the-money then that would reduce the size of inventory. The variable λ defined in chapter 2 makes it very simple to model inventory equations.

Next, we write the cash balance constraint at node n.

Cash Balance Constraint at node n

$$\begin{aligned} C_n = {} & C_{n-}e^{r_n \Delta t} + \sum_{i\in\mathcal{I}} x^-_{in} v_{in}(1-\chi^-) - \sum_{i\in\mathcal{I}} x^+_{in} v_{in}(1+\chi^+) \\ & - \sum_{j=t(n^+),k} [c_n(j,k)O^c_n(j,k) + p_n(j,k)O^p_n(j,k)](1+\chi^+_o) \\ & + \sum_{j=t(n),k} k[p_{n-}(j,k)\lambda^p_n(j,k)](1-\chi_o) - \sum_{j=t(n),k} k[c_{n-}(j,k)\lambda^c_n(j,k)](1+\chi_o) \end{aligned} \tag{3.12}$$

This equation reflects the cash inflow and outflow due to the purchase and sale of underlying assets and options and options expiry which is leading to inventory modification. The $j = t(n^+)$ and $j = t(n)$ refer expiries at the next node and the current node respectively. We buy options that are expiring in the next node (as per our assumption) and we exercise options at the current node. If the call option expires in-the-money, this would lead to cash outflow as inventory size is going to be increased. It is reflected in the equation as strike price times the number of options purchased. Similarly, put options expiring in-the-money reflect cash in-flow.

The wealth at node n is the sum of the market value times the units of all the assets and derivatives held.

$$W_n = C_n + \sum_{i\in\mathcal{I}} x_{in}v_{in} + \sum_{j=t(n^+),k} [c_n(j,k)O^c_n(j,k) + p_n(j,k)O^p_n(j,k)], \forall\, n \in \mathcal{N} \tag{3.13}$$

where $S_n = v_{1n}$ is the price of the stock index $(i = 1)$

The model includes constraints on the upper and lower bounds on investment in underlying security through equation 3.14. Let ϕ_L and ϕ_U be the set of lower and upper bounds on the underlying. Constraints on options in this model would be different from those discussed in Chapter 2. Now, the aim is to update inventory using call and put options. Buying put options give the right to sell the underlying at the strike price, therefore, we cannot have more physical put options in our portfolio than the number of owned underlying assets. This is reflected in the equation 3.16. It is important to note that we have only at-the-money strike options expiring at the next decision stage, this leaves us with exactly one type of option in the portfolio.

As for the call options, it gives us the right to buy the underlying at the strike price. Given the self-financing portfolio, we are optimizing, we need to limit the number of call options we can buy to add underlying securities to our inventory at a decision stage, we don't want to run into a situation where we do not have enough cash to buy the underlying in case of in-the-money expiry of the call option. It is not the case where there is a choice of going for a cash settlement or physical delivery. Only physical options are considered in this model. Equation 3.17 ensures this, the product of strike price of the option and number of options is less than the cash available in that node (C_n). Equation 3.20 is the cash constraint, γ_C is the fractional wealth that is allowed to be kept in the cash account.

Policy Constraints:

$$\phi_L W_n \leq x_{in} v_{in} \leq \phi_U W_n, \ \phi_L = \{\phi_{iL}\}', \ \phi_{iL} \in [0,1], \tag{3.14}$$

$$\phi_U = \{\phi_{iU}\}', \ \phi_{iU} \in [0,1], \ \forall\, i \in \mathcal{I} \tag{3.15}$$

$$p_n(j,k) \leq x_{in}, i = 1 \tag{3.16}$$

$$k.c_n(j,k) \leq C_n, i = 1, \tag{3.17}$$

Non-negativity constraints:

$$x_{in}^{+} \geq 0, \ x_{in}^{-} \geq 0, \ x_{in} \geq 0, \ \forall\, i \in \mathcal{I}, \ \forall\, n \in \mathcal{N}_{0,T-1} \tag{3.18}$$

$$c_n(j,k) \geq 0, \ p_n(j,k) \geq 0, \ \forall\, n \in \mathcal{N}_{0,T-1} \tag{3.19}$$

Cash Constraint:

$$0 \leq C_n \leq \gamma_C W_n, \ \forall n \in \mathcal{N}, \ \gamma_C \in [0,1] \tag{3.20}$$

The following constraint imposes no decision on leaf node:

$$x_{in}^{+} = 0; \ x_{in}^{-} = 0; \ c_n = 0, \ p_n = 0, \ \forall\, n \in \mathcal{N}_T, \ \forall\, i \in \mathcal{I} \tag{3.21}$$

3.2 Possible Extension

Let us consider a trader/broker who holds shares of a company and is willing to add more shares of the same firm to his portfolio at a price lower than the market price while avoiding risk pf any possible downside movement in the underlying's price. A possible solution to this problem would be to use call options to increase the inventory and to use put options to hedge the underlying against downside movement. This is a case where we mix of cash and physically settled contracts. In this case, the equations (9 and 10) would change, the new formulation would be:

$$x_{in} = x_{in-} + x_{in}^{+} - x_{in}^{-} + c_{n-}(j,k)\lambda_n^c(j,k),\ \forall\, i \in \mathcal{I} = \{1\}, n \in \mathcal{N} - \{0\} \tag{3.22}$$

Compared to equation (9), in equation (21) there is no reduction in inventory size from the position in the put options. Hence, only call options adding to the inventory.

$$\begin{aligned} C_n = C_{n-}e^{r_n \Delta t} &+ \sum_{i\in\mathcal{I}} x_{in}^{-} v_{in}(1-\chi^{-}) - \sum_{i\in\mathcal{I}} x_{in}^{+} v_{in}(1+\chi^{+}) \\ &- \sum_{j=t(n^+),k} [c_n(j,k)O_n^c(j,k) + p_n(j,k)O_n^p(j,k)](1+\chi_o^-) \\ &+ \sum_{j=t(n),k} [p_{n-}(j,k)max(0,k-v_{in})](1-\chi_o^-) - \sum_{j=t(n),k} k[c_{n-}(j,k)\lambda_n^c(j,k)] \end{aligned} \tag{3.23}$$

In equation (22) put options expiring in the money adds cash to the portfolio while the cash outflow due to in-the-money expiry of the call options remains the same as in the equation (10).

In another scenario where a trader/broker wants to reduce his inventory, they can formulate the model another way. In equation (23) only put option appears in the inventory equation.

$$x_{in} = x_{in-} + x_{in}^{+} - x_{in}^{-} - p_{n-}(j,k)\lambda_n^c(j,k),\ \forall\, i \in \mathcal{I} = \{1\}, n \in \mathcal{N} - \{0\} \tag{3.24}$$

In the cash constraint, call options contract is now cash settled, while the put options are still physical settled.

$$\begin{aligned} C_n = C_{n-}e^{r_n \Delta t} &+ \sum_{i\in\mathcal{I}} x_{in}^{-} v_{in}(1-\chi^{-}) - \sum_{i\in\mathcal{I}} x_{in}^{+} v_{in}(1+\chi^{+}) \\ &- \sum_{j=t(n^+),k} [c_n(j,k)O_n^c(j,k) + p_n(j,k)O_n^p(j,k)](1+\chi_o^-) \\ &+ \sum_{j=t(n),k} [c_{n-}(j,k)max(0,v_{in}-k)](1+\chi_o^-) + \sum_{j=t(n),k} k[p_{n-}(j,k)\lambda_n^c(j,k)] \end{aligned} \tag{3.25}$$

Chapter 4

Scenario Generation

A stochastic programming (SP) problem is a mathematical programming problem, with values of some parameters replaced by distributions. SP can handle only discrete samples of limited size, so we need to approximate the distribution. The approximation is called a scenario tree.

4.1 Scenario Generation: Underlying Assets

Scenario tree is an information flow along the planning horizon. Scenario generation is a part of the stochastic optimization process where all the underlying forecast processes for assets and risk factors are translated to form a scenario tree that imposes information constraint on the decisions. The information flow is modelled by a filtration of sigma fields $\mathcal{A}_t, t = 1.....T$, which is associated to a stochastic input process $\xi = (\xi_1^T)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Typically, it is required that the $\sigma-$ field is generated by the random vector $\xi_1, \xi_2, ...\xi_T$. Then, the information or the non-anticipative constraint means measurability of the decisions x_t with respect to $\mathcal{A}_\sqcup$ for every $t = 1, 2, 3...T$. $t = 1$ refers to the present or the root node of the scenario tree, therefore, $\mathcal{A}_1 = \{\emptyset, \Omega\}$. Figure **??** presents a sample scenario tree.

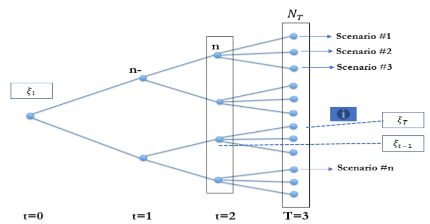

FIGURE 4.1: A Sample Scenario Tree

Any scenario-based approximation of the underlying probability distribution P of ξ has to reflect the growth of the $\sigma-$fields. Hence, the scenarios need to be tree-structured. In general, there are two ways to generate scenario trees, namely, (i) a tree-structure is prescribed and scenarios are generated via conditional distributions for increasing t starting with a root at t = 1, or (ii) in a first step a number of scenarios is generated for the whole horizon $t = 1, ...T$ based on the distribution P and according to some method, such as Monte Carlo simulation, Quasi-Monte Carlo, Quadrature rules using sparse grids and optimal quantization of probability measure. Secondly, a tree structure is generated successively by bundling scenarios. In our research, we rely on the first technique, we start from the root node, time $t = 1$

and produce information using the underlying pricing models to develop an information flow from root to leaf ($\mathcal{N}_T$) of the scenario tree. The sequences of realization are called *scenarios at stage t* $\xi_1, \xi_2, ...\xi_t$ given the conditional probability distribution of ξ_t conditioned by past realizations $\xi_1, \xi_2....\xi_{t-1}$

Having described what a scenario tree is and how the information flow is structured, we now move to the scenario generation of the underlying assets in our portfolio. As we are modeling an optimization problem from the perspective of an investor who wishes to maximize his wealth at the horizon, which is six months, the assets in his portfolio are equity index, fixed income index and S&P GSCI index (formerly known as*Goldman Sachs Commodity Index*). Options contracts are available on the Equity index. So, we need to introduce forecasting model for these assets.

We follow a two-layer approach to forecasting time series. The first layer is of the risk factors, we identify risk factors that describe the underlying assets. The second layer is an asset return formulation of the underlying time series derived from the risk factors and some exogenous variables. We identify short rate (r_n), long rate (l_n) and inflation rate (π_n) as risk factors, $j = 1, 2, 3$ respectively. The second layer of the process is to compute the values of the underlying processes B_n^k where $k = 1, 2, 3$ denote equity, fixed-income and commodity indexes respectively. We assume Cox-Ingersoll-Ross (CIR) dynamics for these risk factors. The coefficients α^j, $\omega^{j,*}$ and σ^j denote, respectively, the mean reversion coefficients, the long-term equilibrium and the standard deviation for each process, whereas, $t_n - t_{n-}$ is the differential time step between two consecutive nodes n and $n-$.

Correlation is introduced directly on the realizations e_n^r of three standard normal variables via the Cholesky elements c_j^r of the correlation matrix. Given the initial states $\omega^j(0)$ as ω_0^j for $t \in \mathcal{T}, n \in \mathcal{N}$, we have

$$\begin{aligned} \omega_n^j = \omega_{n-}^j + \alpha^j(\omega_n^{j,*} - \omega_{n-}^j)(t_n - t_{n-}) \\ + \sigma^j \sqrt(\omega_{n-}^j) \sqrt(t_n - t_{n-})(\sum_{r=1,2,3} c_{j,r} e_n^r) \end{aligned} \tag{4.1}$$

The coefficients of the CIR process are estimated using *Maximum Likelihood Method*. In Table 4.1 we report the coefficients that are generated using this mechanism.

