Nonlinear Model Predictive Control
Towards New Challenging Applications
Model Predictive Control (MPC) is an area in rapid development with respect to both theoretical and application aspects. The former petrochemical applications of MPC were ‘easy’, in the sense that they involved only a small number of rather similar problems, most of which required only control near steady-state conditions. Further control performance specifications were not very challenging. The improving of technology and control theory enabled the application of MPC in new problems often requiring Nonlinear MPC because of the large transients involved, as it has been already seen even in the chemical process industry for the control of product grade changes. There is now a great interest in introducing MPC in many process and non-process applications such as paper-making, control of many kinds of vehicles, including marine, air, space, road and off-road. Some interesting biomedical applications are also very promising. Finally, the interest in the control of complex systems and networks is significantly increasing.

The new applications frequently involve tight performance specifications, model changes or adaptations because of changing operating points, and, perhaps more significantly, safety-criticality. MPC formulations which offer guarantees of stability and robustness feasibility are expected to be of great importance for the deployment of MPC in these applications. The significant effort in developing efficient solutions of the optimisation problem both using an explicit and a numerical approach is of paramount importance for a wider diffusion of NMPC.

In order to summarize these recent developments, and to consider these new challenges, on September 5-9, 2008, we organized an international workshop entitled “International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control” (NMPC08) which was held in Pavia, Italy. In the spirit of the previous successful workshops held in Ascona, Switzerland, in 1998, and in Freudenstadt-Lauterbad, Germany in 2005, internationally recognized researchers from all over the world, working in the area of nonlinear model predictive control, were joined together. The number of participants has sensibly increased with respect to the previous editions and 21 countries from 4 continents were represented. The aim of this workshop was to lead to an open and critical exchange of ideas and to lay the foundation for new research directions and future international collaborations, facilitating the practical and theoretical advancement of NMPC technologies.
This volume contains a selection of papers presented at the workshop that cover the following topics: stability and robustness, control of complex systems, state estimation, tracking, control of stochastic systems, algorithms for explicit solution, algorithms for numerical solutions and applications. The high quality of the papers has been guaranteed by a double careful peer-review process.

We would like to thank all authors for their interesting contributions. Likewise, we are grateful to all of the involved reviewers for their invaluable comments.

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Lalo Magni
Davide Martino Raimondo
Frank Allgöwer
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Input-to-State Stability: A Unifying Framework for Robust Model Predictive Control

D. Limon, T. Alamo, D.M. Raimondo, D. Muñoz de la Peña, J.M. Bravo, A. Ferramosca, and E.F. Camacho

Abstract. This paper deals with the robustness of Model Predictive Controllers for constrained uncertain nonlinear systems. The uncertainty is assumed to be modeled by a state and input dependent signal and a disturbance signal. The framework used for the analysis of the robust stability of the systems controlled by MPC is the well-known Input-to-State Stability. It is shown how this notion is suitable in spite of the presence of constraints on the system and of the possible discontinuity of the control law.

For the case of nominal MPC controllers, existing results on robust stability are extended to the ISS property, and some novel results are presented. Afterwards, stability property of robust MPC is analyzed. The proposed robust predictive controller uses a semi-feedback formulation and the notion of sequence of reachable sets (or tubes) for the robust constraint satisfaction. Under mild assumptions, input-to-state stability of the predictive controller based on nominal predicted cost is proved. Finally, stability of min-max predictive controllers is analyzed and sufficient conditions for the closed-loop system exhibits input-to-state practical stability property are stated. It is also shown how using a modified stage cost can lead to the ISS property. It is remarkable that ISS of predictive controllers is preserved in case of suboptimal solution of the minimization problem.

Keywords: Robust Nonlinear Model Predictive Control, Input-to-State Stability, Robust Constraint Satisfaction.
1 Introduction

Model predictive control (MPC) is one of the few control techniques capable to cope with constrained system providing an optimal control for a certain performance index. This control technique has been widely used in the process industry and studied in academia [51, 6, 34]. The theoretical development in issues such as stability, constraint satisfaction and robustness has recently matured. The main features of this problem are the following [42]: (i) stability must be ensured considering that the control law is a nonlinear (and maybe discontinuous [45]) function of the state; (ii) recursive feasibility must be ensured to guarantee that the control law is well-posed, (iii) constraints must be robustly fulfilled along the evolution of the closed-loop system and (iv) performance and domain of attraction of the closed-loop system should be optimized.

For the nominal control problem, the Lyapunov theory combined with the invariant set theory provide a suitable theoretical framework to deal with the stability problem [42]. When model mismatches and/or disturbances exist, some different stability frameworks, such as robust stability, ultimately bounded evolution or asymptotic gain property, are used [8, 59, 7, 14, 20, 54, 56]. The problem to assure recursive feasibility and constraint satisfaction in presence of uncertainties is more involved, especially for the case of nonlinear prediction models [4, 21, 41, 54, 39].

