



## On a sharp lemma of Cassels and Montgomery on manifolds

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### Abstract

Let  $(\mathcal{M}, g)$  be a  $d$ -dimensional compact connected Riemannian manifold and let  $\{\varphi_m\}_{m=0}^{+\infty}$  be a complete sequence of orthonormal eigenfunctions of the Laplace–Beltrami operator on  $\mathcal{M}$ . We show that there exists a positive constant  $C$  such that for all integers  $N$  and  $X$  and for all finite sequences of  $N$  points in  $\mathcal{M}$ ,  $\{x(j)\}_{j=1}^N$ , and positive weights  $\{a_j\}_{j=1}^N$  we have

$$\sum_{m=0}^X \left| \sum_{j=1}^N a_j \varphi_m(x(j)) \right|^2 \geq \max \left\{ CX \sum_{j=1}^N a_j^2, \left( \sum_{j=1}^N a_j \right)^2 \right\}.$$

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Let  $(\mathcal{M}, g)$  be a  $d$ -dimensional compact connected Riemannian manifold, with normalized Riemannian measure  $\mu$  such that  $\mu(\mathcal{M}) = 1$ , and Riemannian distance  $d(x, y)$ . Let  $\{\lambda_m^2\}_{m=0}^{+\infty}$  be the sequence of eigenvalues of the (positive) Laplace–Beltrami operator  $\Delta$ , listed in increasing order with repetitions, and let  $\{\varphi_m\}_{m=0}^{+\infty}$  be an associated sequence of orthonormal eigenfunctions. In particular  $\varphi_0 \equiv 1$  and  $\lambda_0 = 0$ .

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This allows to define the Fourier coefficients of  $L^1(\mathcal{M})$  functions as

$$\widehat{f}(\lambda_m) = \int_{\mathcal{M}} f(x) \overline{\varphi_m(x)} d\mu(x)$$

and the associated Fourier series

$$\sum_{m=0}^{+\infty} \widehat{f}(\lambda_m) \varphi_m(x).$$

The main result of this paper is the following theorem.

**Theorem 1** *There exists a positive constant  $C$  such that for all integers  $N$  and  $X$  and for all finite sequences of  $N$  points in  $\mathcal{M}$ ,  $\{x(j)\}_{j=1}^N$ , and positive weights  $\{a_j\}_{j=1}^N$  we have*

$$\sum_{m=0}^X \left| \sum_{j=1}^N a_j \varphi_m(x(j)) \right|^2 \geq \max \left\{ CX \sum_{j=1}^N a_j^2, \left( \sum_{j=1}^N a_j \right)^2 \right\}. \quad (1)$$

Notice that the estimate

$$\sum_{m=0}^X \left| \sum_{j=1}^N a_j \varphi_m(x(j)) \right|^2 \geq \left( \sum_{j=1}^N a_j \right)^2$$

is immediately obtained since for  $m = 0$  one has  $\varphi_0(x) = 1$  for all  $x$  in  $\mathcal{M}$ . The essential part of the theorem is therefore the estimate

$$\sum_{m=0}^X \left| \sum_{j=1}^N a_j \varphi_m(x(j)) \right|^2 \geq CX \sum_{j=1}^N a_j^2. \quad (2)$$

Since for any  $m$  the expected value of  $\left| \sum_{j=1}^N a_j \varphi_m(x(j)) \right|^2$  is  $\sum_{j=1}^N a_j^2$  (see the proof of Proposition 2 below) the above estimate means that independently of how the points are chosen, there is a positive proportion of values of  $m$  between 0 and  $X$  for which  $\left| \sum_{j=1}^N a_j \varphi_m(x(j)) \right|^2$  cannot be essentially smaller than its expected value.

When  $\mathcal{M}$  is the one-dimensional torus, the above theorem is classical and goes back to the work of Cassels [5]. He was interested in estimates on exponential sums, and their relation to Dirichlet's approximation theorem. More precisely, as part of the proof of a slightly weaker version of Dirichlet's theorem, in [5, page 288] he showed that for any choice of  $N$  real numbers  $x(1), \dots, x(N)$  and for any integer  $p \geq 1$ ,

$$\sum_{m=1}^X \left| \sum_{j=1}^N e^{2\pi i m x(j)} \right|^{2p} \geq \frac{1}{2}(X+1)C - \frac{1}{2}N^{2p}, \quad (3)$$

where  $C$  is the sum of the squares of the coefficients in the usual multinomial expansion of  $(x(1) + \dots + x(N))^p$ . When  $p = 1$ , we have  $C = N$  and Cassels estimate reduces to a version of our estimate (1) for the torus and for  $a_j = 1$ . In [14, Theorem 8, Chapter 5], Montgomery gave a version of (3) (with  $p = 1$ ) with positive weights,

$$\sum_{m=1}^X \left| \sum_{j=1}^N a_j e^{2imx(j)} \right|^2 \geq \frac{1}{2}(X + 1) \sum_{j=1}^N a_j^2 - \frac{1}{2} \left( \sum_{j=1}^N a_j \right)^2.$$

He also proved the following version of Cassels inequality on the two dimensional torus  $\mathbb{T}^2$  ([14, Theorem 12 in Chapter 5]). For any  $\mathbf{x}(1), \dots, \mathbf{x}(N)$  in  $\mathbb{T}^2$  and for any  $X_1, X_2$

$$\sum_{|m_1| \leq X_1, |m_2| \leq X_2, \mathbf{m} \neq \mathbf{0}} \left| \sum_{j=1}^N e^{2\pi i \mathbf{m} \cdot \mathbf{x}(j)} \right|^2 \geq NX_1X_2 - N^2. \tag{4}$$

The proof of these results was inspired by Siegel’s analytic proof of Minkowski’s convex body theorem [15]. Indeed, given a symmetric convex body  $\mathcal{C}$ , there exists a non-negative trigonometric polynomial  $T$  such that  $\widehat{T}$  is also non-negative, the support of  $\widehat{T}$  is contained in  $\mathcal{C}$ ,  $T(\mathbf{0}) \geq \text{area}(\mathcal{C})/4$  and  $\widehat{T}(\mathbf{0}) = 1$ . It follows that

$$\begin{aligned} \sum_{\mathbf{m} \in \mathcal{C}} \left| \sum_{j=1}^N e^{2\pi i \mathbf{m} \cdot \mathbf{x}(j)} \right|^2 &\geq \sum_{\mathbf{m} \in \mathcal{C}} \widehat{T}(\mathbf{m}) \left| \sum_{j=1}^N e^{2\pi i \mathbf{m} \cdot \mathbf{x}(j)} \right|^2 \\ &= \sum_{j=1}^N \sum_{k=1}^N \sum_{\mathbf{m} \in \mathcal{C}} \widehat{T}(\mathbf{m}) e^{2\pi i \mathbf{m} \cdot (\mathbf{x}(j) - \mathbf{x}(k))} = \sum_{j=1}^N \sum_{k=1}^N T(\mathbf{x}(j) - \mathbf{x}(k)) \\ &\geq NT(\mathbf{0}) \geq N \text{area}(\mathcal{C})/4. \end{aligned} \tag{5}$$

The construction of the trigonometric polynomial  $T$  is the following. Let  $\bar{\mathbf{x}}$  such that  $\sharp((\frac{1}{2}\mathcal{C} - \bar{\mathbf{x}}) \cap \mathbb{Z}^2) \geq \text{area}(\mathcal{C})/4$ . Then one can simply take

$$T(\mathbf{x}) = \frac{1}{\sharp((\frac{1}{2}\mathcal{C} - \bar{\mathbf{x}}) \cap \mathbb{Z}^2)} \sum_{\mathbf{m}, \mathbf{k} \in \frac{1}{2}\mathcal{C} - \bar{\mathbf{x}}} e^{2\pi i(\mathbf{m} - \mathbf{k}) \cdot \mathbf{x}}. \tag{6}$$

The applications that Montgomery had in mind for this type of inequalities was to the theory of irregularities of distribution (see [14, Chapter 6]. See also [20]). Let  $\mathcal{C}$  be a subset of  $\mathbb{T}^2$  and for any collection of  $N$  points  $\mathbf{x}(1), \dots, \mathbf{x}(N)$  define the discrepancy function

$$D_{\mathcal{C}}(\mathbf{x}) = \sum_{j=1}^N \chi_{\mathcal{C}}(\mathbf{x} + \mathbf{x}(j)) - \text{area}(\mathcal{C}).$$

By Parseval's identity, its  $L^2$  norm is

$$\int_{\mathbb{T}^2} |D_{\mathcal{C}}(\mathbf{x})|^2 d\mathbf{x} = \sum_{\mathbf{m} \neq \mathbf{0}} |\widehat{\chi}_{\mathcal{C}}(\mathbf{m})|^2 \left| \sum_{j=1}^N e^{2\pi i \mathbf{m} \cdot \mathbf{x}(j)} \right|^2.$$

The idea is now to estimate this quantity from below by means of inequality (4). In general, due to the zeros of the function  $\widehat{\chi}_{\mathcal{C}}$ , a lower bound for  $|\widehat{\chi}_{\mathcal{C}}(\mathbf{m})|^2$  is not available. However, it is sometimes possible to let  $\mathcal{C}$  vary in some class of subsets of  $\mathbb{T}^2$ , in such a way that the average of the Fourier coefficients  $|\widehat{\chi}_{\mathcal{C}}(\mathbf{m})|$  over this collection is bounded below by some decreasing function of  $|\mathbf{m}|$ . For example, one can take two disks of radius  $1/4$  and  $1/2$ , or one can take dilations of a square, or rotations and dilations of a smooth convex set. In the case of two disks  $\mathcal{C}_1$  and  $\mathcal{C}_2$  of radius  $1/4$  and  $1/2$  respectively, Montgomery showed that

$$|\widehat{\chi}_{\mathcal{C}_1}(\mathbf{m})|^2 + |\widehat{\chi}_{\mathcal{C}_2}(\mathbf{m})|^2 \geq c \frac{1}{|\mathbf{m}|^3}.$$

Thus,

$$\int_{\mathbb{T}^2} |D_{\mathcal{C}_1}(\mathbf{x})|^2 d\mathbf{x} + \int_{\mathbb{T}^2} |D_{\mathcal{C}_2}(\mathbf{x})|^2 d\mathbf{x} \geq c \sum_{\mathbf{m} \neq \mathbf{0}} \frac{1}{|\mathbf{m}|^3} \left| \sum_{j=1}^N e^{2\pi i \mathbf{m} \cdot \mathbf{x}(j)} \right|^2.$$

Applying now inequality (4) with  $X_1 = X_2 = (2N)^{1/2}$ , this is bounded below by  $cNX_1X_2(\sqrt{2}X_1)^{-3} = cN^{1/2}$ . This means that for any choice of  $N$  points there is a disk of radius  $1/4$  or  $1/2$  for which the discrepancy is greater than  $cN^{1/4}$ .