We adopt the models developed by Consigli et al., 2012 to model equity and bond index using these risk factors. Consider an equity benchmark that includes constant volatility σ_n, while the drift μ_n is random and depends on the market prices of risk λ_n and the interest rates r_n. The coefficient λ_n is market-specific and reflect a varying risk aversion in the market. This is assumed to depend on the long interest rate l_n, inflation rate π_n and the recent market performance $(B_n/\bar{B})$.

$$\mu_n = r_n + \sigma\lambda_n, \tag{4.2}$$

$$\lambda_n = \beta_0 + \beta_1 l_n + \beta_2 \pi_n + \beta_3(B_n/\bar{B}) + e_n \tag{4.3}$$

$\bar{B}$ represents the constant average price over a given time while e_n is the realization of the standard normal variable. So, for given initial benchmark value B_0^1, the following price transition along the tree are derived:

$$B_n^k = B_0^k(1 + \mu_{n-}(t_n - t_{n-}) + \sigma\sqrt{t_n - t_{n-}}), \forall k = 1 \tag{4.4}$$

Using the risk factors we derive the fixed income benchmark tree model employing a duration-convexity approximation. Let $\bar{D}$ and $\bar{C}$ be the duration and convexity of the fixed income benchmark respectively, the evolution of the fixed income benchmark is determined through the following equation:

$$B_n^k = B_{n-}^k(1 - \bar{D}(l_n - l_{n-}) + 0.5\bar{C}(l_n - l_{n-})^2 + l_n(t_n - t_{n-})), \forall k = 2 \tag{4.5}$$

This completes our model for a fixed income benchmark. Next, we introduce a model for the pricing of GSCI commodity index. Instead of using an established model for forecasting a commodity index, we adopt the econometrics technique to develop a model for the GSCI index. This requires identifying the key factors that best approximate the performance of the GSCI index. We reviewed the literature for this to know the drivers of a commodity index.

Historically, commodities have shown a positive correlation with the inflation rate and change in the inflation rate both in the short-run and in the long-run. Some studies have taken a longer-run perspective. Gorton and Rouwenhorst, 2006 find that correlations between commodity futures returns and inflation tend to rise and become statistically significant as the horizon lengthens. Adams et al., 2008 also concluded that correlations between commodities, measured using the GSCI excess returns, and U.S. inflation rises with the investment horizon, although these positive correlations do not hold consistently for inflation in the euro area and Asia. Worthington and Pahlavani, 2007 presented evidence of the long-run hedging properties of gold based on a positive long-run relationship between gold and U.S. inflation in the post-war period.

Becker and Finnerty, 2000 attempted to incorporate futures leverage into the analysis by constructing levered indexes, which scale futures returns by a multiplier. They find that commodity futures serve as an inflation hedge, with the degree of protection increases as the commodity futures are levered. This gives us the motivation to include inflation and change in the inflation rate as the factors for developing a model for the GSCI index.

Commodities have also shown a negative correlation with stocks and bond markets and therefore, been a good financial instrument for diversifying a portfolio. A high negative correlation has existed between stock and commodity prices over the past 140 years, Zapata, Detre, and Hanabuchi, 2012.Some early observers of commodity markets Bannister and Forward, 2002; Rogers, 2007 note that the history of U.S. stock and commodity prices has been characterized by recurring supercycles that last several decades. These observations make it evident to include the performance of the stock and bond market as drivers for the commodity index. Since commodities prices are heavily driven by demand and supply in the physical market, it is necessary to include lagged variables of the dependent variable in order to derive the return dynamics.

We reviewed different approaches to estimate this model, Vector Autoregressive method (VAR), Vector Error Correction Mechanism (VECM), and the third method where we formulate an autoregressive distributed lagged (ARDL) model.

The classical VAR method and VECM method are used to generate scenarios. The key concern here is the number of parameters estimated in VAR and VECM methods that the model could actually be inefficient and in that case it owuld be vulnerable to type II error. Conventional regression estimators, including VARs, have good properties when applied to covariance-stationary time series, but encounter difficulties when applied to non-stationary or integrated processes which is the case here. These difficulties were illustrated by Granger and Newbold, 1974 when they introduced the concept of spurious regressions. If we have two independent random walk processes, a regression of one on the other will yield a significant coefficient, even though they are not related in any way. VECM method is the extension of VAR method where two or more time series are co-integrated. The model is fit to the first differences of the non-stationary variables, but a lagged error-correction term is added to the relationship. This addition of term leads to loss in degree of freedom.

Next, we consider ARDL *(autoregressive distributed lags)* method where we include one lagged variable of GSCI index with a constant term, we include equity market performance and fixed income market performance as exogenous variables and inflation rate as a risk factor. The equation for modelling GSCI index returns $(j = 4)$ is the following:

$$\begin{aligned} \omega_n^j &= \beta_0^j + \beta_1^j (B_n^1 - B_{n-}^1)/B_{n-}^1 + \beta_2^j (B_n^2 - B_{n-}^2)/B_{n-}^2 + \beta_3^j \omega_n^{j=3} + \beta_4^j \omega_{n-}^j \\ &+ \beta_5 (\omega_n^{j=3} - \omega_{n-}^{j=3}) + \sigma^j \sqrt{t_n - t_{n-}} e_n \end{aligned} \tag{4.6}$$

$$B_n^k = B_{n-}^k (1 + \omega_n^{j=4}) \tag{4.7}$$

We estimated the parameters of the above model using the OLS method. In this chapter, we present the model coefficients for the period May 2008 to May 2014, first we transform the monthly data to annual data points by taking annual returns at a monthly frequency on a rolling basis. It turns out that the performance of the equity index, changes in inflation rate and lagged variable of the first order of GSCI index returns are significant in approximating GSCI returns. The model has an impressive R-squared value and the error terms did not show autocorrelation up to 12th order that is significant at 99% confidence level.

GSCI Model: OLS, using observations 2008:05–2014:05 (T = 73)
Dependent variable: GSCI

	Coefficient	Std. Error	t-ratio	p-value
SP500	0.173856	0.0542959	3.2020	0.0021
dinfl	7.03558	1.87614	3.7500	0.0004
GSCI_1	0.813304	0.0394863	20.5971	0.0000

Mean dependent var	−0.010112	S.D. dependent var	0.269500
Sum squared resid	0.496064	S.E. of regression	0.084182
R^2	0.905274	Adjusted R^2	0.902568
$F(3,70)$	222.9916	P-value(F)	9.65e–36
Log-likelihood	78.60761	Akaike criterion	−151.2152
Schwarz criterion	−144.3438	Hannan–Quinn	−148.4768
$\hat{\rho}$	−0.119793	Durbin's h	−1.087260

LM test for autocorrelation up to order 12 –
Null hypothesis: no autocorrelation
Test statistic: LMF = 1.62913
with p-value = $P(F(12,58) > 1.62913)$ = 0.108659

```
Breusch-Godfrey test for autocorrelation up to order 12
OLS, using observations 2008:05-2014:05 (T = 73)
Dependent variable: uhat

              coefficient    std. error    t-ratio    p-value
  -----------------------------------------------------------
  SP500       -0.0618944     0.0710367    -0.8713     0.3872
  dinfl        1.65489       2.37431       0.6970     0.4886
  GSCI_1       0.0338920     0.0580509     0.5838     0.5616
  uhat_1      -0.124469      0.164143     -0.7583     0.4513
  uhat_2       0.0160500     0.152508      0.1052     0.9165
  uhat_3      -0.111580      0.140176     -0.7960     0.4293
  uhat_4      -0.108847      0.139960     -0.7777     0.4399
  uhat_5       0.173034      0.135852      1.274      0.2079
  uhat_6       0.149434      0.130343      1.146      0.2563
  uhat_7      -0.136355      0.128837     -1.058      0.2943
  uhat_8      -0.228446      0.136237     -1.677      0.0990   *
  uhat_9       0.00359541    0.134938      0.02664    0.9788
  uhat_10      0.154117      0.131051      1.176      0.2444
  uhat_11     -0.363626      0.138877     -2.618      0.0113   **
  uhat_12      0.0181454     0.152357      0.1191     0.9056

  Unadjusted R-squared = 0.252091

Test statistic: LMF = 1.629127,
with p-value = P(F(12,58) > 1.62913) = 0.109

Alternative statistic: TR^2 = 18.402627,
with p-value = P(Chi-square(12) > 18.4026) = 0.104

Ljung-Box Q' = 23.6291,
with p-value = P(Chi-square(12) > 23.6291) = 0.0228
```

We calibrate and run the above models for the period from May-2008 to May-2014. The figures 4.2- 4.7 below show the effectiveness of the models presented. Table 4.1 presents the estimates of the parameters obtained by fitting the Cox-Ingersoll-Ross model. These parameters are used in the set of equations 4.1. Starting from 15th

TABLE 4.1: Estimates of the CIR model fitted on the risk factors

	Cholesky Matrix			Speed	Level	Sigma
Short Rate	1.1417	0	0	0.8265	0.6885	0.5538
Long Rate	0.6558	0.4583	0	0.6019	3.3989	0.3954
Inflation Rate	0.4665	-0.1847	1.4898	0.01	0.2057	0.0476

May 2014, we simulate rates and prices for risk factors and indexes for the next six months at a monthly time step by taking into account history from May 2008. We adopt [1 10 3 3 3 3 3] tree structure for the scenario generation process. The fan plot shows the distribution of the rates or index prices over the next six months. The red zone in the fan plot is the region where most of the scenarios are realized. The yellow ones have low regional density. The dark black line in the plots is the mean scenario which is defined as the 50th percentile of the distribution of the terminal stage values. The blue dotted line is the actual market dynamics, it extends from 12 data points in the past (one-year history) to the next six data points in the future (monthly values).

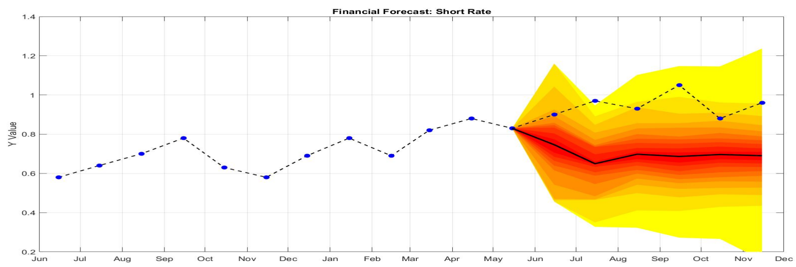

FIGURE 4.2: Short Rate

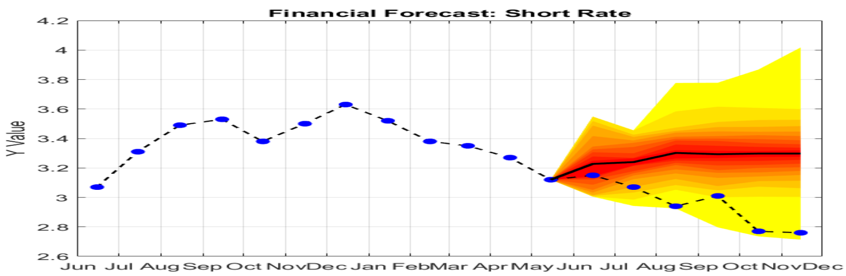

FIGURE 4.3: Long Rate

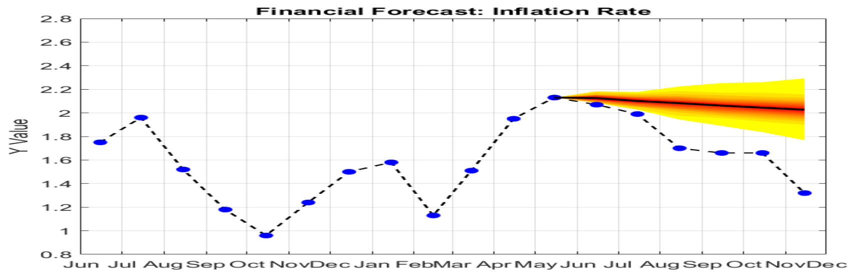

FIGURE 4.4: Inflation Rate

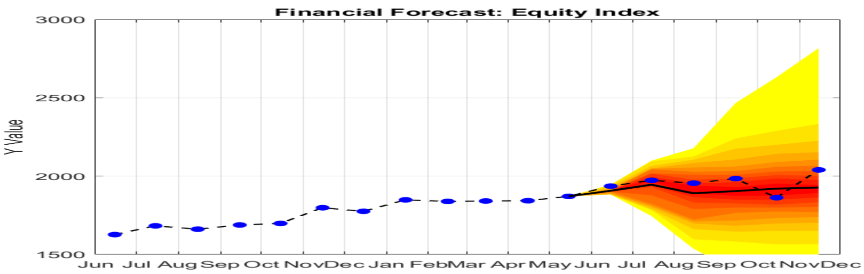

FIGURE 4.5: S&P500 Equity Index

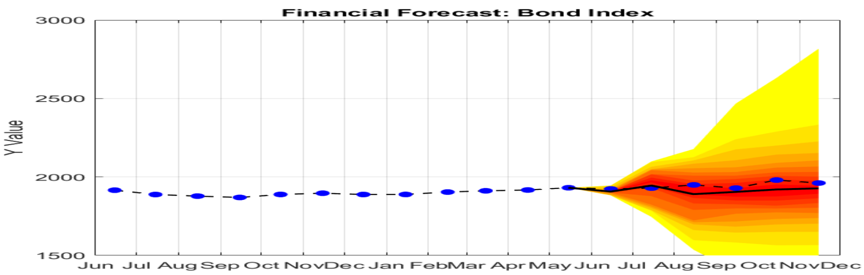

FIGURE 4.6: US-AGG Bond Index

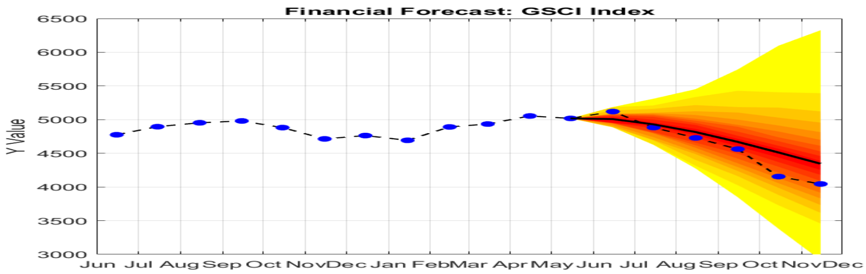

FIGURE 4.7: GSCI Index

What is more relevant from the above figures is that the first-month value was captured in all the simulations for all the risk factors and as well as for all the indexes.

4.2 Scenario Generation: Options

To price options on a scenario tree has been a challenging and debatable topic for researchers for many years. Starting from the Black-Scholes formulation to price options in a risk neutral environment the literature has expanded to many empirical studies. Several empirical studies show that Black-Scholes mis-price the deep OTM options, Rubinstein, 1985.

Dempster and his collaborators (Dempster, Hutton, and Richards, 1998; Dempster and Hutton, 1999; Dempster and Richards, 2000) developed and tested many techniques to price American options within the Black-Scholes framework. So far the volatility had been treated as a constant parameter. The structure of asset prices is then calibrated using binomial and trinomial lattices. Rubinstein, 1994, Jackwerth and Rubinstein, 1996 and Derman, Kani, and Chriss, 1996 worked on these approaches. While in the late 90s when stochastic optimization was becoming popular a little work had been done price derivatives on a multinomial scenario tree.

Topaloglou, Vladimirou, and Zenios, 2008b talked about two techniques for pricing options on a scenario tree. Starting with a multinomial tree where an optimization program needs to be run to calculate the risk-neutral probabilities. Hence, it gets computationally inefficient as the size of the underlying tree expands. The second method, however, is an empirical approach that extends the Black-Scholes framework to take into account higher moments (skewness and kurtosis). So, the calculation of the option prices becomes much simpler. As the price of an option can be expressed in a simple linear equation.

So far in the literature, where stochastic optimization techniques have been used on a portfolio that includes options, either the multinomial tree approach has been used or the Black-Scholes model or moment matching methods have been used. Most of the researchers have considered options expiring at the next decision stage and therefore, it becomes less relevant to price option, as there is no worth of the options at the subsequent decision stage, either they expire in-the-money adding cash to the cash account or they expire worthlessly.

It becomes more relevant to price options in a multi-stage stochastic programming framework when options are not expiring at the next stage. As they carry a value that needs to be correctly calculated to avoid any spurious profits or losses.

We use here a simple yet effective approach to price options on a scenario tree. We utilize the Greeks information of options to price them. The Greeks are the quantities representing the sensitivity of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent. We use Delta-Gamma approximation, Estrella and Kambhu, 1997 and it works well for pricing options at a one-month time step which is consistent with the discrete-time steps of planning horizon of our investment planning problem.

In general, in-the-money options will move more than out-of-the-money options, and short-term options will react more than longer-term options to the same price change in the stock. This fluctuation of prices with maturity and moneyness add more complexity to the option pricing. Since OTM options have low premiums, so even a small absolute change in the option price value would reflect a higher change in relative terms. An OTM option trading at $2.5 if has a positive absolute change of $0.5 which means it has sored up 20%, while the absolute change is small. Such options can prove to be very lucrative if the price of the underlying security moves in a favorable direction. However, pricing of such options on scenario tree can generate spurious profits and loss.

The delta-gamma approximation is used to estimate option price movements if the underlying stock price changes. This approach is better than the delta approximation approach which is linear and since the option price is a non-linear function of the stock price we need to introduce another sensitivity parameter that can tackle this non-linearity. To take account of this we can use gamma to make our option price estimate more precise. Delta-gamma makes our approximation non-linear.

The delta-gamma approximation for call options is can be expressed with the following equation.

$$c(S_{T+1}) = c(S_T) + \Delta(S_T)(S_{T+1} - S_T) + 0.5\Gamma(S_T)(S_{T+1} - S_T)^2 \tag{4.8}$$

where, $\Delta(S_T)$ is the delta of the stock option on the underlying series S at time T

where, $\Gamma(S_T)$ is the gamma of the stock option on the underlying series S at time T

$\Delta(S_T)$ is approximated using Black-Scholes probabilities $d1$ and $d2$, where $N(d1)$ is the delta of the call option and $N(d2)$ is the probability that option would expire in the money.

Gamma of call option is expressed as:

$$\Gamma = K \exp^{-rt} \phi(d2)/(S^2\sigma\sqrt{t}) \tag{4.9}$$

The same formula can also be used for put options, the delta of a put option is negative, so if the price of the underlying would increase means the price of the put

option would decrease. We tested the reliability of this approach on options market data and found it quite consistent. We present below some results to make it clear for the readers that this approach can be used in the scenario tree nodal framework. Then we move on to the nodal formulation of these equations to make it consistent with the optimization models presented earlier.
We test the accuracy of this approach on real market data. The method is applied to call and put options, both in-the-money and out-of-the-money of different strikes and maturities. The figure below shows the market price of call and put options and the model price, the second subplot in the figures shows tracking error with respect market price. Each bar in the figures is named 'Call12142000', which corresponds to the European type call option expiring in December 2014 that has strike price 2000. The first four characters correspond to the type of European option, the next two characters correspond month of expiry, then the following two characters correspond to the year of expiry and the last four characters are the strike price of the option.

In figure 4.8, call option prices are predicted on 15th July 2014 for 15th August 2014. Since we have a planning horizon of six months with monthly rebalancing stages, so it becomes more relevant to check the accuracy of this method at monthly frequency. It can be seen from the figure that options that are deep out-of-the-money have the highest tracking error in price prediction. We define tracking error as the difference between the current market price and the model price divided by the current market price. As explained earlier, when the price value is small then even a small absolute change would be reflected as a high relative change, this is clearly seen in the OTM options price prediction here. Options trading at USD 2 is predicted to have a price of around USD 3.5, which is actually 75% more than the market price. While in the case of at-the-money or in-the-money options the tracking error remains quite small. As the price of the option increases the accuracy of the delta-gamma price approximation increases.

The same behavior in price prediction (fig 4.9) is observed in call options price prediction on 25th August 2014 on 24th September 2014. Higher the option premium, lower the tracking error. The same is observed with put options prices (figures 4.10 and 4.11). It is interesting to note that in most of the cases tracking error remains lower than 5%, the delta-gamma approximation may be a rough approach to option pricing but still comparable to the approaches discussed by Topaloglou, Vladimirou, and Zenios, 2008b, we observe more error when we price out-of-the-money options.