This type of arguments have a straightforward extension to the higher dimensional torus.

If the collection of points  $\{\mathbf{x}(j)\}_{j=1}^N$  in  $\mathbb{T}^d$  is not evenly distributed, one has to expect that the lower bound in (5) is not achieved (e.g., let all points be the same). This suggests that the exponential sums

$$\sum_{\mathbf{m} \in \mathcal{C}} \left| \sum_{j=1}^N e^{2\pi i \mathbf{m} \cdot \mathbf{x}(j)} \right|^2$$

can be used as a measure of the regularity of the collection of points. See [13] where this is related with Riesz type energy functionals for the torus.

A spherical analog of Cassels–Montgomery inequality has also been used by Bilyk and Dai [2, formula (4.9)] to prove a lower bound for the discrepancy in the  $d$ -dimensional sphere.

Recently, Bilyk, Dai, Steinerberger [3] extended Cassels–Montgomery inequality to the case of smooth compact  $d$ -dimensional Riemannian manifolds without boundary. More precisely they showed that there exists a positive constant  $C$  such that for all

integers  $N$  and  $X$  and for all finite sequences of  $N$  points in  $\mathcal{M}$ ,  $\{x(j)\}_{j=1}^N$ , and positive weights  $\{a_j\}_{j=1}^N$ ,

$$\sum_{m=0}^X \left| \sum_{j=1}^N a_j \varphi_m(x(j)) \right|^2 \geq C \frac{X}{(\log X)^{d/2}} \sum_{j=1}^N a_j^2.$$

This result should be compared with the following simple proposition.

**Proposition 2** *Let  $X$  and  $N$  be positive integers. For all positive weights  $\{a_j\}_{j=1}^N$ , there exists a sequence of points  $\{x(j)\}_{j=1}^N$  in  $\mathcal{M}$  such that*

$$\sum_{m=0}^X \left| \sum_{j=1}^N a_j \varphi_m(x(j)) \right|^2 \leq X \sum_{j=1}^N a_j^2 + \left( \sum_{j=1}^N a_j \right)^2.$$

**Proof** Let

$$\Phi(y_1, \dots, y_N) = \sum_{m=1}^X \left| \sum_{j=1}^N a_j \varphi_m(y_j) \right|^2 = \sum_{m=1}^X \sum_{j,k=1}^N a_j a_k \varphi_m(y_j) \overline{\varphi_m(y_k)}.$$

Since for  $m \neq 0$

$$\int_{\mathcal{M}} \varphi_m(y_j) dy_j = 0,$$

if  $j \neq k$  we have

$$\int_{\mathcal{M}} \dots \int_{\mathcal{M}} \varphi_m(y_j) \overline{\varphi_m(y_k)} dy_1 \dots dy_N = \int_{\mathcal{M}} \int_{\mathcal{M}} \varphi_m(y_j) \overline{\varphi_m(y_k)} dy_j dy_k = 0,$$

while

$$\int_{\mathcal{M}} \dots \int_{\mathcal{M}} \varphi_m(y_j) \overline{\varphi_m(y_j)} dy_1 \dots dy_N = \int_{\mathcal{M}} |\varphi_m(y_j)|^2 dy_j = 1.$$

Hence,

$$\begin{aligned} & \int_{\mathcal{M}} \dots \int_{\mathcal{M}} \Phi(y_1, \dots, y_N) dy_1 \dots dy_N \\ &= \sum_{m=1}^X \sum_{j=1}^N a_j^2 \int_{\mathcal{M}} |\varphi_m(y_j)|^2 dy_j = X \sum_{j=1}^N a_j^2. \end{aligned}$$

Therefore there exist points  $\{x(j)\}_{j=1}^N$  such that

$$\Phi(x(1), \dots, x(N)) \leq X \sum_{j=1}^N a_j^2.$$

□

Our goal is therefore to remove the logarithmic loss in the above result of Bilyk, Dai and Steinerberger, thus obtaining a sharp estimate.

The original proof by Montgomery in the case of the torus uses the Fejér kernel, or more in general a trigonometric polynomial as the one described in (6). A direct adaptation of this proof to the case of a general manifold would require to construct a positive kernel of the form

$$\sum_{m=0}^X c_m \varphi_m(x) \overline{\varphi_m(y)},$$

but unfortunately this type of kernel is not available in a general manifold. One could therefore withdraw, for example, the requirement that the spectrum of the kernel be contained in the set  $\{\lambda_0^2, \dots, \lambda_X^2\}$ . This is the strategy followed by Bilyk, Dai and Steinerberger which use the heat kernel. Our strategy here is on the contrary to use a kernel which is positive up to a negligible error, without dropping the spectrum condition. The existence of such type of kernel can be proved by means of the Hadamard parametrix for the wave operator on the manifold. In the next section we introduce this construction.

A simple consequence of Theorem 1 is the following estimate on the maximum degree  $X$  of linear combinations of eigenfunctions of the Laplacian up to the eigenvalue  $\lambda_X$  that a quadrature rule can integrate exactly. This is a well known result for equal weights, see e.g. [8, Proposition 1], or [18, Theorem 2] where one can find an estimate of the constant  $C$  that depends only on the dimension of the manifold. See also [18, Theorem 1] for a result with general weights.

**Corollary 3** *Let  $X$  be a positive integer and assume there exist points  $\{x(j)\}_{j=1}^N$  and weights  $\{a_j\}_{j=1}^N$  such that for every polynomial*

$$P(x) = \sum_{m=0}^X c_m \varphi_m(x)$$

we have

$$\int_{\mathcal{M}} P(x) dx = \sum_{j=1}^N a_j P(x(j)). \quad (7)$$

Then there exists a constant  $C > 0$  independent of  $X$  and  $N$  such that

$$1 \geq CX \sum_{j=1}^N a_j^2.$$

In particular

$$CX \leq N.$$

**Proof** Since  $\varphi_0(x) \equiv 1$  we must have  $\sum_{i=1}^N a_i = 1$ . Let

$$P(x) = \sum_{m=0}^X \sum_{i=1}^N a_i \overline{\varphi_m(x(i))} \varphi_m(x),$$

then

$$\begin{aligned} \int_{\mathcal{M}} P(x) dx &= \int_{\mathcal{M}} \sum_{m=0}^X \sum_{i=1}^N a_i \overline{\varphi_m(x(i))} \varphi_m(x) dx \\ &= \sum_{m=0}^X \sum_{i=1}^N a_i \overline{\varphi_m(x(i))} \int_{\mathcal{M}} \varphi_m(x) dx = \sum_{i=1}^N a_i = 1. \end{aligned}$$

On the other hand

$$\begin{aligned} \sum_{j=1}^N a_j P(x(j)) &= \sum_{j=1}^N a_j \sum_{m=0}^X \sum_{i=1}^N a_i \overline{\varphi_m(x(i))} \varphi_m(x(j)) \\ &= \sum_{m=0}^X \left| \sum_{j=1}^N a_j \varphi_m(x(j)) \right|^2 \geq CX \sum_{j=1}^N a_j^2. \end{aligned}$$

Hence

$$1 \geq CX \sum_{j=1}^N a_j^2.$$

Applying Cauchy–Schwarz inequality to  $1 = \sum_{i=1}^N a_i$  we easily obtain

$$\sum_{j=1}^N a_j^2 \geq 1/N$$

and therefore  $N \geq CX$ . □

When the manifold  $\mathcal{M}$  is the  $d$  dimensional sphere and the weights  $a_j$  are all equal, the existence of point distributions (spherical designs)  $\{x(j)\}_{j=1}^{N(X)}$  for which (7) holds for some  $N(X) \leq cX$  is the famous conjecture of Korevaar and Meyers [12], recently proved by Bondarenko, Radchenko, Viazovska [4]. The analogous result for a general manifold has been proved in [8]. See also [6] for the case of general weights.