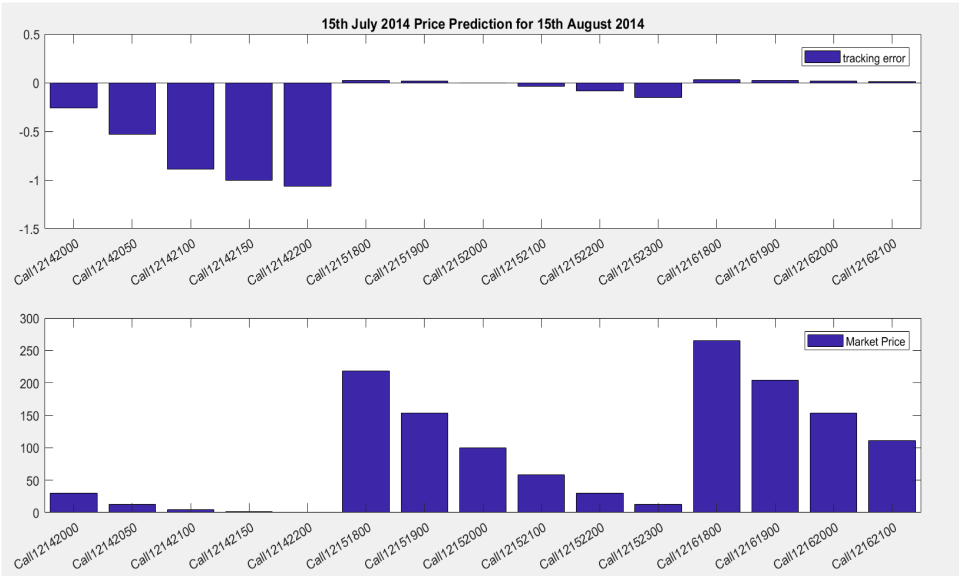

FIGURE 4.8: Call Option Price Prediction

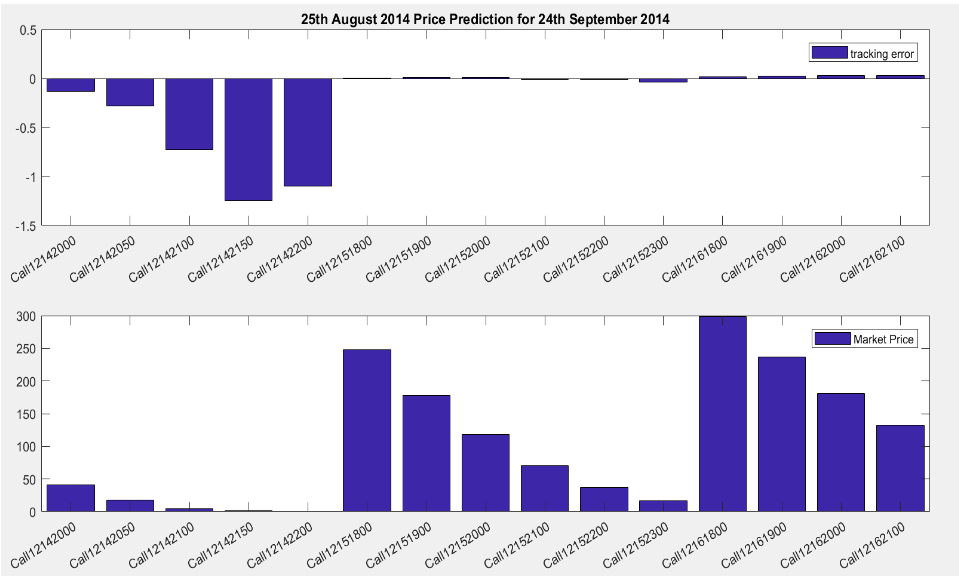

FIGURE 4.9: Call Option Price Prediction

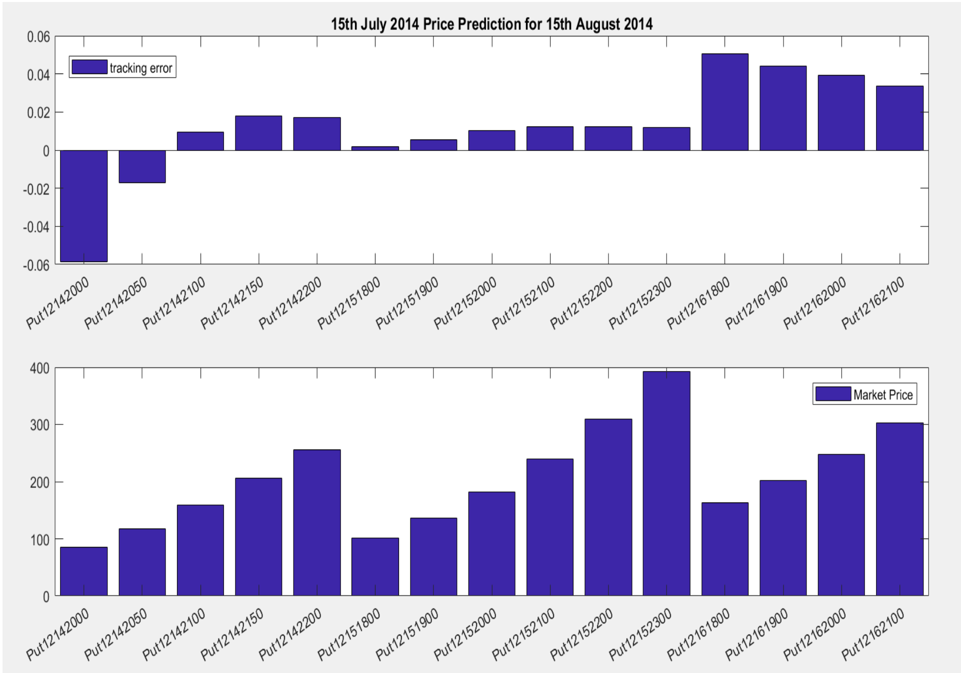

FIGURE 4.10: Put Option Price Prediction

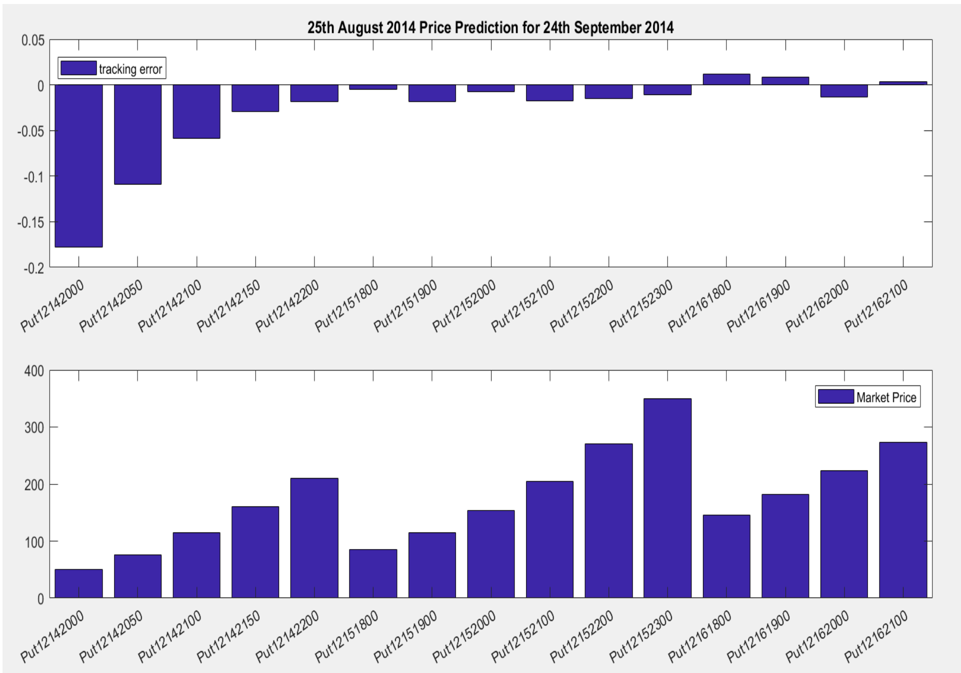

FIGURE 4.11: Put Option Price Prediction

A less relevant analysis in this context is the option price prediction on daily basis, this however, has shown impressive results. Fig 4.12 and Fig 4.13 shows market price and the model price in the upper subplot, on the right y-axis is the performance of the underlying S&P500 index. In the lower subplot, we plot the tracking error and it is found to be within 5% from the actual price. This motivates that shorter the price prediction time step, more accurate the price would be. This also motivates to follow this approach where we have nested simulation in the event tree or when optimization problem has frequent rebalancing stages.

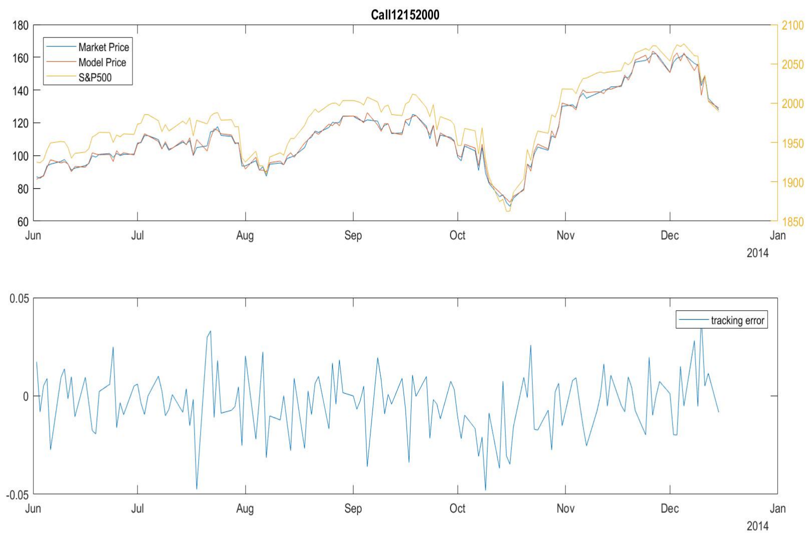

FIGURE 4.12: Daily price prediction using Delta-Gamma Approximation

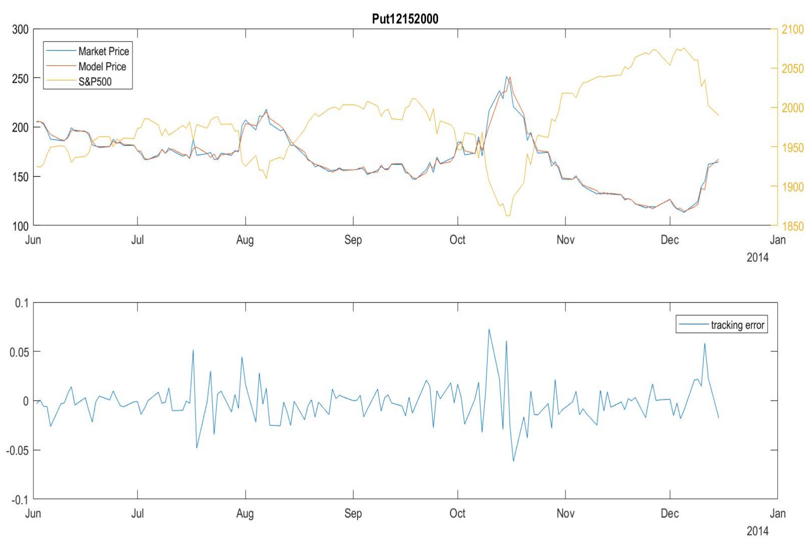

FIGURE 4.13: Daily price prediction using Delta-Gamma Approximation

4.2.1 Nodal Representation of Delta-Gamma Approximation for Option Pricing

It is important to map the delta-gamma approximation in a logical conformable way to the description of the underlying process and the discrete time dynamic stochastic optimization program in order to complete the definition of a stochastic program.

Let $c_{in}(j,k)$ be the price of call option on asset i in node n having maturity j and strike k. Let $\Delta^c_{in}(j,k)$ and $\Gamma^c_{in}(j,k)$ be the delta and gamma Greeks of that option in node n. The equation then can be rewritten as:

$$c_{in}(j,k) = c_{in-}(j,k) + \Delta^c_{in}(j,k)(v_{in} - v_{in-}) + 0.5\Gamma^c_{in}(j,k)(v_{in} - v_{in-})^2 \quad (4.10)$$

Similarly, the equation for pricing put option would be:

$$p_{in}(j,k) = p_{in-}(j,k) + \Delta^p_{in}(j,k)(v_{in} - v_{in-}) + 0.5\Gamma^p_{in}(j,k)(v_{in} - v_{in-})^2 \quad (4.11)$$

4.3 Arbitrage Free Pricing and In-sample Stability Analysis

It is important to check the scenarios for arbitrage opportunities. Any such opportunity would generate spurious profits in the wealth distribution obtained from the optimization program, hence, making it difficult to analyze actual realized returns, we may see unrealistic gains in the that make no sense in the real world. No-arbitrage scenario generation has been discussed by many researchers. Some famous studies were, Klaassen, 1997,Klaassen, 1998, Dupačová, Consigli, and Wallace, 2000,Høyland and Wallace, 2001 and Consiglio, Carollo, and Zenios, 2016.

We adopt Klaassen, 2002 approach for precluding arbitrage opportunities in multistage scenario tree. In Ingersoll, 1987, two types of arbitrage opportunities are discussed, *type 1*, where it is possible to construct a zero investment portfolio that has non-negative payoff in all states of the world and *type 2* arbitrage opportunities, where we construct a portfolio with negative wealth and end up with non-negative payoff in at least one state of the world. The inclusion or exclusion of type 1 arbitrage opportunity does not imply inclusion or exclusion of type 2 arbitrage opportunity. Hence, both opportunities should be checked while generating scenario trees. Klaassen, 2002 presented a single check by which it is possible to check both types of arbitrage opportunities at the same time. If the set of equations 4.12 has a strictly positive solution then arbitrage opportunities of type 1 and type 2 do not exist, where N is the number of children nodes at the subsequent decision stage, R_n is the return in children nodes. If there exists a solution ($X > 0$) to this system of the linear equation then scenario tree between time t and $t+1$ is arbitrage-free. It is important to note that the method assumes equally probable scenarios in the future states of the world and this may not be the case in the real world. Consiglio, Carollo, and Zenios, 2016, discussed two types of system of equations where they considered scenarios with equal and different probabilities. We do not get into this approach and limit ourselves to equally probable scenarios and adopt Klaassen, 2002 approach.

The set of equations 4.12, guarantees arbitrage-free scenario tree from time t to $t+1$. In a multi-stage setting, this needs to be checked for all the stages before the horizon. The methodology we follow is shown in the pseudo-code 1. We generate a scenario tree at first node by generating risk factors and the price of the underlying securities discussed in section 4.1, we check for arbitrage opportunities, if exist, we generate the scenario tree again until we get arbitrage-free scenario tree. Once we obtain an arbitrage-free scenario tree at the first node, we proceed to the next node and generate arbitrage-free for its children nodes. We do this until we have populated all the nodes of the scenario tree for all the underlying assets. Once, we have generated the underlying asset scenario tree, we can compute option prices in all the nodes using delta-gamma approximation discussed earlier. Another important aspect here is to discuss any arbitrage opportunities due to options in the investment universe. This can be checked while solving the optimization problem, we set no constraints on the investment in options and if this gives us an unbounded solution then it means arbitrage opportunities exist.

$$\sum_{n=1}^{N} v_n(1 + R^n_{i,t+1}) = 1, \ \forall\, i = 1, 2, 3.. \tag{4.12}$$

Algorithm 1 Arbitrage-free scenario tree generation (Underlying assets)

given $\alpha^j, \omega^{j,*}, \sigma^j$
given $\omega^j(0)$ and $B^k(0)$ $\forall j = 1, 2, 3$ and $k = 1, 2, 3$
$\mathcal{N}_{0,T-1}$: Set of nodes from stage 1 to stage T-1
procedure NOARBITRAGETREE($\mathcal{N}_{0,T}$)

    for each item n in $\mathcal{N}_{0,T-1}$ **do**
        $N \leftarrow$ children nodes of n
        while No arbitrage **do**
            given $\omega^j(n-)$ and $B^k(n-)$ $\forall j = 1, 2, 3$ and $k = 1, 2, 3$
            compute $\omega^j(n)$ and $B^k(n)$ $\forall j = 1, 2, 3$ and $k = 1, 2, 3$
            Solve equations 4.12
            if $X > 0$ **then**
                No arbitrage = True
            end if
        end while
    end for
end procedure

4.3.1 In-sample Stability

In the optimization framework arbitrage-free event trees are required to produce realistic results by avoiding any spurious profits arising from arbitrage opportunities. However, the solution to these optimization problems depends on the distribution of the scenario tree. Different distribution can give different results, therefore, we must generate enough scenarios such that going beyond a certain number of scenarios the optimal solution does not change significantly. Hence, the in-sample stability of these models is required.