## 1 The Hadamard parametrix for the wave equation

Following [11, III, §17.4], for  $\nu = 0, 1, 2, \dots$ , let us call  $E_\nu(t, x)$  the distribution defined as the inverse Fourier-Laplace transform on  $\mathbb{R}^{d+1}$  of  $\nu!(|\xi|^2 - \tau^2)^{-\nu-1}$ ,

$$E_\nu(t, x) = \nu!(2\pi)^{-d-1} \int_{\text{Im } \tau = c < 0} e^{i(x \cdot \xi + t\tau)} (|\xi|^2 - \tau^2)^{-\nu-1} d\xi d\tau.$$

Note that for  $\nu = 0$ , this is exactly the fundamental solution of the wave operator, see [11, I, §6.2]. The next Proposition (see [11, III, Lemma 17.4.2]) gives more information about the distributions  $E_\nu$ .

**Proposition 4** (i)  $E_\nu$  is a homogeneous distribution of degree  $2\nu - d + 1$  supported in the forward light cone  $\{(t, x) \in \mathbb{R}^{1+d} : t > 0, t^2 \geq |x|^2\}$ .

(ii) Moreover

$$E_\nu(t, x) = 2^{-2\nu-1} \pi^{(1-d)/2} \chi_+^{\nu+(1-d)/2} (t^2 - |x|^2), \quad t > 0,$$

and  $E_\nu(t, x)$  can be regarded as a smooth function of  $t > 0$  with values in  $\mathcal{D}'(\mathbb{R}^d)$ . In particular if  $\psi \in C_0^\infty(\mathbb{R}^{1+d})$  then

$$\langle E_\nu, \psi \rangle = 2^{-2\nu-1} \pi^{(1-d)/2} \int_0^{+\infty} \langle \chi_+^{\nu+(1-d)/2} (t^2 - |\cdot|^2), \psi(t, \cdot) \rangle dt.$$

Also

$$\partial_t^k E_\nu(0^+, \cdot) = 0 \quad \text{for } k \leq 2\nu$$

and

$$\partial_t^{2\nu+1} E_\nu(0^+, \cdot) = \nu! \delta_0.$$

(iii) Finally, setting

$$\check{E}_\nu, \varphi := \langle E_\nu, \check{\varphi} \rangle,$$

where

$$\check{\varphi}(t, x) = \varphi(-t, x),$$



the distributions  $(E_\nu - \check{E}_\nu)(t, x)$  and  $\partial_t(E_\nu - \check{E}_\nu)(t, x)$  can be regarded as continuous radial functions of  $x$  with values in  $\mathcal{D}'(\mathbb{R})$ . With a small abuse of notation we will write  $(E_\nu - \check{E}_\nu)(\cdot, |x|)$  and  $\partial_t(E_\nu - \check{E}_\nu)(\cdot, |x|)$ .

Let us clarify the meaning of the objects that appear in this proposition. Let  $\alpha \in \mathbb{C}$  be such that  $\text{Re } \alpha > -1$  and for every test function  $\varphi \in C_0^\infty(\mathbb{R})$  define the distribution  $\chi_+^\alpha$  as

$$\langle \chi_+^\alpha, \varphi \rangle = \frac{1}{\Gamma(\alpha + 1)} \int_0^{+\infty} x^\alpha \varphi(x) dx.$$

Integration by parts immediately gives

$$\langle \chi_+^\alpha, \varphi \rangle = -\langle \chi_+^{\alpha+1}, \varphi' \rangle$$

so that  $\chi_+^\alpha$  can be extended to all  $\alpha$  with  $\text{Re } \alpha > -2$ , and, repeating the argument, to the whole complex plane (see [11, I, §3.2] for the details).

Also, since the function  $f(x, t) = t^2 - |x|^2$  is a submersion of  $\mathbb{R}^{d+1} \setminus \{0\}$  in  $\mathbb{R}$ , then the pull-back  $\chi_+^\alpha(t^2 - |x|^2) := f^*(\chi_+^\alpha) \in \mathcal{D}'(\mathbb{R}^{d+1} \setminus \{0\})$  is defined by the identity

$$\langle f^*(\chi_+^\alpha), \varphi \rangle := \langle \chi_+^\alpha, \int_{f^{-1}(\cdot)} \frac{\varphi(x, t)}{\|\nabla f(x, t)\|} d\sigma(x, t) \rangle.$$

We observe that by [11, I, Theorem 3.23] the distribution  $\chi_+^{\nu+(1-d)/2}(t^2 - |x|^2)$  can be uniquely extended to  $\mathcal{D}'(\mathbb{R}^{d+1})$  for  $\nu = 0, 1, \dots$

Recall that distributions in  $\mathcal{D}'(\mathcal{M})$  can always be written as  $u = \sum_{m=0}^{+\infty} c_m \varphi_m$ , where the sequence  $\{c_m\}$  is slowly increasing. Their action on smooth functions is given by

$$\langle u, \phi \rangle = \sum_{m=0}^{+\infty} c_m \int \phi \varphi_m.$$

Consider the continuous linear map  $\mathcal{K}_t : \mathcal{D}(\mathcal{M}) \rightarrow \mathcal{D}'(\mathcal{M})$  defined by

$$\phi \mapsto \mathcal{K}_t \phi = \sum_m \cos(\lambda_m t) \widehat{\phi}(\lambda_m) \varphi_m.$$

Observe that  $\mathcal{K}_t \phi$  is in fact a smooth function and it is the solution of the following Cauchy problem for the wave equation

$$\begin{cases} \left( \frac{\partial^2}{\partial t^2} + \Delta_x \right) w(t, x) = 0 \\ w(0, x) = \phi(x) \\ \frac{\partial w}{\partial t}(0, x) = 0. \end{cases}$$

By the Schwartz kernel Theorem (see [11, I, Theorem 5.2.1]), there exists one and only one distribution  $\cos(t\sqrt{\Delta})(x, y) \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$ , called kernel of the map  $\mathcal{K}_t$ , such that

$$\begin{aligned} \langle \cos(t\sqrt{\Delta})(x, y), \eta(x)\phi(y) \rangle &= \langle \mathcal{K}_t\phi, \eta \rangle = \sum_m \cos(\lambda_m t) \widehat{\phi}(\lambda_m) \langle \varphi_m, \eta \rangle \\ &= \sum_m \cos(\lambda_m t) \int \phi(y) \overline{\varphi_m(y)} dy \int \eta(x) \varphi_m(x) dx. \end{aligned}$$

This immediately implies that

$$\cos(t\sqrt{\Delta})(x, y) = \sum_m \cos(\lambda_m t) \varphi_m(x) \overline{\varphi_m(y)},$$

and the identity is of course in the sense of distributions in  $\mathcal{D}'(\mathcal{M} \times \mathcal{M})$ .

Hadamard's construction of the parametrix for the wave operator allows to describe for small values of time  $t$  the singularities of  $\cos(t\sqrt{\Delta})(x, y)$ .

**Theorem 5** (see [16, Theorem 3.1.5]) *Given a  $d$ -dimensional Riemannian manifold  $(\mathcal{M}, g)$ , there exists  $\varepsilon > 0$  and functions  $\alpha_\nu \in \mathcal{C}^\infty(\mathcal{M} \times \mathcal{M})$ , so that if  $Q > d + 3$  the following holds. Let*

$$K_Q(t, x, y) = \sum_{\nu=0}^Q \alpha_\nu(x, y) \partial_t (E_\nu - \check{E}_\nu)(t, d(x, y))$$

and

$$R_Q(t, x, y) = \cos(t\sqrt{\Delta})(x, y) - K_Q(t, x, y),$$

then  $R_Q \in \mathcal{C}^{Q-d-3}([-\varepsilon, \varepsilon] \times \mathcal{M} \times \mathcal{M})$  and

$$|\partial_{t,x,y}^\beta R_Q(t, x, y)| \leq C|t|^{2Q+2-d-|\beta|}.$$

Furthermore  $\alpha_0(x, y) > 0$  in  $\mathcal{M} \times \mathcal{M}$ .

Observe that  $K_Q(t, x, y)$ , by Proposition 4 (iii), defines a distribution on  $\mathbb{R} \times \mathcal{M} \times \mathcal{M}$  via the identity

$$\langle K_Q, \varphi \rangle = \iint_{\mathcal{M} \times \mathcal{M}} \langle K_Q(\cdot, x, y), \varphi(\cdot, x, y) \rangle dx dy.$$

However this distribution describes the singularities of the kernel  $\cos(t\sqrt{\Delta})(x, y)$  only for small time.

## 2 Notations and Fourier transforms

Let us introduce some notation. If  $f$  and  $g$  are integrable functions on  $\mathbb{R}^d$ , we shall denote their convolution by

$$f *_d g(x) = \int_{\mathbb{R}^d} f(x - y)g(y)dy.$$

We define the cosine transform of smooth even functions on  $\mathbb{R}$  as

$$\mathcal{C}f(t) = \int_0^\infty f(s) \cos(st)ds$$

with inverse

$$\mathcal{C}^{-1}f(s) = \frac{2}{\pi} \int_0^\infty f(t) \cos(st)dt.$$

For smooth functions on  $\mathbb{R}^d$  we will use a slightly different normalization, and we define the Fourier transform and its inverse as

$$\begin{aligned} \mathcal{F}_d f(\xi) &= \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} dx, \\ \mathcal{F}_d^{-1} f(x) &= \int_{\mathbb{R}^d} f(\xi)e^{2\pi i x \cdot \xi} d\xi. \end{aligned}$$

For radial functions  $f(x) = f_0(|x|)$ , the above Fourier transform reduces essentially to the Hankel transform, given by (see [17, Chapter 4, Theorem 3.3])

$$\begin{aligned} \mathcal{F}_d f(\xi) &= 2\pi |\xi|^{-\frac{d-2}{2}} \int_0^\infty f_0(s) J_{\frac{d-2}{2}}(2\pi |\xi|s) s^{\frac{d}{2}} ds, \\ \mathcal{F}_d^{-1} f(x) &= 2\pi |x|^{-\frac{d-2}{2}} \int_0^\infty f_0(s) J_{\frac{d-2}{2}}(2\pi |x|s) s^{\frac{d}{2}} ds. \end{aligned} \tag{8}$$

In the future, with an abuse of notation, we will identify the function  $f$  with its radial profile  $f_0$  and write  $\mathcal{F}_d f(|\xi|)$  instead of  $\mathcal{F}_d f(\xi)$ . One can easily show that

$$\mathcal{F}_1 f(t) = 2\mathcal{C}f(2\pi t).$$

In the proof of Theorem 1 we need the inverse cosine transform of the distribution  $\partial_t(E_v - \check{E}_v)$ . By Proposition 4 (iii),  $\partial_t(E_v - \check{E}_v)(t, z)$  can be seen as a continuous function of  $z$  into  $\mathcal{D}'(\mathbb{R})$ . In the following Lemma we compute for every fixed  $z$  the inverse cosine transform of this distribution.