TABLE 4.2: Value of the objective function vs number of scenarios

Scenarios	Model Status	Value of the Objective Function	% Change in Obj Function Value
972	1	37318	
1215	1	37536	0.005824688
1458	1	38274	0.019470342
1701	1	37714	0.014739436
1944	1	37726	0.000318134
2187	1	37635	0.002415043

We run the optimization program for different scenarios. Investment universe includes S&P 500 equity index, US Aggregate bond index, S&P GSCI commodity index and at-the-money call and put options on an equity index, expiring in the subsequent decision stages. We test the model discussed in the chapter 2.

Table 4.2 shows how the value of the objective function changes with the number of scenarios. We adopt [1 4 3 3 3 3 3] branching structure. At each iteration, we increase the value of the first stage branching degree by 1. The first column shows the number of scenarios, increasing from 972 to 2187 scenarios. The second column has got model status, its value equal to 1 means the optimal solution has been achieved. The third column has the value of the objective function at optimality and the last column has the percentage change in the value of the objective function (wrt previous value) when we increase the number of scenarios. As we move from 972 number of scenarios to 2187, we see that the value of the objective function does not change much when we reach 1701 number of scenarios. At 2187 scenarios, the value of the objective function has changed 0.2% and this change is acceptable given a reasonably large number of scenarios. Next, we run the optimization program 50 times for a different set of scenarios with the branching structure [1,9,3,3,3,3,3].

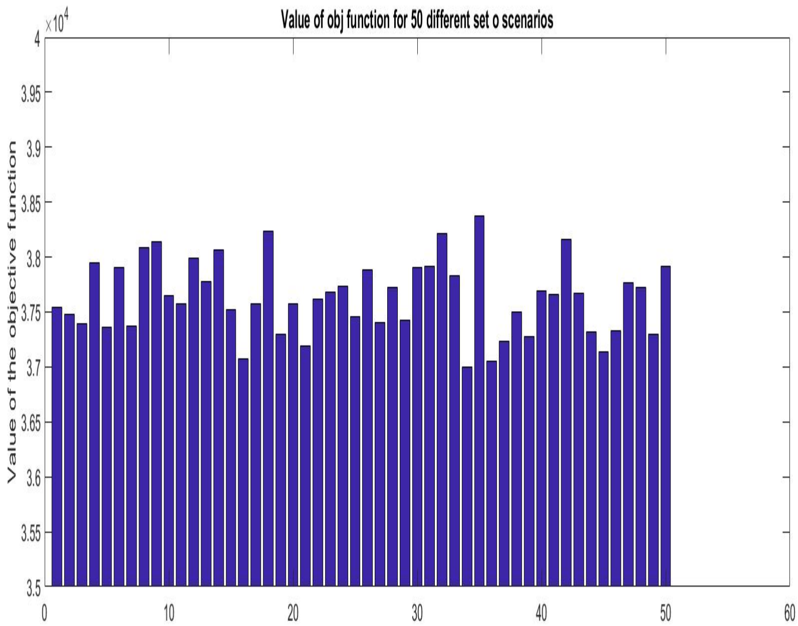

FIGURE 4.14: Value of the objective function

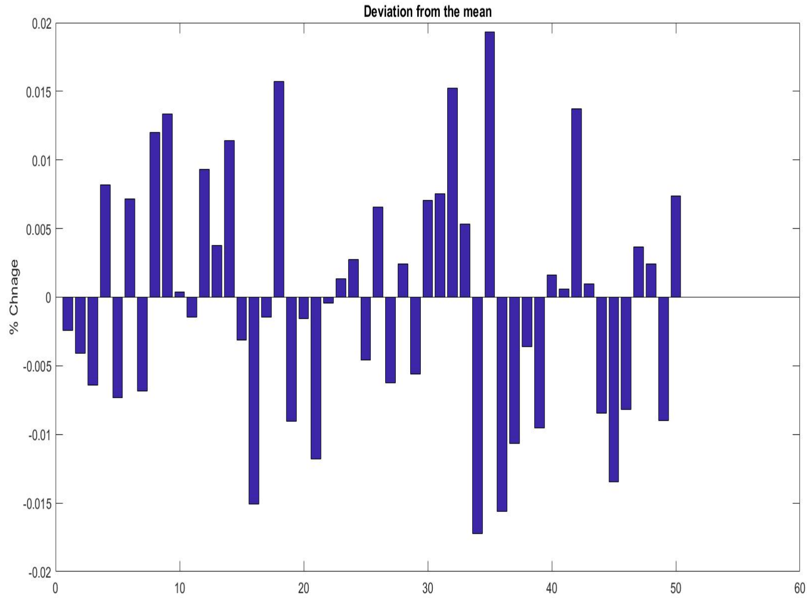

FIGURE 4.15: Value of the objective function

Figure 4.14 shows the value of the objective function for 50 different sets of scenarios. It is found that the optimal value does not change significantly. Figure 4.15 shows the deviation of each of the optimal objective function value from the mean of the 50 objective function value. It can be seen that the deviation is within the range of 2% and it confirms that in-sample results are pretty stable.

In the next chapter, we discuss results of the optimization models discussed in the previous chapter using the scenario generation techniques discussed here.

Chapter 5

Numerical Results

In this chapter we present results generated from the models discussed in the previous chapters, we collect some evidence from them. We start with the model where options are expiring in the next stage. We study the in-sample results of this model, then we check its performance on three-year data (June 2014 - March 2017, in out-of-sample analysis case). We then present results for the model where options selling is allowed before their expiry and for the model where the short position in options contracts are considered. Finally, we produce results for the inventory update model using derivatives and close the chapter with the summary of this research work and its possible extensions in the future.

5.1 Multistage Model-Options Expiring at the next stage

We first present results of the multi-stage model where options are assumed to be expiring on the subsequent decision stages. We model the problem from the perspective of an investor who wishes to maximize his wealth at the end of planning horizon which is six months. Investor is open to invest in equity, bond, commodity and options on equity index. We use the multi-stage model discussed in chapter 2. We go by doing the statistical analysis to generate scenario tree, branching structure considered for the scenario tree is [1 9 3 3 3 3 3], that generates 2187 scenarios for equity, bond, commodity, and options. The model used for pricing these securities is discussed in chapter 4. Target set at the end of six months is 110000 USD. The objective function used is the maximization of wealth penalized by a risk measure(expected average shortfall), risk aversion coefficient equal to 0.5. We use monthly data to develop statistical models. The available cash at the beginning is 100000$ and there is no position in any security. Data from May 2008-March 2017, is collected from Datastream5.1. We divide the data into two parts to carry out the in-sample and out-sample analysis. We truncate the data at May 2014 to see what adopted strategies have yielded in future data. All the computation is done on Matlab 2013a and GAMS 2.7 platforms.

We consider a simple case where only equity and equity options are there in the portfolio. We see how wealth distribution changes when we adopt different strategies. We test long straddle, strap, strip and protective put strategy. These strategies have been discussed by Topaloglou, Vladimirou, and Zenios, 2011 in a stochastic programming framework. The model here we adopt is the one discussed in section 2.3.1, which is equivalent to the model presented by Yin and Han, 2013b. Table 5.1 shows the portfolio composition in the worst-case scenario. We constraint the number of the call and put options in our portfolio to not be more than the units of the

TABLE 5.1: Equity and long straddle on equity options

Month	1	2	3	4	5	6	7
Cash	0	4.59	9802.78	0	0	0	1847.2
Equity	99980.00131	99975.4133	90176.23932	99976.1059	99975.9037	99975.39972	98128.01214
Call Option	596.843014	601.8528273	542.1821531	613.1618504	607.1774831	609.0012182	0
Put Option	554.7901763	549.9272523	496.6752633	539.1974147	544.9805089	543.2367488	0
Wealth	101131.6345	101131.7834	101017.8767	101128.4652	101128.0617	101127.6377	99975.21214

underlying equity index.

Table 5.1 shows the portfolio composition in the worst scenario, we identify the worst scenario corresponding to the 0th percentile of wealth distribution at the leaf nodes (terminal stage). The first row in the table is the cash held at each decision stage, the second row gives the position in equity, 3rd and 4th row give the position in call and put option, respectively, and the last row is the wealth of the portfolio over time. Month 0 corresponds to decision taken at time 0, then subsequent decision stages are followed at one-month frequency. We observe from the table that wealth decreases by 0.025% in the worst scenario over six months time, i.e. in the worst case if we are not able to make then at least we are not losing a significant part of the wealth. Let us also have a look at the distribution of wealth at the final stage. We plot cumulative distribution function and histogram to see the frequency distribution. Fig 5.1 and 5.2 show CDF and histogram plot respectively.

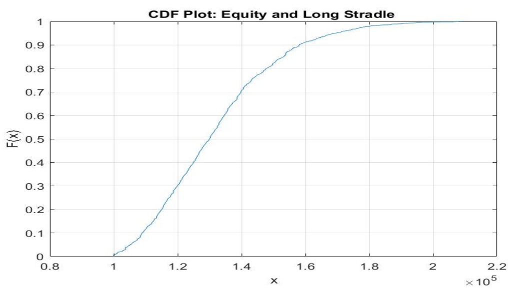

FIGURE 5.1: CDF plot: Equity Index and Long Straddle

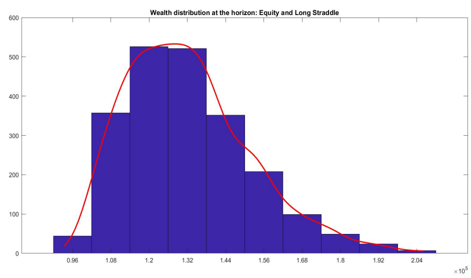

FIGURE 5.2: Wealth Distribution: Equity Index and Long Straddle

The distribution shows profit in most of the cases at the horizon, maximum loss observed was 0.025%, however, the probability of this event is very small. Most of the scenarios are centered around 25% gain. Distribution show more upside potential than downside risk. We present here the next portfolio composition and wealth corresponding distribution and histograms in different cases, when strap, strip, and protective put strategies are considered.

TABLE 5.2: Equity and strap strategy on equity options

Month	1	2	3	4	5	6	7
Cash	0	4.59	9802.78	0	0	0	1847.2
Equity	99980.00131	99975.4133	90176.23932	99976.1059	99975.9037	99975.39972	98128.01214
Call Option	1193.686028	1203.705655	1084.364306	1226.423516	1214.256813	1217.905601	0
Put Option	554.7901763	549.9272523	496.6752633	539.1974147	544.9805089	543.2367488	0
Wealth	101728.4775	101733.6362	101560.0589	101741.7268	101735.141	101736.5421	99975.21214

TABLE 5.3: Equity and strip strategy on equity options

Month	1	2	3	4	5	6	7
Cash	0	0	0	0	0	0	266.89
Equity	99980.00131	100118.1976	104033.0241	104901.5653	105248.5099	110088.7424	109955.2816
Call Option	1193.686028	1224.945622	1260.171555	1265.481166	1262.063972	1322.614478	0
Put Option	1109.580353	1083.059334	1137.222063	1151.8253	1162.857949	1213.891629	0
Wealth	102283.2677	102426.2025	106430.4177	107318.8718	107673.4318	112625.2485	110222.1716

TABLE 5.4: Equity and protective put strategy on equity index

Month	1	2	3	4	5	6	7
Cash	0	4.59	9802.78	0	0	0	1847.2
Equity	99980.00131	99975.4133	90176.23932	99976.1059	99975.9037	99975.39972	98128.01214
Call Option	0	0	0	0	0	0	0
Put Option	554.7901763	549.9272523	496.6752633	539.1974147	544.9805089	543.2367488	0
Wealth	100534.7915	100529.9306	100475.6946	100515.3033	100520.8842	100518.6365	99975.21214

It can be seen from the figures 5.1-5.8, that wealth distribution in each case is more concentrated in the upper region where wealth is higher than the starting wealth level which is 100000. The probability of facing losses is small. Almost, in all the scenarios wealth is seen to be increasing, maybe because of arbitrage, we discuss later. We present a worst-case scenario for the protective put case. We observe a decrease in wealth by 0.025%, it is however not less in magnitude as compared to other cases where loss was around 0.025% in the worst case. A protective put strategy is a sort of insurance against any down-trending market. The maximum loss in this strategy is the net premium paid to buy a put option. Next, we plot distributions in various cases. Interestingly, when we double the number of put options as compared to call options (strip strategy) we see significant changes in the portfolio wealth in the worst case. In the worst case, when the equity market is bearish, put options are proving profitable. We present a comparative analysis of all these strategies in fig 5.9 and 5.10.