**Lemma 6** Let  $0 \leq \nu < d/2$ . For every  $z \in \mathbb{R}^d$ ,  $\mathcal{C}^{-1}(\partial_t(E_\nu - \check{E}_\nu)(\cdot, z))$  is a function and for all  $t \in \mathbb{R}$

$$\mathcal{C}^{-1}(\partial_t(E_\nu - \check{E}_\nu)(\cdot, z))(t) = \pi^{-d/2} 2^{-\nu-d/2} |t|^{-2\nu-1+d} \frac{J_{-\nu+d/2-1}(t|z|)}{(t|z|)^{-\nu+d/2-1}}. \quad (9)$$

**Proof** Since by Proposition 4 (i) and (iii)

$$E_\nu(t, z) = 2^{-2\nu-1} \pi^{(1-d)/2} \chi_+^{\nu+(1-d)/2}(t^2 - |z|^2) \quad t > 0,$$

if  $(d-1)/2 < \nu < d/2$ , then

$$\partial_t(E_\nu - \check{E}_\nu)(t, z)$$

is an even, locally integrable function in  $t$ , vanishing at  $\infty$ , so that its cosine transform is (see [7, Formula 11, Table 1.3, Chapter 1, page 12]),

$$\begin{aligned} & \mathcal{C}^{-1}(\partial_t(E_\nu - \check{E}_\nu)(\cdot, z))(s) \\ &= \frac{2}{\pi} \int_0^{+\infty} \partial_t E_\nu(t, z) \cos(st) dt \\ &= \frac{2^{-2\nu+1}}{\pi^{(d+1)/2}} \frac{1}{\Gamma(\nu + (1-d)/2)} \int_{|z|}^{+\infty} t(t^2 - |z|^2)^{\nu+(1-d)/2-1} \cos(st) dt \\ &= \pi^{-d/2} 2^{-\nu-d/2} |s|^{-2\nu-1+d} \frac{J_{-\nu+d/2-1}(s|z|)}{(s|z|)^{-\nu+d/2-1}}. \end{aligned}$$

Observe now that the distribution  $\chi_+^{\nu+(1-d)/2}$  in  $\mathcal{D}'(\mathbb{R})$  is entire in the variable  $\nu$ , and so is the distribution  $\partial_t(E_\nu - \check{E}_\nu)(t, z)$  in  $\mathcal{D}'(\mathbb{R})$  for fixed  $z$ . This implies that also the cosine transform

$$\mathcal{C}^{-1}(\partial_t(E_\nu - \check{E}_\nu)(\cdot, z))(s)$$

can be analytically extended to all complex values of  $\nu$  (see [9, Note 1, page 171]). This analytic extension coincides therefore with the analytic extension of the distribution

$$\pi^{-d/2} 2^{-\nu-d/2} |s|^{-2\nu-1+d} \frac{J_{-\nu+d/2-1}(s|z|)}{(s|z|)^{-\nu+d/2-1}}.$$

Observe that this is the product of the locally integrable function  $|s|^{-2\nu-1+d}$  (recall that  $\nu < d/2$ ) with the smooth function  $\pi^{-d/2} 2^{-\nu-d/2} \frac{J_{-\nu+d/2-1}(s|z|)}{(s|z|)^{-\nu+d/2-1}}$ , which is analytic in  $\nu \in \mathbb{C}$ .

Thus, the identity

$$\mathcal{C}^{-1}(\partial_t(E_\nu - \check{E}_\nu)(\cdot, z))(s) = \pi^{-d/2} 2^{-\nu-d/2} |s|^{-2\nu-1+d} \frac{J_{-\nu+d/2-1}(s|z|)}{(s|z|)^{-\nu+d/2-1}}$$

holds for all  $v < d/2$ . □

### 3 Proof of the main result

It suffices to show the main inequality (2) for any positive integer  $N$  and for any integer  $X$  sufficiently large. Indeed, if  $1 \leq X < X_0$

$$\begin{aligned} \sum_{m=0}^X \left| \sum_{j=1}^N a_j \varphi_m(x(j)) \right|^2 &\geq \left| \sum_{j=1}^N a_j \varphi_0(x(j)) \right|^2 = \left( \sum_{j=1}^N a_j \right)^2 \\ &\geq \sum_{j=1}^N a_j^2 \geq \frac{1}{X_0} X \sum_{j=1}^N a_j^2. \end{aligned}$$

Let  $\kappa$  be a positive integer that we will choose later and let  $Y = \kappa X$ . By [10, Theorem 2], we can split the manifold  $\mathcal{M}$  into  $Y$  disjoint regions  $\{\mathcal{R}_i\}_{i=1}^Y$  with measure  $|\mathcal{R}_i| = 1/Y$  and such that each region contains a ball of radius  $c_1 Y^{-1/d}$  and is contained in a ball of radius  $c_2 Y^{-1/d}$ , for appropriate values of  $c_1$  and  $c_2$  independent of  $Y$ . Let us call  $\{\mathcal{B}_r\}_{r=1}^R$  the sequence of all the regions of the above collection  $\{\mathcal{R}_i\}_{i=1}^Y$  which contain at least one of the points  $x(j)$ . We call  $K_r$  the cardinality of the set  $\{j = 1, \dots, N : x(j) \in \mathcal{B}_r\}$  and  $S_r$  the sum of the weights  $\{a_j\}$  corresponding to points  $x(j) \in \mathcal{B}_r$ . Without loss of generality we can assume that

$$S_1 \geq S_2 \geq \dots \geq S_R > 0.$$

We rename the sequence  $\{x(j)\}_{j=1}^N$  as

$$\{x_{r,j}\}_{\substack{r=1,\dots,R \\ j=1,\dots,K_r}}$$

with  $x_{r,j} \in \mathcal{B}_r$  for all  $j = 1, \dots, K_r$ , and the sequence  $\{a_j\}_{j=1}^N$  as

$$\{a_{r,j}\}_{\substack{r=1,\dots,R \\ j=1,\dots,K_r}}$$

Observe that

$$S_r = \sum_{j=1}^{K_r} a_{r,j}.$$

Inequality (2) is an immediate consequence of the following

$$\sum_{m=0}^X \left| \sum_{r=1}^R \sum_{j=1}^{K_r} a_{r,j} \varphi_m(x_{r,j}) \right|^2 \geq CX \sum_{r=1}^R \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2. \tag{10}$$

Let  $\psi$  be a smooth radial function on  $\mathbb{R}^d$  compactly supported in the ball  $B(0, 1/2) = \{x \in \mathbb{R}^d : |x| \leq 1/2\}$  such that  $\|\psi\|_2 = 1$  and  $\int \psi > 0$ , and set  $H(x) = \psi *_d \psi(x)$ . Then clearly  $H$  is radial, compactly supported in  $B(0, 1)$ ,  $H(x) \leq 1$  for all  $x \in \mathbb{R}^d$ , and  $H(0) = 1$ . Moreover its Fourier transform is  $\mathcal{F}_d H(\xi) = (\mathcal{F}_d \psi(\xi))^2 \geq 0$  for all  $\xi \in \mathbb{R}^d$ , and has fast decay at infinity with all its derivatives.

If we now identify  $H(x)$  with its radial profile, we can write

$$\begin{aligned} & \sum_{m=0}^X \left| \sum_{r=1}^R \sum_{j=1}^{K_r} a_{r,j} \varphi_m(x_{r,j}) \right|^2 \\ & \geq \sum_{m=0}^{+\infty} H\left(\frac{\lambda_m}{\lambda_X}\right) \left| \sum_{r=1}^R \sum_{j=1}^{K_r} a_{r,j} \varphi_m(x_{r,j}) \right|^2 \\ & = \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \left( \sum_{m=0}^{+\infty} H\left(\frac{\lambda_m}{\lambda_X}\right) \varphi_m(x_{r,j}) \overline{\varphi_m(x_{s,i})} \right). \end{aligned} \tag{11}$$

Let us define the kernel

$$F_X(x, y) := \sum_{m=0}^{+\infty} H\left(\frac{\lambda_m}{\lambda_X}\right) \varphi_m(x) \overline{\varphi_m(y)}. \tag{12}$$

We will estimate  $F_X(x, y)$  using the parametrix for the wave operator described in the previous section. For this, one would need that the Fourier cosine transform of  $H(\frac{\cdot}{\lambda_X})$  have small support. This of course cannot be achieved, having  $H$  itself compact support. For this reason we pick  $\eta = \mathcal{F}_d \phi$  where  $\phi(\xi)$  is a nonnegative smooth radial function supported in  $B(0, \varepsilon/2\pi)$  and such that  $\phi(\xi) = 1$  in  $B(0, \varepsilon/4\pi)$  and define

$$\tilde{H}(x) = H\left(\frac{\cdot}{\lambda_X}\right) *_d \eta(x).$$

The reason for taking a  $d$ -dimensional convolution will be clarified in Lemma 8 where we use the fact that  $\mathcal{F}_d \tilde{H} \geq 0$ .