Figures 5.9 and 5.10 summarize the four strategies, it is found that on the same set of scenarios protective put strategy has the least risk but at the same time it has the lowest profitability. Other strategies have outperformed protective put, strip gives the best returns in most of the scenarios, however, it is a scenario dependent phenomenon, had the scenario of equity been increasing then the strip was going to perform better. It is evident from the plots that protective put does give good

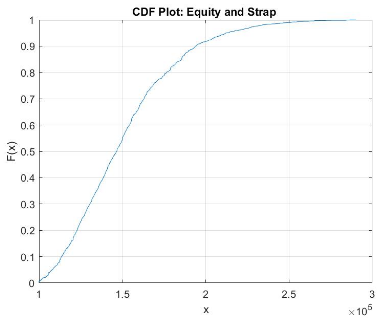

FIGURE 5.3: CDF plot: Equity and Strap Strategy

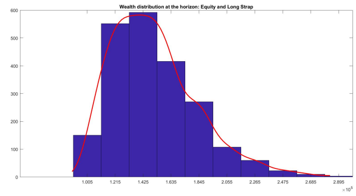

FIGURE 5.4: Histogram kernel fit:Equity and Strap Strategy

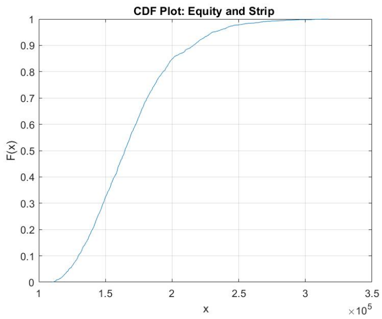

FIGURE 5.5: CDF plot: Equity and Strip Strategy

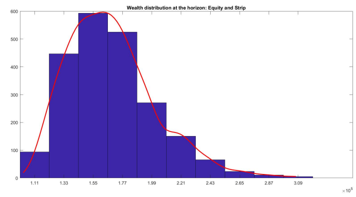

FIGURE 5.6: Histogram kernel fit:Equity and Strip Strategy

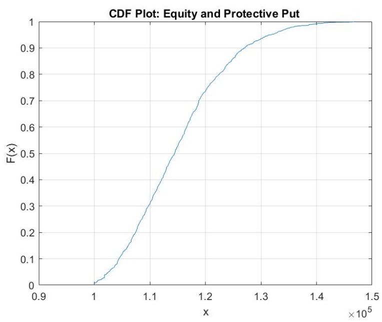

FIGURE 5.7: CDF plot: Equity and Protective Put

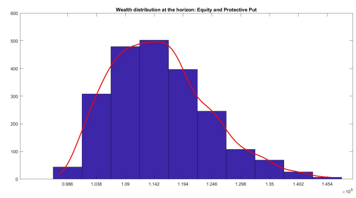

FIGURE 5.8: Histogram kernel fit:Equity and Protective Put

protection but limits the upside potential, hence a mix of strategies could be more profitable here.

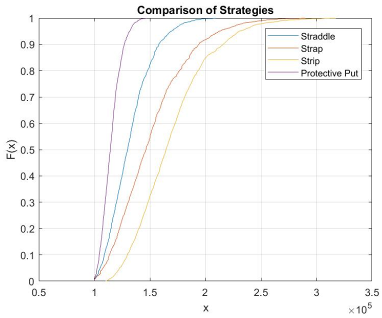

FIGURE 5.9: CDF plot for all the strategies on a portfolio of equity and equity options

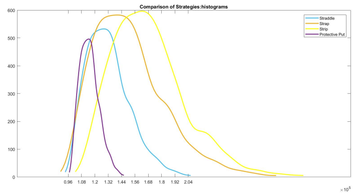

FIGURE 5.10: Histogram kernel fitting for all the strategies on a portfolio of equity and equity options

Now, we analyze a portfolio where we have equity, bond and commodity indexes and options available on an equity index. We consider a minimum 60% fixed income securities in our portfolio, commodities are capped at 15% and equity is free to take any value between 0 to 40%. The cash account is also set free, there is no minimum cash requirement or liquidity issues. Table 5.5 shows the portfolio composition in the worst scenario where no options were included in the portfolio. Figures 5.11 and 5.12 plot the wealth distributions. We then consider cases where options are considered in the portfolio through straddle, strap, strip and protective put strategy using call and put options on the equity index.

TABLE 5.5: Portfolio of equity, bond and commodity indexes (Worst scenario)

Month	1	2	3	4	5	6	7
Cash	0	24503.9	0	0	0	0	0
Equity	39980.00837	0	0	0	39953.15358	0	0
Bond	60000.00862	58809.34899	97674.37646	99230.25875	59929.73094	99025.86802	98921.67495
Commodity	0	14702.36145	0	0	0	0	0
Call Option	0	0	0	0	0	0	0
Put Option	0	0	0	0	0	0	0
Wealth	99980.017	98015.61044	97674.37646	99230.25875	99882.88452	99025.86802	98921.67495

TABLE 5.6: Equity, Bond, Commodity and equity options straddle strategy (worst scenario)

Month	1	2	3	4	5	6	7
Cash	0	0	0	0	0	103.54	0
Equity	39980.00837	40475.54377	0	0	0	37386.07574	35200.85535
Bond	60000.00862	60713.32762	98850.87892	99804.42491	59956.76313	61503.08741	60783.29137
Commodity	0	0	0	0	0	0	0
Call Option	238.6701948	239.8675324	0	0	0	227.1170084	0
Put Option	221.8537812	226.2348624	0	0	0	203.666453	0
Wealth	100440.541	101654.9738	98850.87892	99804.42491	59956.76313	99423.48661	95984.14672

TABLE 5.7: Equity, Bond, Commodity and Strap Strategy on equity options (worst scenario)

Month	1	2	3	4	5	6	7
Cash	0	0	0	0	0	103.73	0
Equity	39980.00837	40549.94618	0	0	0	37454.78275	35265.54643
Bond	60000.00862	60824.92141	99032.56676	99987.86537	60066.95793	61616.12253	60895.00359
Commodity	0	0	0	0	0	0	0
Call Option	477.3403896	480.7514527	0	0	0	455.1034566	0
Put Option	221.8537812	226.6609168	0	0	0	204.0996016	0
Wealth	100679.2112	102082.28	99032.56676	99987.86537	60066.95793	99833.83834	96160.55002

TABLE 5.8: Equity, bond, commodity and strip strategy on equity options (worst scenario)

Month	1	2	3	4	5	6	7
Cash	0	0	0	0	0	197.96	0
Equity	39980.00837	40475.54377	39754.18136	39498.36149	40332.52468	35740.20808	33651.18884
Bond	60000.00862	60713.32762	59631.2688	59247.53969	60498.79243	67109.34652	66323.93812
Commodity	0	0	0	0	0	0	0
Call Option	238.6701948	239.8675324	239.5366994	239.340033	242.44296	217.1667524	0
Put Option	443.7075624	452.5762384	436.8645321	431.5268514	444.3539371	389.4005556	0
Wealth	100662.3948	101881.3152	100061.8514	99416.76807	101518.114	103654.0819	99975.12696

TABLE 5.9: Equit, Bond, Commodity and Protective Put strategy using equity options (worst scenario)

Month	1	2	3	4	5	6	7
Cash	0	0	0	0	0	105.22	0
Equity	39980.00837	40401.1604	0	0	39897.7177	37992.44971	35771.78669
Bond	60000.00862	60601.73383	98669.19108	99620.98445	59846.56833	62500.63061	61769.15991
Commodity	0	0	0	0	0	0	0
Call Option	0	0	0	0	0	0	0
Put Option	221.8537812	225.8088081	0	0	219.7500106	206.9583821	0
Wealth	100201.8708	101228.703	98669.19108	99620.98445	99964.03604	100805.2587	97540.9466

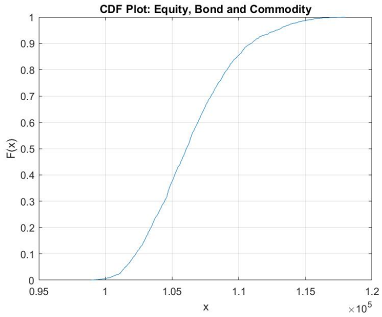

FIGURE 5.11: CDF: Equity, Bond & Commodity

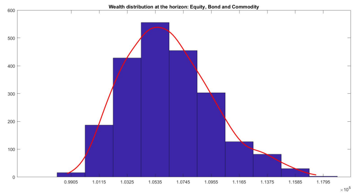

FIGURE 5.12: Hist: Equity, Bond & Commodity

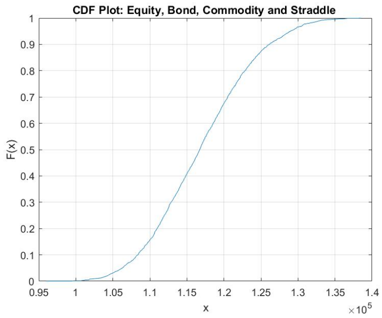

FIGURE 5.13: CDF: Equity, Bond, Commodity & Straddle Strategy on Equity Options

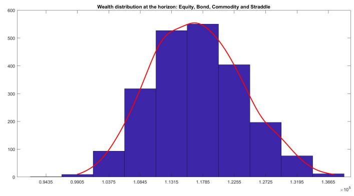

FIGURE 5.14: Hist: Equity, Bond, Commodity & Straddle Strategy on Equity Options

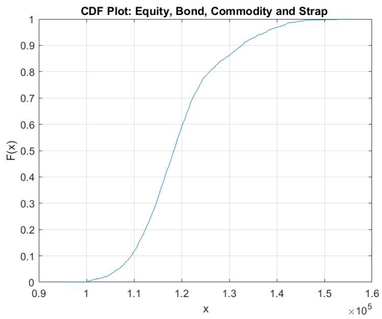

FIGURE 5.15: CDF: Equity, Bond, Commodity & Strap Strategy on Equity Options

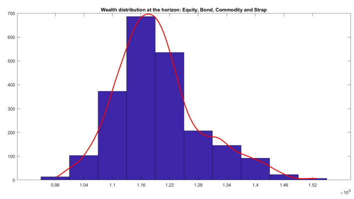

FIGURE 5.16: Hist: Equity, Bond, Commodity & Strap Strategy on Equity Options

Figures 5.11 -5.20 show optimal wealth distribution when options are used through straddle, strap, strip and protective put strategy in the portfolio. In all the distributions, wealth is increasing significantly at the planning horizon. However, this may not be the case, in reality, we talk about this in the latter part of this chapter. We also present a comparative analysis of all these strategies. Figures 5.21 and 5.22 show the wealth distribution in various strategies. It can be clearly seen that no option portfolio is trailing behind all other portfolios while maintaining a close gap with protective put strategy. While the left tail of the no-option portfolio is longer than the protective put portfolio. Other strategies however have shown a different but expected behavior. Straddle, Strip and Strap tend to outperform the other two portfolios. It is to be noted that these three strategies have longer left tails than the protective put strategy. It is because of the fact that these strategies work under certain market conditions. For instance, a straddle would have profit potential if the market experiences high volatility. If not, the investment in options to achieve a straddle strategy would be worthless. The same applies to other strategies. It is to be noted

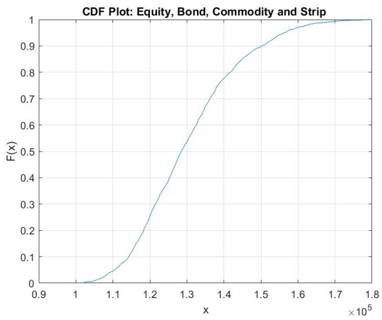

FIGURE 5.17: CDF: Equity, Bond, Commodity & Strip Strategy on Equity Options

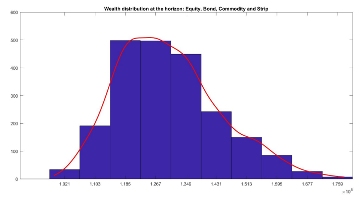

FIGURE 5.18: Hist: Equity, Bond, Commodity & Strip Strategy on Equity Options

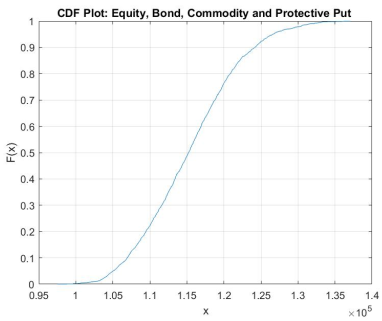

FIGURE 5.19: CDF: Equity, Bond, Commodity & Protective Put Strategy on Equity Options

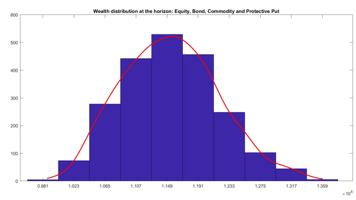

FIGURE 5.20: Hist: Equity, Bond, Commodity & Protective Put Strategy on Equity Options

that options are considered only on the equity index, whilst we have other two securities in our portfolio as well. Considering options on individual security would definitely yield better results. Since the equity fraction in this portfolio is capped at 40%, the effects of including options are minor but significant from a risk management perspective.