Observe that  $\text{supp } \mathcal{F}_d \tilde{H} \subset B(0, \varepsilon/2\pi)$ . It is remarkable that the cosine transform of  $\tilde{H}$  has support in  $[0, \varepsilon]$  and is nonnegative.

**Lemma 7**  $C^{-1} \tilde{H}(\rho) \geq 0$  for  $\rho \geq 0$  and  $C^{-1} \tilde{H}(\rho) = 0$  for  $\rho > \varepsilon$ .

**Proof** It is known (see [19, eq. (3.9)]) that for  $d > d' \geq 1$ ,

$$\mathcal{F}_{d'}(\mathcal{F}_d g)(s) = c_{d,d'} \int_s^{+\infty} (r^2 - s^2)^{(d-d')/2-1} r g(r) dr. \tag{13}$$

Let now  $g(r) = \mathcal{F}_d \tilde{H}(r)$ . Since  $\tilde{H}(s) = \mathcal{F}_d g(s)$  and the cosine transform is essentially  $\mathcal{F}_1$  we obtain

$$C^{-1} \tilde{H}(\rho) = \frac{1}{\pi} \mathcal{F}_1 \mathcal{F}_d g \left( \frac{\rho}{2\pi} \right)$$

and the thesis follows immediately from (13), the fact that  $g(r) \geq 0$  and the fact that  $g(r) = 0$  for  $r > \varepsilon/2\pi$ . □

Let us go back to the kernel  $F_X$ ,

$$\begin{aligned} F_X(x, y) &= \sum_{m=0}^{+\infty} H \left( \frac{\lambda_m}{\lambda_X} \right) \varphi_m(x) \overline{\varphi_m(y)} \\ &= \sum_{m=0}^{+\infty} \tilde{H}(\lambda_m) \varphi_m(x) \overline{\varphi_m(y)} + \sum_{m=0}^{+\infty} \left( H \left( \frac{\lambda_m}{\lambda_X} \right) - \tilde{H}(\lambda_m) \right) \varphi_m(x) \overline{\varphi_m(y)} \end{aligned}$$

Since

$$\tilde{H}(\lambda) = \int_0^{+\infty} C^{-1} \tilde{H}(t) \cos(\lambda t) dt,$$

and  $C^{-1} \tilde{H}(t)$  is supported in  $[-\varepsilon, \varepsilon]$ , by Theorem 5, we can write

$$\begin{aligned} &\sum_{m=0}^{+\infty} \tilde{H}(\lambda_m) \varphi_m(x) \overline{\varphi_m(y)} \\ &= \sum_{m=0}^{+\infty} \int_0^{+\infty} C^{-1} \tilde{H}(t) \cos(\lambda_m t) dt \varphi_m(x) \overline{\varphi_m(y)} \\ &= \frac{1}{2} \sum_{m=0}^{+\infty} \left\langle \cos(\lambda_m \cdot) \varphi_m(x) \overline{\varphi_m(y)}, C^{-1} \tilde{H} \right\rangle \\ &= \frac{1}{2} \left\langle \cos(\cdot \sqrt{\Delta})(x, y), C^{-1} \tilde{H} \right\rangle \\ &= \frac{1}{2} \sum_{\nu=0}^Q \alpha_\nu(x, y) \Omega_\nu(x, y) + \int_0^\varepsilon R_Q(t, x, y) C^{-1} \tilde{H}(t) dt \end{aligned}$$

where we set

$$\Omega_\nu(x, y) = \left\langle \partial_t (E_\nu - \check{E}_\nu)(\cdot, d(x, y)), C^{-1} \tilde{H} \right\rangle.$$

We can therefore decompose the kernel  $F_X$  as follows

$$F_X(x, y) = \sum_{n=1}^5 F_n(x, y),$$

where

$$\begin{aligned}
 F_1(x, y) &= \frac{1}{2} \alpha_0(x, y) \Omega_0(x, y), \\
 F_2(x, y) &= \frac{1}{2} \sum_{1 \leq v < d/2} \alpha_v(x, y) \Omega_v(x, y), \\
 F_3(x, y) &= \frac{1}{2} \sum_{d/2 \leq v \leq Q} \alpha_v(x, y) \Omega_v(x, y), \\
 F_4(x, y) &= \int_0^\varepsilon R_Q(t, x, y) C^{-1} \tilde{H}(t) dt, \\
 F_5(x, y) &= \sum_{m=0}^{+\infty} \left( H\left(\frac{\lambda_m}{\lambda_X}\right) - \tilde{H}(\lambda_m) \right) \varphi_m(x) \overline{\varphi_m(y)}.
 \end{aligned}$$

Recalling (11) and (12) we have

$$\begin{aligned}
 \sum_{m=0}^X \left| \sum_{r=1}^R \sum_{j=1}^{K_r} a_{r,j} \varphi_m(x_{r,j}) \right|^2 &= \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} F_X(x_{r,j}, x_{s,i}) \\
 &= \sum_{n=1}^5 \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} F_n(x_{r,j}, x_{s,i}).
 \end{aligned}$$

We start estimating the term with  $F_1$  which is the positive part of the kernel and gives the main contribution.

**Lemma 8** *For  $\kappa$  large enough there exist  $X_0 > 0$  and  $C > 0$  such that for every  $X > X_0$*

$$\sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} F_1(x_{r,j}, x_{s,i}) \geq CX \sum_{r=1}^R \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2.$$

**Proof** First of all we show that  $\Omega_0(x, y)$  is positive. Indeed, by Lemma 6 and (8), for every  $x, y \in \mathcal{M}$ ,

$$\begin{aligned}
 \Omega_0(x, y) &= \left\langle C^{-1} \left( \partial_t \left( E_0 - \check{E}_0 \right) (\cdot, d(x, y)) \right), \tilde{H} \right\rangle \\
 &= 2 \int_0^{+\infty} C^{-1} \left( \partial_t \left( E_0 - \check{E}_0 \right) (\cdot, d(x, y)) \right) (t) \tilde{H}(t) dt \\
 &= \frac{2}{d(x, y)^{d/2-1} (2\pi)^{d/2}} \int_0^{+\infty} J_{d/2-1} \left( 2\pi \frac{d(x, y)}{2\pi} t \right) \tilde{H}(t) t^{d/2} dt
 \end{aligned}$$



$$\begin{aligned} &= \frac{2}{(2\pi)^d} \mathcal{F}_d \tilde{H} \left( \frac{d(x, y)}{2\pi} \right) \\ &= \frac{2}{(2\pi)^d} \lambda_X^d \mathcal{F}_d H \left( \frac{\lambda_X d(x, y)}{2\pi} \right) \mathcal{F}_d \eta \left( \frac{d(x, y)}{2\pi} \right) \geq 0. \end{aligned}$$

Since also  $\alpha_0(x, y)$  is positive, we can disregard off-diagonal terms,

$$\begin{aligned} &\sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} F_1(x_{r,j}, x_{s,i}) \\ &= \frac{1}{2} \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \alpha_0(x_{r,j}, x_{s,i}) \Omega_0(x_{r,j}, x_{s,i}) \\ &\geq \frac{1}{2} \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{i=1}^{K_r} a_{r,j} a_{r,i} \alpha_0(x_{r,j}, x_{r,i}) \Omega_0(x_{r,j}, x_{r,i}) \\ &= \frac{\lambda_X^d}{(2\pi)^d} \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{i=1}^{K_r} a_{r,j} a_{r,i} \alpha_0(x_{r,j}, x_{r,i}) \mathcal{F}_d H \left( \frac{\lambda_X d(x_{r,j}, x_{r,i})}{2\pi} \right) \mathcal{F}_d \eta \left( \frac{d(x_{r,j}, x_{r,i})}{2\pi} \right). \end{aligned}$$

By Weyl’s estimate (see e.g. [11, III, Corollary 17.5.8])  $\lambda_X \sim X^{1/d}$ . Thus, if  $x, y \in \mathcal{B}_r$  then

$$\lambda_X d(x, y) \leq \lambda_X 2c_2(\kappa X)^{-1/d} \leq c_3 \kappa^{-1/d}.$$

Let  $\kappa$  large enough so that if  $x, y \in \mathcal{B}_r$

$$\mathcal{F}_d H \left( \frac{\lambda_X d(x, y)}{2\pi} \right) = (\mathcal{F}_d \psi)^2 \left( \frac{\lambda_X d(x, y)}{2\pi} \right) \geq \frac{(\mathcal{F}_d \psi)^2(0)}{2} > 0$$

and

$$\frac{d(x, y)}{2\pi} \leq \frac{\varepsilon}{4\pi},$$

so that

$$\mathcal{F}_d \eta \left( \frac{d(x, y)}{2\pi} \right) = 1.$$

Finally,

$$\begin{aligned} & \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} F_1(x_{r,j}, x_{s,i}) \\ & \geq CX \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{i=1}^{K_r} a_{r,j} a_{r,i} = CX \sum_{r=1}^R \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2. \end{aligned}$$

□

The following lemmas show that the contributions given by the terms with  $F_2, F_3, F_4, F_5$  are negligible.

**Lemma 9** *There exist  $C > 0$  and  $X_0 > 0$  such that for every  $X > X_0$*

$$\left| \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} F_2(x_{r,j}, x_{s,i}) \right| \leq CX^{1-2/d} \sum_{r=1}^R \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2.$$

**Proof** We will show that for every integer  $\nu, 1 \leq \nu < d/2,$

$$\left| \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \alpha_\nu(x_{r,j}, x_{s,i}) \Omega_\nu(x_{r,j}, x_{s,i}) \right| \leq CX^{1-2\nu/d} \sum_{r=1}^R \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2.$$

By Lemma 6, for every  $x, y \in \mathcal{M},$

$$\begin{aligned} \Omega_\nu(x, y) &= \left\langle C^{-1} \left( \partial \cdot (E_\nu - \check{E}_\nu)(\cdot, d(x, y)) \right), \tilde{H} \right\rangle \\ &= 2 \int_0^{+\infty} \pi^{-d/2} 2^{-\nu-d/2} |t|^{-2\nu-1+d} \frac{J_{-\nu+d/2-1}(d(x, y)t)}{(d(x, y)t)^{-\nu+d/2-1}} \tilde{H}(t) dt \\ &= \pi^{\nu-d} 2^{1-d} \mathcal{F}_{d-2\nu} \tilde{H} \left( \frac{d(x, y)}{2\pi} \right) \\ &= \pi^{\nu-d} 2^{1-d} \mathcal{F}_{d-2\nu} \left( H \left( \frac{\cdot}{\lambda_X} \right) *_d \eta \right) \left( \frac{d(x, y)}{2\pi} \right). \end{aligned}$$

Using (13) and the fast decay at infinity of  $\mathcal{F}_d \psi,$  for any positive  $M$  there exist positive constants  $C$  and  $G$  such that for every  $\rho \geq 0$

$$\begin{aligned} & \left| \mathcal{F}_{d-2\nu} \left( H \left( \frac{\cdot}{\lambda_X} \right) *_d \eta \right) (\rho) \right| = \lambda_X^d \left| \mathcal{F}_{d-2\nu} \left( \mathcal{F}_d \left( (\mathcal{F}_d \psi)^2 (\lambda_X \cdot) \mathcal{F}_d \eta \right) \right) (\rho) \right| \\ & = c_{d,d-2\nu} \lambda_X^d \int_\rho^{+\infty} (r^2 - \rho^2)^{\nu-1} r (\mathcal{F}_d \psi (\lambda_X r))^2 \mathcal{F}_d \eta(r) dr \end{aligned}$$

$$\begin{aligned}
 &\leq C\lambda_X^d \int_{\rho}^{+\infty} (r^2 - \rho^2)^{\nu-1} r (\mathcal{F}_d \psi(\lambda_X r))^2 dr \\
 &\leq C\lambda_X^d \int_{\rho}^{+\infty} (r^2 - \rho^2)^{\nu-1} r \frac{C}{(1 + \lambda_X r)^G} dr \\
 &\leq C\lambda_X^{d-2\nu} \int_{\lambda_X \rho}^{+\infty} (u^2 - (\lambda_X \rho)^2)^{\nu-1} u \frac{C}{(1 + u)^G} du \leq \frac{C\lambda_X^{d-2\nu}}{(1 + |\lambda_X \rho|)^M}.
 \end{aligned}$$

Therefore, using the symmetry of  $\Omega_{\nu}(x, y)$ , for any integer  $\nu$  with  $1 \leq \nu < d/2$  we obtain

$$\begin{aligned}
 &\left| \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \alpha_{\nu}(x_{r,j}, x_{s,i}) \Omega_{\nu}(x_{r,j}, x_{s,i}) \right| \\
 &\leq 2C \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=r}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \left| \Omega_{\nu}(x_{r,j}, x_{s,i}) \right| \\
 &\leq C \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=r}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \lambda_X^{d-2\nu} \frac{1}{(1 + \lambda_X d(x_{r,j}, x_{s,i}))^M}.
 \end{aligned}$$

In order to estimate the above sum recall that every region  $\mathcal{B}_r$  is contained in a ball centered at a point  $z_r \in \mathcal{B}_r$  of radius  $c_2 Y^{-1/d}$  and let  $c_3 = 10c_2$ . For every fixed  $r = 1, \dots, R$  we will consider separately the contribution of those values of  $s$  for which the  $\mathcal{B}_s$  is near  $\mathcal{B}_r$ , in the sense that  $\mathcal{B}_s$  is contained in the ball centered at  $z_r$  and with radius  $c_3 Y^{-1/d}$ , and the contribution of the remaining values of  $s$ , for which we will say that  $\mathcal{B}_s$  is far from  $\mathcal{B}_r$ . Notice that there are at most

$$\frac{|B(z_r, c_3 Y^{-1/d})|}{Y^{-1}} \leq \frac{C(c_3 Y^{-1/d})^d}{Y^{-1}} \leq Cc_3^d$$

regions  $\mathcal{B}_s$  near  $\mathcal{B}_r$ . Thus, using again that  $\lambda_X \sim X^{1/d}$  and that for  $r \leq s$  we have  $\sum_{j=1}^{K_r} a_{r,j} \geq \sum_{i=1}^{K_s} a_{s,i}$ , we obtain

$$\begin{aligned}
 &\sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=r}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \lambda_X^{d-2\nu} \frac{1}{(1 + \lambda_X d(x_{r,j}, x_{s,i}))^M} \\
 &\leq CX^{1-2\nu/d} \sum_{r=1}^R \sum_{\substack{s=r \\ \mathcal{B}_s \text{ near } \mathcal{B}_r}}^R \sum_{j=1}^{K_r} a_{r,j} \sum_{i=1}^{K_s} a_{s,i} \\
 &\quad + CX^{1-2\nu/d} \sum_{r=1}^R \sum_{\substack{s=r \\ \mathcal{B}_s \text{ far from } \mathcal{B}_r}}^R \sum_{j=1}^{K_r} a_{r,j} \sum_{i=1}^{K_s} a_{s,i} (\lambda_X d(x_{r,j}, x_{s,i}))^{-M}
 \end{aligned}$$

$$\begin{aligned} &\leq CX^{1-2\nu/d} \sum_{r=1}^R \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2 \\ &\quad + CX^{1-2\nu/d} \sum_{r=1}^{R-1} \sum_{\substack{s=r+1 \\ B_s \text{ far from } B_r}}^R \sum_{j=1}^{K_r} a_{r,j} \sum_{i=1}^{K_s} a_{s,i} \left( X^{1/d} d(x_{r,j}, x_{s,i}) \right)^{-M}. \end{aligned}$$

Using again that for  $r \leq s$  we have  $\sum_{j=1}^{K_r} a_{r,j} \geq \sum_{i=1}^{K_s} a_{s,i}$ ,

$$\begin{aligned} &\sum_{r=1}^{R-1} \sum_{\substack{s=r+1 \\ B_s \text{ far from } B_r}}^R \sum_{j=1}^{K_r} a_{r,j} \sum_{i=1}^{K_s} a_{s,i} \left( X^{1/d} d(x_{r,j}, x_{s,i}) \right)^{-M} \\ &= \sum_{r=1}^{R-1} \sum_{j=1}^{K_r} a_{r,j} \sum_{\ell=0}^{\infty} \sum_{\substack{s>r: \\ 2^{\ell-1}c_3Y^{-1/d} \leq d(z_r, z_s) \leq 2^{\ell}c_3Y^{-1/d}}} \sum_{i=1}^{K_s} a_{s,i} \left( X^{1/d} d(x_{r,j}, x_{s,i}) \right)^{-M} \\ &\leq C \sum_{r=1}^{R-1} \sum_{j=1}^{K_r} a_{r,j} \sum_{\ell=0}^{\infty} 2^{-\ell M} \sum_{\substack{s>r: \\ d(z_r, z_s) \leq 2^{\ell}c_3Y^{-1/d}}} \sum_{i=1}^{K_s} a_{s,i} \\ &\leq C \sum_{r=1}^{R-1} \sum_{j=1}^{K_r} a_{r,j} \sum_{\ell=0}^{\infty} 2^{-\ell M} \frac{(2^{\ell}Y^{-1/d})^d}{Y^{-1}} \sum_{j=1}^{K_r} a_{r,j} \\ &\leq C \sum_{r=1}^{R-1} \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2 \sum_{\ell=0}^{\infty} 2^{-\ell(M-d)} \leq C \sum_{r=1}^{R-1} \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2. \end{aligned}$$

□

**Lemma 10** *There exist  $C > 0$  and  $X_0$  such that for every  $X > X_0$*

$$\left| \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} F_3(x_{r,j}, x_{s,i}) \right| \leq C \sum_{r=1}^R \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2.$$

**Proof** We will show that for every integer  $\nu \geq d/2$ ,

$$\begin{aligned} &\left| \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \alpha_{\nu}(x_{r,j}, x_{s,i}) \Omega_{\nu}(x_{r,j}, x_{s,i}) \right| \\ &\leq CX^{1-2\nu/d} \sum_{r=1}^R \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2. \end{aligned}$$

Observe that for  $\nu \geq d/2$ , the distribution  $\partial_t(E_\nu - \check{E}_\nu)(t, d(x, y))$  can be identified with the locally integrable function

$$C_\nu |t| \left( t^2 - d(x, y)^2 \right)_+^{\nu-1+(1-d)/2},$$

for an appropriate value of  $C_\nu$ . Therefore, using the symmetry of  $\Omega_\nu(x, y)$ ,

$$\begin{aligned} & \left| \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \alpha_\nu(x_{r,j}, x_{s,i}) \Omega_\nu(x_{r,j}, x_{s,i}) \right| \\ & \leq C \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=r}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \int_{d(x_{r,j}, x_{s,i})}^{+\infty} t \left( t^2 - d(x_{r,j}, x_{s,i})^2 \right)^{\nu-1+(1-d)/2} C^{-1} \tilde{H}(t) dt, \end{aligned}$$

where we use the fact that  $C^{-1} \tilde{H}(t) \geq 0$ , by Lemma 7.