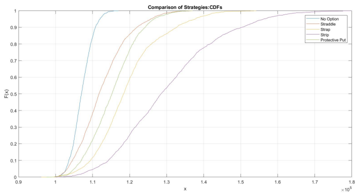

FIGURE 5.21: CDF plot: Equity, Bond, Commodity & Options Portfolio (options expiring at the next decision stage)

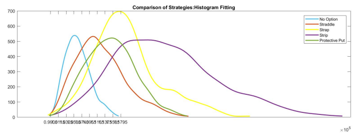

FIGURE 5.22: Histogram kernel fitting: Equity, Bond, Commodity & Options Portfolio (options expiring at the next decision stage)

Having seen the performance of this model on simulated data, its now time to check the out-of-sample performance of the multistage model where options are expiring at the next decision stage. Keeping the above constraints and initial parameters the same, we run this algorithm for the period of May 2014- March 2017. Figure 5.23 shows the portfolio performance using different strategies.

We test straddle, strip, strap and protective put strategies, we report their performance along with the equity, bond and fixed-mix strategy (where equity constitutes 25%, bond, 60%, and commodity 15% of the portfolio). We also add one more strategy where we allow speculation on call options on the basis of event trees generated

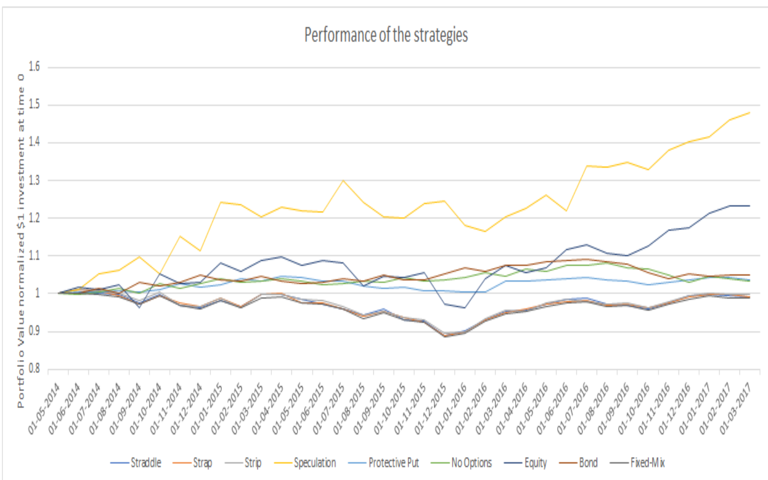

FIGURE 5.23: Out-of-sample: Portfolio performance of Equity, Bond, Commodity and Equity Options

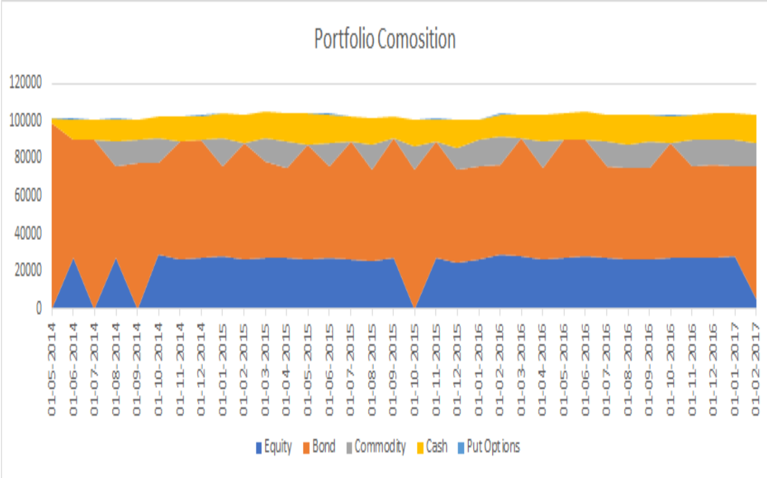

FIGURE 5.24: Out-of-sample: Portfolio composition of Equity, Bond, Commodity and Equity Options

at each rebalancing stage, we restrict investment in options at 1 basis point wealth of the portfolio.

Call options, when used for speculation, seem more profitable but also make the portfolio more volatile. Portfolio, where call options were bought for speculation, has more volatility than the volatility of the equity index. However, the returns of this portfolio are correlated to equity returns. Showing that, even a small amount of wealth used in call option made the portfolio more exposed to its underlying. While the bond index has a minimum 60% weight in the portfolio, the returns are seen to be far from the fixed income benchmark. Please see figure 5.24. On the other hand, a protective put strategy tends to give good protection to the portfolio. The wealth level over three years remain more or less the same, it increased due to the price increase of the underlying instruments. Speculation through options can have multi-fold advantages for traders depending on the risk-reward profile of the traders/investors. Figure 5.24 shows the portfolio composition in the protective put strategy. The option amount is not reflected in the chart, as it is really small compared to the investments in equity, bond and commodity indexes. It is important to note that straddle, strip and strap strategy have not given positive returns as found in the in-sample analyses. It is because of the fact that call and put options were not bought at every decision stage, a decision is made on the basis of forecasted data on the planning horizon and it doesn't seem to be a very accurate forecast here. These strategies were also discussed by Topaloglou, Vladimirou, and Zenios, 2011 and they reported positive performance of the portfolio using these strategies.

5.2 Multistage model, Trading long options positions

So far, we have discussed models for optimization where options were expiring at the subsequent decision stage, in this section, we are going to present results on trading long position on options, we have already discussed and validated a model for this in chapter 2. We assume the same initial conditions as in the cases discussed earlier in this chapter. The only dimension added is that the investor is now able to sell options before their expiry. For this, we consider that ATM equity options of

1-month, 3-month, and 6-month expiry are available at time 0. We run the optimization model using the scenario generation models discussed in the previous chapters. We consider three different cases to understand the buy/sell effects on the portfolio. To study this it is important to allow the portfolio to invest in options at different levels. Since we have some position in the equity, we assume that only those many options contracts (both call and put) can be traded as the number of underlying equity units in the portfolio, that's the first case, in the second case, we increase this number to 5 times and then in the third case we increase this number to 10 times. Table 5.10shows the portfolio composition in the mean scenario when as many options contracts are available for trading as the units of equity in the portfolio. Table 5.11 and 5.12 show the 2nd and 3rd case respectively. Figures 5.25 - 5.30 plot wealth distributions in these three cases.

TABLE 5.10: Options Buy/Sell (Mean Scenario)

Month	1	2	3	4	5	6	7
Cash	100000	40029.83	0	0	106293.49	0	0
Equity	0	0	39920.52577	0	0	41187.28114	42249.15858
Bond	0	59850.47151	59924.89573	105331.7368	0	64894.04202	65839.46927
Commodity	0	0	0	0	0	0	0
Call_Equity_1Month	0	0	0	0	0	0	0
Call_Equity_3Month	0	0	0	0	0	0	0
Call_Equity_6Month	0	0	29.40630226	0	0	0	0
Put_Equity_1Month	0	0	0	0	0	0	0
Put_Equity_3Month	0	0	0	0	0	0	0
Put_Equity_6Month	0	0	0	0	0	0	0
Wealth	100000	99880.30151	99874.8278	105331.7368	106293.49	106081.3232	108088.6279

TABLE 5.11: options Buy Sell (constrained to less than 5 times of underlying units)

Month	1	2	3	4	5	6	7
Cash	0	0	0	0	0	0	0
Equity	37332.21308	37664.17547	41838.16268	44136.91916	36361.9765	32336.69082	31057.21995
Bond	59880.2356	58978.59533	62922.42605	67679.59641	73901.04421	80073.52605	79334.22409
Commodity	0	0	0	0	0	0	0
Call_Equity_1Month	0	0	0	0	0	0	0
Call_Equity_3Month	2587.95272	0	0	0	0	0	0
Call_Equity_6Month	0	0	110.1298039	0	0	0	0
Put_Equity_1Month	0	0	0	0	0	0	0
Put_Equity_3Month	0	1654.898604	0	0	0	0	0
Put_Equity_6Month	0	0	0	982.8007893	12905.39231	12524.02817	18988.81613
Wealth	99800.4014	98297.6694	104870.7185	112799.3164	123168.413	124934.245	129380.2602

TABLE 5.12: Options Buy Sell (constrained to less than 10 times of underlying units)

Month	1	2	3	4	5	6	7
Cash	0	0	0	0	0	0	0
Equity	35059.37354	43164.46825	47218.32734	46833.79426	43651.20269	34762.22767	35058.12733
Bond	59880.2356	65048.86604	72149.96867	75857.7575	82027.63057	97525.48715	98014.29226
Commodity	0	0	0	0	0	0	0
Call_Equity_1Month	0	0	0	0	0	0	0
Call_Equity_3Month	4860.789229	0	0	0	0	0	0
Call_Equity_6Month	0	0	881.6431919	3738.045668	11033.88732	20246.50842	23000.1167
Put_Equity_1Month	0	0	0	0	0	0	0
Put_Equity_3Month	0	201.4413795	0	0	0	0	0
Put_Equity_6Month	0	0	0	0	0	0	0
Wealth	99800.39837	108414.7757	120249.9392	126429.5974	136712.7206	152534.2232	156072.5363

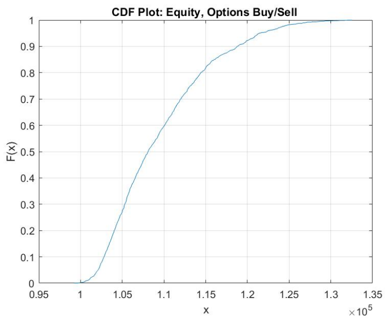

FIGURE 5.25: CDF plot: Equity, Bond, Commodity and Options of different expiries

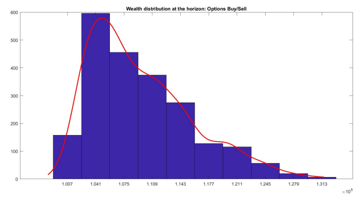

FIGURE 5.26: Hist: Equity, Bond, Commodity and options of different expiries

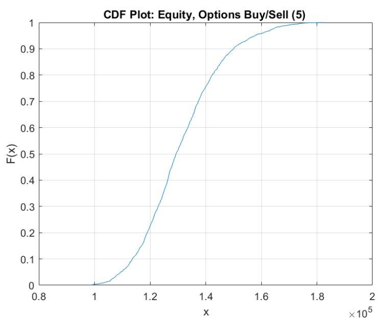

FIGURE 5.27: CDF plot: Equity, Bond, Commodity & trading long positions in options

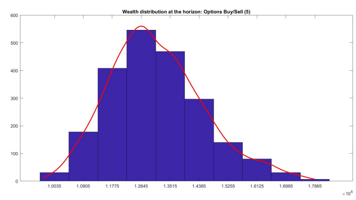

FIGURE 5.28: Histogram fitting: Equity, Bond, Commodity & long Options (5x underlying)

From the tables 5.10-5.12 we see that wealth increases sharply as we increase the amount invested in options. In the first case, a return of 9% was observed, in the second case when the amount invested in options increased to five times, the profit rises to 29% and finally, in the third case, it hits 56%. It should be noted that when we are increasing the investment in options by 5 or 10 times, the absolute change is investment is still relatively small compared to the portfolio value. In the three cases, higher volatility is observed with the increase in options investments. We now see the comparative results of these cases.