Assume first that  $d = 1$  and let  $D = d(x_{r,j}, x_{s,i})$ , then

$$\begin{aligned} & \int_D^{+\infty} t(t^2 - D^2)^{\nu-1+(1-d)/2} C^{-1} \tilde{H}(t) dt \\ & = \int_D^{+\infty} t(t^2 - D^2)^{\nu-1} \frac{1}{\pi} \mathcal{F}_1 \tilde{H} \left( \frac{t}{2\pi} \right) dt \\ & = \lambda_X \int_D^{+\infty} t(t^2 - D^2)^{\nu-1} \frac{1}{\pi} \left( \mathcal{F}_1 \psi \left( \lambda_X \frac{t}{2\pi} \right) \right)^2 \mathcal{F}_1 \eta \left( \frac{t}{2\pi} \right) dt \\ & \leq c \lambda_X \int_{\frac{D}{2\pi}}^{+\infty} u^\nu \left( u - \frac{D}{2\pi} \right)^{\nu-1} (\mathcal{F}_1 \psi(\lambda_X u))^2 \mathcal{F}_1 \eta(u) du. \end{aligned}$$

A similar estimate can be obtained for  $d \geq 2$ . Indeed, by formula (13)

$$\begin{aligned} C^{-1} \tilde{H}(t) & = \frac{2}{\pi} C \tilde{H}(t) \\ & = \frac{1}{\pi} \mathcal{F}_1 \mathcal{F}_d \mathcal{F}_d \tilde{H} \left( \frac{t}{2\pi} \right) \\ & = c \int_{\frac{t}{2\pi}}^{+\infty} \left( u^2 - \left( \frac{t}{2\pi} \right)^2 \right)^{(d-1)/2-1} u \mathcal{F}_d \tilde{H}(u) du \\ & = c \lambda_X^d \int_{\frac{t}{2\pi}}^{+\infty} \left( u^2 - \left( \frac{t}{2\pi} \right)^2 \right)^{(d-1)/2-1} u (\mathcal{F}_d \psi(\lambda_X u))^2 \mathcal{F}_d \eta(u) du, \end{aligned}$$

so that, by Fubini's theorem,

$$\begin{aligned} & \int_D^{+\infty} t(t^2 - D^2)^{v-1+(1-d)/2} C^{-1} \tilde{H}(t) dt \\ &= c \lambda_X^d \int_{\frac{D}{2\pi}}^{+\infty} u (\mathcal{F}_d \psi(\lambda_X u))^2 \mathcal{F}_d \eta(u) \\ & \quad \times \left( \int_D^{2\pi u} 2t(t^2 - D^2)^{v-d/2-1/2} \left( u^2 - \left( \frac{t}{2\pi} \right)^2 \right)^{(d-3)/2} dt \right) du. \end{aligned}$$

Since

$$\begin{aligned} & \int_D^U t(t^2 - D^2)^{v-d/2-1/2} (U^2 - t^2)^{(d-3)/2} dt \\ & \leq C U^{(d-3)/2} \int_D^U t^{v-d/2+1/2} (t - D)^{v-d/2-1/2} (U - t)^{(d-3)/2} dt \\ & \leq C U^{(d-3)/2} U^{v-d/2+1/2} \\ & \quad \times \int_0^1 (z(U - D))^{v-d/2-1/2} ((1 - z)(U - D))^{(d-3)/2} (U - D) dz \\ & = C U^{v-1} (U - D)^{v-1} \int_0^1 z^{v-d/2-1/2} (1 - z)^{(d-3)/2} dz, \end{aligned}$$

we obtain

$$\begin{aligned} & \int_D^{+\infty} t(t^2 - D^2)^{v-1+(1-d)/2} C^{-1} \tilde{H}(t) dt \\ & \leq c \lambda_X^d \int_{\frac{D}{2\pi}}^{+\infty} u^v \left( u - \frac{D}{2\pi} \right)^{v-1} (\mathcal{F}_d \psi(\lambda_X u))^2 \mathcal{F}_d \eta(u) du. \end{aligned}$$

Thus, for all  $d \geq 1$ ,

$$\begin{aligned} & \int_D^{+\infty} t(t^2 - D^2)^{v-1+(1-d)/2} C^{-1} \tilde{H}(t) dt \\ & \leq c \lambda_X^d \int_{\frac{D}{2\pi}}^{+\infty} u^v \left( u - \frac{D}{2\pi} \right)^{v-1} (\mathcal{F}_d \psi(\lambda_X u))^2 \mathcal{F}_d \eta(u) du \\ & \leq c \lambda_X^{d-1} \int_{\lambda_X \frac{D}{2\pi}}^{+\infty} \left( \frac{v}{\lambda_X} \right)^v \left( \frac{v}{\lambda_X} - \frac{D}{2\pi} \right)^{v-1} (\mathcal{F}_d \psi(v))^2 \mathcal{F}_d \eta \left( \frac{v}{\lambda_X} \right) dv \\ & \leq c \lambda_X^{d-2v} \int_{\lambda_X \frac{D}{2\pi}}^{+\infty} v^v \left( v - \frac{D \lambda_X}{2\pi} \right)^{v-1} (\mathcal{F}_d \psi(v))^2 dv \end{aligned}$$

$$\begin{aligned} &\leq c\lambda_X^{d-2v} \int_{\lambda_X \frac{D}{2\pi}}^{+\infty} v^{2v-1} (\mathcal{F}_d \psi(v))^2 dv \\ &\leq c\lambda_X^{d-2v} \frac{1}{(1 + \lambda_X D)^M}. \end{aligned}$$

Finally, arguing as in the previous lemma,

$$\begin{aligned} &\sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=r}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \int_{d(x_{r,j}, x_{s,i})}^{+\infty} t \left(t^2 - d(x_{r,j}, x_{s,i})^2\right)^{v-1+(1-d)/2} C^{-1} \tilde{H}(t) dt \\ &\leq C \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=r}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \lambda_X^{d-2v} \frac{1}{(1 + \lambda_X d(x_{r,j}, x_{s,i}))^M} \\ &\leq C X^{1-2v/d} \sum_{r=1}^R \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2. \end{aligned}$$

□

**Lemma 11** *There exist  $C > 0$  and  $X_0 > 0$  such that for every  $X > X_0$*

$$\left| \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} F_4(x_{r,j}, x_{s,i}) \right| \leq C \sum_{r=1}^R \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2.$$

**Proof** Recall that for every  $x, y \in \mathcal{M}$ ,

$$F_4(x, y) = \int_0^\varepsilon R_Q(t, x, y) C^{-1} \tilde{H}(t) dt.$$

As before, if  $d = 1$ , then

$$C^{-1} \tilde{H}(t) = \frac{1}{\pi} \mathcal{F}_1 \tilde{H} \left( \frac{t}{2\pi} \right) = \frac{\lambda_X}{\pi} \left( \mathcal{F}_1 \psi \left( \lambda_X \frac{t}{2\pi} \right) \right)^2 \mathcal{F}_1 \eta \left( \frac{t}{2\pi} \right)$$

and, by Theorem 5,

$$\begin{aligned} |F_4(x, y)| &\leq \frac{\lambda_X}{\pi} \int_0^\varepsilon |R_Q(t, x, y)| \left( \mathcal{F}_1 \psi \left( \lambda_X \frac{t}{2\pi} \right) \right)^2 \mathcal{F}_1 \eta \left( \frac{t}{2\pi} \right) dt \\ &\leq c\lambda_X^{1-2Q-2} \int_0^{+\infty} t^{2Q+1} (\mathcal{F}_1 \psi(t))^2 dt. \end{aligned}$$

A similar estimate holds for  $d \geq 2$ . Indeed, as in the previous lemma,

$$C^{-1} \tilde{H}(t) = c\lambda_X^d \int_{\frac{t}{2\pi}}^{+\infty} \left( u^2 - \left( \frac{t}{2\pi} \right)^2 \right)^{(d-1)/2-1} u (\mathcal{F}_d \psi(\lambda_X u))^2 \mathcal{F}_d \eta(u) du$$

so that, again by Theorem 5,

$$\begin{aligned}
& |F_4(x, y)| \\
&= c \left| \lambda_X^d \int_0^\varepsilon R_Q(t, x, y) \int_{\frac{t}{2\pi}}^{+\infty} \left( u^2 - \left( \frac{t}{2\pi} \right)^2 \right)^{(d-1)/2-1} u (\mathcal{F}_d \psi(\lambda_X u))^2 \mathcal{F}_d \eta(u) du dt \right| \\
&\leq c \lambda_X^d \int_0^{+\infty} u (\mathcal{F}_d \psi(\lambda_X u))^2 \mathcal{F}_d \eta(u) \\
&\quad \times \int_0^{\min(2\pi u, \varepsilon)} |R_Q(t, x, y)| \left( u^2 - \left( \frac{t}{2\pi} \right)^2 \right)^{(d-1)/2-1} dt du \\
&\leq c \lambda_X^d \int_0^{+\infty} u (\mathcal{F}_d \psi(\lambda_X u))^2 \mathcal{F}_d \eta(u) \int_0^{2\pi u} t^{2Q+2-d} \left( u^2 - \left( \frac{t}{2\pi} \right)^2 \right)^{(d-1)/2-1} dt du \\
&\leq c \lambda_X^d \int_0^{+\infty} (\mathcal{F}_d \psi(\lambda_X u))^2 \mathcal{F}_d \eta(u) u^{2Q+1} du. \\
&\leq c \lambda_X^{d-2Q-2} \int_0^{+\infty} (\mathcal{F}_d \psi(v))^2 v^{2Q+1} dv.
\end{aligned}$$