We plot CDFs for four different cases when there are no options in the portfolio and the other three cases are where options trading is allowed and we gradually increase the investment amount in that case. The wealth distribution plot of the no-option portfolio is clearly trailing by other portfolios where options are allowed to buy/sell before expiry. As we increase the number of options available for trade the wealth distribution stretches towards right side, significantly. On the simulated data, it has outperformed the previous model where options were bought and exercised at the very next decision stage. Please see figure 5.32. Interestingly, the left tail of the distributions also moves towards the right. However, high investment in options is seen to make portfolios more volatile. This is evident from the wealth trajectory in the mean scenarios.

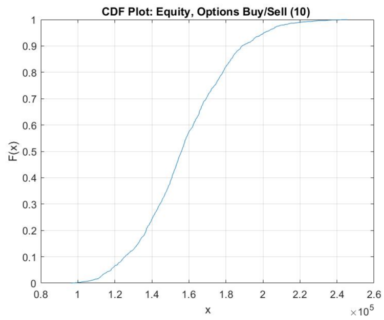

FIGURE 5.29: CDF: Equity, Bond, Commodity & Long Options (10x underlying)

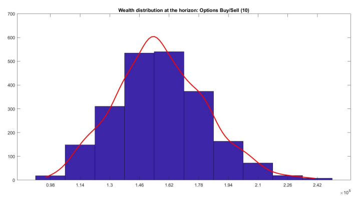

FIGURE 5.30: Hist: Equity, Bond, Commodity & Long Options

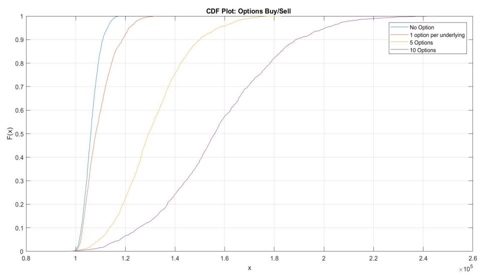

FIGURE 5.31: CDF Plots: Portfolio of Equity, Bond, Commodity and Long Options

5.3 Multistage model- Trading short options positions

In this section, we present results where we run a multistage model that takes into account short selling of options contracts. Since short selling of options can lead to unlimited losses, so we restrict ourselves from entering naked short positions on options contracts. Therefore, we implement strategies like bull call spread (as discussed in chapter 2) and put bear spread. We keep all the initial conditions the same as in the cases discussed earlier in this chapter. Since we test the model for different levels of moneyness to achieve bull call spreads. We take 5%, 10% and 15% ITM and OTM options to achieve bull call spread strategy.

Table 5.5 shows the performance of the portfolio when no options are available for long/short selling. Table 5.13 shows portfolio composition and performance when call bull spread strategy is considered with options 15% moneyness levels, we build a call bull spread strategy using ITM and OTM options that are 15% in-the-money and 15% out-of-the-money. Figure 5.33 and 5.34 show wealth distributions in various cases. It is observed in all the cases that when options included in the portfolio its performance has improved significantly. Portfolio return in the mean scenario is increasing as the moneyness level of ITM and OTM options increases. At the same time, the volatility of the portfolio increases with an increase in moneyness levels of options, it is evident from the fact that OTM options have high volatility in prices. We present a comparative analysis of all these strategies, figure 5.33 and 5.34 show wealth distribution of various strategies, portfolio without options, bull call spread of 5-10-15 % moneyness levels and portfolio that has access to both bull call spread and put bear spread.

TABLE 5.13: Portfolio of equity, bond, commodity and call bull spread on equity options, moneyness of the options 15% (mean scenario)

Month	1	2	3	4	5	6	7
Cash	499.42	0	0	0	0	0	0
Equity	34539.93204	0	0	35031.8528	35045.36272	33457.23263	35146.99602
Bond	59581.18928	99470.19898	99247.52086	59914.09494	60911.45718	66090.49009	66152.98496
Commodity	0	0	0	0	0	0	0
Call Bull Spread	5180.861736	0	0	4910.875948	5562.287721	7265.203518	8954.981897
Wealth	99801.40305	99470.19898	99247.52086	99856.82369	101519.1076	106812.9262	110254.9629

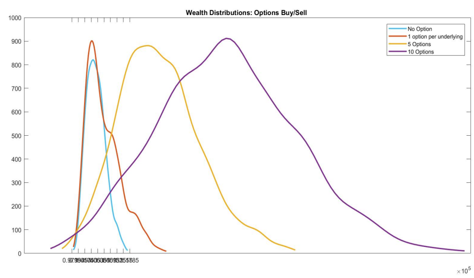

FIGURE 5.32: Wealth Distributions: Portfolio of Equity, Bond, Commodity and Long Options

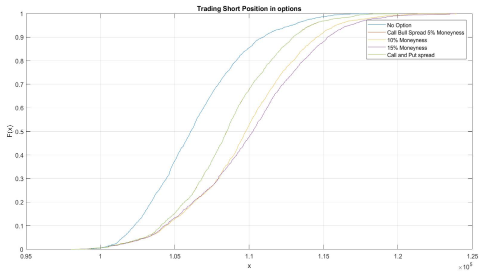

FIGURE 5.33: CDF Plots: Portfolio of Equity, Bond, Commodity and Long and Short position in Options through bull and bear spreads

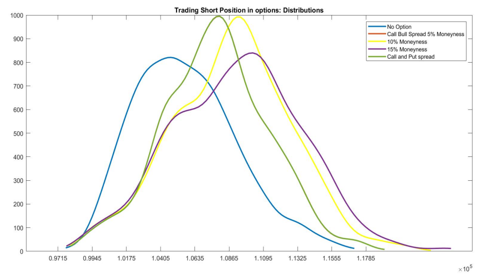

FIGURE 5.34: Wealth Distribution: Portfolio of Equity, Bond, Commodity and Options

All the portfolios with options outperform the portfolio that has no access to trade options. A clear pattern is observed in the CDF plot when we increase the moneyness levels of the options, distribution curves shift towards the right. In the strategy, where put bear spread and call bear spread both are considered have outperformed all the strategies. As it gives access to vertical spreads on both call and put options and hence able to make a profit out of bullish and bearish market conditions.

5.4 Multi-stage Model to update inventory using options

In this section, we present results for the inventory update model. We consider a problem where the investor is willing to increase his holdings in the equity index at a price lower than the market price. His planning horizon is six months, monthly rebalancing is allowed on the portfolio. We apply the multistage model developed in chapter 4. We run the algorithm for the period of May 2014- March 2017 and see how it has performed.

Figure 5.35 shows the months where the algorithm achieved to update inventory using options. The y-axis is the buying cost with respect to the market price, WAP is the weighted average price of the equity index, as we have two possibilities to buy the equity index, one directly at the market price, second, by exercising call options. WAP is the weighted average price of the two. It can be seen from the figure that sometimes, algorithm bought equity at 6% lower than the market price. The maximum, saving observed was 10% in December'14. On average, 1.75% less price was paid compared to the market price. We have considered call option in this case study, the profitability of this strategy relies on market momentum. If the market is bullish, it is going to be profitable.

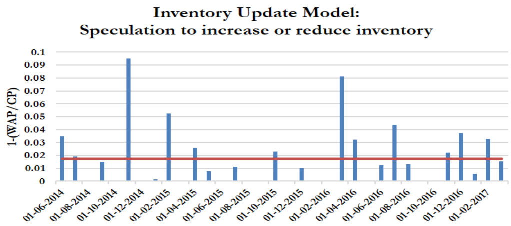

FIGURE 5.35: Inventory Update using Options

In this chapter, we have seen numerical evidence that supports the intuition behind developing this research. We started with a simple 1-month expiry options case and increased the complexity by introducing various features such as option expiring beyond 1-month horizon, buying, selling before expiry and short positions on them and finally, implementation of classical strategies in risk mitigation and profit maximization. With this, we conclude all the chapters of this thesis and conclude our research in the next section.

Conclusion and Future Research

In this research, we discussed multi-stage stochastic programming techniques, cons and pros of including options in a portfolio and their interaction. We reviewed the recent developments in the field of stochastic programming and options in general. Our motivation for the study was to explore both cash and physical settled options contracts within the framework of stochastic programming. We thoroughly reviewed work done Blomvall and Lindberg, 2003, Topaloglou, Vladimirou, and Zenios, 2011 and Yin and Han, 2013b and Davari-Ardakani, Aminnayeri, and Seifi, 2016 who developed single and multiple stage models to include options in a portfolio. We take some big steps in extending the work presented by these researchers. So far, in the literature, only cash-settled options contracts were considered, we extended it to physically settled contracts, where we actually show through out-of-sample results that inventory can actually be updated using call and options, depending on the objective of the investor.

We explored cash-settled options contracts in detail. Other researchers had only considered buying and exercising features of the options, while we extend the literature to trade both long and short positions on call and put options. We have shown it through multiple models in this research report. We start with a single-stage model where options expire at the horizon, then we extend the model to consider options expiring in the very next decision stage, a model similar to the model developed by Yin and Han, 2013b. Then we extend the model to consider options that are expiring at any decision stage along the planning horizon.

Difficulty to introduce such models lies in options pricing on scenario trees, many researchers have expressed their concern about this. We reviewed the work done by Topaloglou, Vladimirou, and Zenios, 2008b to price options on a scenario tree, where they talk about multinomial tree and numerical extension of the Black-Scholes formula. To address this issue of option pricing on scenario trees, we relied on a relatively simple technique. We introduce delta-gamma approximation for option pricing, as options payoff are non-linear, the non-linearity is captured by the gamma of the options in pricing formula, the technique sounds naive but has performed well in this context.

After formulating an option pricing technique on the scenario tree, we went on introducing more complexities to stochastic optimization models to play with options contracts. We then extended the model where we allow selling decision variables on options inventory, giving more flexibility to the investor. We then finally present a model where we consider short selling of options, something that has not been touched by any researcher, to the best of our knowledge. We call it a generic model and we verify the model theoretically that how it reduces to multi-stage models or single-stage models with options by setting a few variables equal to 0. We also present numerical validation for each model, where we actually verify that the desired objective is achieved through the equations implemented in the models.

Finally, we present some numerical results collected from these models. To run the models, we needed a statistical model for the underlying, we referred to Consigli et al., 2012 for equity and bond index model and we rely on econometrics technique to develop a first-order autoregressive distributed lag model where we considered inflation, equity and performance of the bond index as exogenous variables. On the basis of these models, we generate scenarios for the optimization program and collect some evidence.

Our research finds that options can increase the profitability potential of a portfolio many-fold if included in a portfolio in the right amount. Options in all the portfolio optimization models have shown good performance, portfolio with options have outperformed portfolio without options. The protective put strategy has provided protection to the portfolio, while speculating through options have generated huge profits, at the same time they made the portfolio more volatile. Hence, it is still a question if there can be an optimal amount that can be assigned for speculation through options. We have seen in the out-of-sample results that minor speculation through options outperformed all other strategies significantly.

We also saw that options premiums are very lucrative and buying and selling them before expiry could be a profitable strategy. We saw that as we allocate more wealth for buying and selling in options our profit increases by up t0 50%, while the amount that is invested in the options is still relatively small compared to the portfolio value. We then finally introduce bull call spread and bear put spread strategies through a general model that we developed. As we have mentioned before, we avoid taking naked short positions in options. We implemented vertical spread strategies and found that they have performed quite well compared to the portfolio where no options were considered. We considered both call bull spread and put bear spread and saw how well the portfolio performed, as it was protected from shocks in the market in either direction. In summary, our finding is that options should be included in a portfolio to improve profitability and to provide protection to portfolios.

We have taken some big steps forward from the existing literature, however, there are still many questions to be answered. The contribution of this Ph.D. dissertation is in the modeling of dynamic stochastic programming models that can tackle options of different expiries and maturities. Most of the results that we presented here are in-sample results and we do not discuss any arbitrage opportunities arising from including options in the portfolio. In-sample results presented make a profit in almost all the scenarios, it is because of the strict constraints we are using on various asset classes and options, as a result, the in-sample results were not in sync with the out-sample results we observed. The other side of the problem is the proper pricing of options which we have not touched in this thesis. The model we adopt should relate the volatility of the underlying process to the option pricing. This was not in the scope of this dissertation, so, we aim to accomplish that in our future research and we expect that with good option pricing method (such as the one recently discussed by Barkhagen and Blomvall, 2016) we should see more realistic results on the out-sample data.

List of Figures

*All the figures produced in this book are the author's work, the source of data is mentioned where plotting market data.

List of Tables

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