Finally, for all  $d \geq 1$ , since  $R \leq Y = \kappa X$ ,

$$\begin{aligned}
& \left| \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \int_0^\varepsilon R_Q(t, x_{r,j}, x_{s,i}) \mathcal{C}^{-1} \tilde{H}(t) dt \right| \\
&\leq c \lambda_X^{d-2Q-2} \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \\
&\leq c X^{1-(2Q+2)/d} \sum_{r=1}^R \sum_{s=1}^R \sum_{j=1}^{K_r} a_{r,j} \sum_{i=1}^{K_s} a_{s,i} \\
&\leq c X^{1-(2Q+2)/d} \sum_{r=1}^R \sum_{s=r}^R \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2 \\
&\leq c X^{1-(2Q+2)/d} \sum_{r=1}^R (R-r+1) \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2 \\
&\leq c \kappa X^{2-(2Q+2)/d} \sum_{r=1}^R \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2,
\end{aligned}$$

and since  $Q > d + 3$ , the exponent  $2 - (2Q + 2)/d$  is negative and the result follows.  $\square$



**Lemma 12** *There exist  $C > 0$  and  $X_0 > 0$  such that for every  $X > X_0$*

$$\left| \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} F_5(x_{r,j}, x_{s,i}) \right| \leq C \sum_{r=1}^R \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2.$$

**Proof** We need to estimate

$$\begin{aligned} & \left| \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} a_{r,j} a_{s,i} \sum_{m=0}^{+\infty} \left( H \left( \frac{\lambda_m}{\lambda_X} \right) - \tilde{H}(\lambda_m) \right) \varphi_m(x_{r,j}) \overline{\varphi_m(x_{s,i})} \right| \\ & \leq 2 \sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=1}^R \sum_{i=1}^{K_s} \sum_{m=0}^{+\infty} a_{r,j} a_{s,i} \left| H \left( \frac{\lambda_m}{\lambda_X} \right) - \tilde{H}(\lambda_m) \right| |\varphi_m(x_{r,j})| |\varphi_m(x_{s,i})|. \end{aligned}$$

Let us first study the term

$$I(x) = H \left( \frac{x}{\lambda_X} \right) - \tilde{H}(x) = \int_{\mathbb{R}^d} \left[ H \left( \frac{x}{\lambda_X} \right) - H \left( \frac{x-y}{\lambda_X} \right) \right] \eta(y) dy.$$

Since  $\eta(y)$  has rapid decay at infinity and  $H(x)$  is supported in  $B(0, 1)$ , if  $|x| \geq 2\lambda_X$  we have

$$\begin{aligned} |I(x)| & \leq \int_{\mathbb{R}^d} \left| H \left( \frac{x-y}{\lambda_X} \right) \eta(y) \right| dy \leq \int_{\{|x-y| \leq \lambda_X\}} \left| H \left( \frac{x-y}{\lambda_X} \right) \eta(y) \right| dy \\ & \leq c \int_{\{|y| \geq |x| - \lambda_X\}} |\eta(y)| dy \leq C (1 + |x| - \lambda_X)^{-M}. \end{aligned}$$

Assume  $|x| < 2\lambda_X$ . By Taylor’s theorem with integral reminder we can write

$$\begin{aligned} & H \left( \frac{x}{\lambda_X} - \frac{y}{\lambda_X} \right) \\ & = H \left( \frac{x}{\lambda_X} \right) + \sum_{1 \leq |\alpha| \leq M-1} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} H}{\partial x^\alpha} \left( \frac{x}{\lambda_X} \right) \left( -\frac{y}{\lambda_X} \right)^\alpha \\ & \quad + \sum_{|\alpha|=M} \frac{M}{\alpha!} \left( -\frac{y}{\lambda_X} \right)^\alpha \int_0^1 (1-t)^{M-1} \frac{\partial^{|\alpha|} H}{\partial x^\alpha} \left( \frac{x}{\lambda_X} - t \frac{y}{\lambda_X} \right) dt \end{aligned}$$

so that

$$\begin{aligned} H \left( \frac{x}{\lambda_X} - \frac{y}{\lambda_X} \right) & = H \left( \frac{x}{\lambda_X} \right) + \sum_{1 \leq |\alpha| \leq M-1} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} H}{\partial x^\alpha} \left( \frac{x}{\lambda_X} \right) \left( -\frac{y}{\lambda_X} \right)^\alpha \\ & \quad + O \left( \left| -\frac{y}{\lambda_X} \right|^M \right). \end{aligned}$$

It follows that

$$\begin{aligned}
 I(x) &= \int_{\mathbb{R}^d} \left[ H\left(\frac{x}{\lambda_X}\right) - H\left(\frac{x-y}{\lambda_X}\right) \right] \eta(y) dy \\
 &= - \sum_{1 \leq |\alpha| \leq M-1} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} H}{\partial x^\alpha} \left(\frac{x}{\lambda_X}\right) \int_{\mathbb{R}^d} \left(-\frac{y}{\lambda_X}\right)^\alpha \eta(y) dy \\
 &\quad + \int_{\mathbb{R}^d} O\left(\frac{|y|^M}{\lambda_X^M}\right) \eta(y) dy
 \end{aligned}$$

and since

$$\int_{\mathbb{R}^d} y^\alpha \eta(y) dy = \mathcal{F}_d(y^\alpha \eta(y))(0) = (-2\pi i)^{-|\alpha|} \frac{\partial^{|\alpha|} \mathcal{F}_d \eta}{\partial \xi^\alpha}(0) = 0$$

we obtain

$$|I(x)| \leq c \lambda_X^{-M}.$$

Using Hormander’s estimates on the  $L^\infty$  norm of the eigenfunctions (see [16, (3.2.2), page 48])

$$\|\varphi_m\|_\infty \leq C(1 + \lambda_m)^{\frac{d-1}{2}},$$

we obtain

$$\begin{aligned}
 &\sum_{r=1}^R \sum_{j=1}^{K_r} \sum_{s=r}^R \sum_{i=1}^{K_s} \sum_{m=0}^{+\infty} a_{r,j} a_{s,i} \left| H\left(\frac{\lambda_m}{\lambda_X}\right) - \tilde{H}(\lambda_m) \right| |\varphi_m(x_{r,j})| |\varphi_m(x_{s,i})| \\
 &\leq c \lambda_X^{-M} \sum_{\lambda_m \leq 2\lambda_X} (1 + \lambda_m)^{d-1} \sum_{r=1}^R \sum_{s=r}^R \sum_{j=1}^{K_r} a_{r,j} \sum_{i=1}^{K_s} a_{s,i} \\
 &\quad + c \sum_{\lambda_m \geq 2\lambda_X} (1 + \lambda_m - \lambda_X)^{-M} \lambda_m^{d-1} \sum_{r=1}^R \sum_{s=r}^R \sum_{j=1}^{K_r} a_{r,j} \sum_{i=1}^{K_s} a_{s,i} \\
 &\leq c \left( \lambda_X^{-M} \sum_{\lambda_m \leq 2\lambda_X} (1 + \lambda_m)^{d-1} + \sum_{\lambda_m \geq 2\lambda_X} \lambda_m^{-M+d-1} \right) \sum_{r=1}^R (R - r + 1) \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2 \\
 &\leq c \kappa X \left( \lambda_X^{-M} \sum_{\lambda_m \leq 2\lambda_X} (1 + \lambda_m)^{d-1} + \sum_{\lambda_m \geq 2\lambda_X} \lambda_m^{-M+d-1} \right) \sum_{r=1}^R \left( \sum_{j=1}^{K_r} a_{r,j} \right)^2.
 \end{aligned}$$

By Weyl’s estimates, which say that the number of eigenvalues  $\lambda_m^2 \leq T^2$  is asymptotic to  $cT^d$ ,

$$\begin{aligned} & \lambda_X^{-M} \sum_{\lambda_m \leq 2\lambda_X} (1 + \lambda_m)^{d-1} + \sum_{\lambda_m \geq 2\lambda_X} \lambda_m^{-M+d-1} \\ & \leq c\lambda_X^{-M} \lambda_X^d \lambda_X^{d-1} + \sum_{k=1}^{+\infty} \sum_{2^k \lambda_X \leq \lambda_m \leq 2^{k+1} \lambda_X} \lambda_m^{-M+d-1} \\ & \leq c\lambda_X^{-M+2d-1} + c \sum_{k=1}^{+\infty} 2^{d(k+1)} \lambda_X^d (2^k \lambda_X)^{-M+d-1} \\ & \leq c\lambda_X^{-M+2d-1} + c\lambda_X^{-M+2d-1} \sum_{k=1}^{+\infty} 2^{(2d-M-1)k} \end{aligned}$$

and taking  $M$  such that  $-M + 2d - 1 < -d$  gives the result. □

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