This paper presents the notion of input-to-state stability (ISS) [60, 18, 19] as a suitable framework for the analysis of the stabilizing properties of model predictive controllers in presence of uncertainties. The use of ISS analysis in the context of nonlinear MPC is not new (see for instance [28, 30, 37, 53, 25, 26]), but this paper aims to show that it can be used as a general framework of robust stability analysis of constrained nonlinear discontinuous systems. Based on this notion, existing robust MPC techniques are studied: firstly, inherent robustness of the nominal MPC is analyzed and novel sufficient conditions for local ISS are presented, extending existing results [59, 29, 14, 46]. It is demonstrated that uniform continuity of the closed-loop model function or of the cost function are sufficient conditions to ensure robustness of the nominal MPC.

Then robust predictive controllers are studied. These controllers must ensure robust constraint satisfaction and recursive feasibility in spite of the uncertainties as well as a suit closed-loop performance. In order to enhance these properties, a semi-feedback parametrization and a tube (i.e. sequence of reachable sets) based approach [7, 28, 32, 5, 44, 55, 43] has been considered. Under suitable novel assumptions on the tube robust constraint satisfaction is proved.

In the case of cost functions based on nominal predictions [7, 47, 28, 32, 5, 44, 55, 43], ISS is proved subject to continuity of some ingredients. In the case of a worst-case cost function (i.e. min-max predictive controllers) [42, 41, 35, 10, 30, 37, 26, 49] the stability property is analyzed following [53] and shows how, under some standard assumptions, min-max predictive controllers are input to state practically stabilizing due to the worst-case based nature of the controller. Moreover, it
is shown, how ISS can be achieved by means of a dual mode approach or by using an $\mathcal{H}_\infty$ formulation.

A remarkable property derived from this analysis is that the ISS property of the closed-loop system is preserved in the case that the solution of the optimization problem does not provide the optimal solution, but the best suboptimal one.

**Notation and basic definitions**

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$ and $\mathbb{Z}_{\geq 0}$ denote the real, the non-negative real, the integer and the non-negative integer numbers, respectively. Given two integers $a, b \in \mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{[a,b]} \triangleq \{ j \in \mathbb{Z}_{\geq 0} : a \leq j \leq b \}$. Given two vectors $x_1 \in \mathbb{R}^a$ and $x_2 \in \mathbb{R}^b$, $(x_1, x_2) \triangleq [x_1', x_2']' \in \mathbb{R}^{a+b}$. A norm of a vector $x \in \mathbb{R}^a$ is denoted by $|x|$. Given a signal $w \in \mathbb{R}^a$, the signal sequence is denoted by $w \triangleq \{w(0), w(1), \cdots\}$ where the cardinality of the sequence is inferred from the context. $\mathbf{0}$ denotes a suitable signal sequence taking a null value. If a sequence depends on a parameter, as $w(x)$, $w(j,x)$ denotes its $j$-th element. The sequence $w_{[\tau]}$ denotes the truncation of sequence $w$, i.e. $w_{[\tau]}(j) = w(j)$ if $0 \leq j \leq \tau$ and $w_{[\tau]}(j) = 0$ if $j > \tau$. For a given sequence, we denote $\|w\| \triangleq \sup_{k \geq 0}\{|w(k)|\}$.

The set of sequences $\mathbf{w}$, whose elements $w(j)$ belong to a set $\mathbf{W} \subseteq \mathbb{R}^n$ is denoted by $\mathcal{M}_\mathbf{w}$. For a compact set $A$, $A^{sup} \triangleq \sup_{a \in A}\{|a|\}$.

Consider a function $f(x,y) : A \times B \to \mathbb{R}^c$ with $A \subseteq \mathbb{R}^a$ and $B \subseteq \mathbb{R}^b$, then $f$ is said to be uniformly continuous in $x$ for all $x \in A$ and $y \in B$ if for all $\varepsilon > 0$, there exists a real number $\delta(\varepsilon) > 0$ such that $|f(x_1,y) − f(x_2,y)| \leq \varepsilon$ for all $x_1, x_2 \in A$ with $|x_1 − x_2| \leq \delta(\varepsilon)$ and for all $y \in B$. For a given set $\hat{A} \subseteq A$ and $y \in B$, the range of the function $w$ r.t. $x$ is $f(\hat{A},y) \triangleq \{f(x,y) : x \in \hat{A}\} \subseteq \mathbb{R}^c$.

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{H}$ (or a “$\mathcal{H}$-function”) if it is continuous, strictly increasing and $\gamma(0) = 0$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{H}_\infty$ if it is a $\mathcal{H}$-function and $\gamma(s) \to +\infty$ as $s \to +\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{H}$ if, for each fixed $t \geq 0$, $\beta(\cdot,t)$ is of class $\mathcal{H}$, for each fixed $s \geq 0$, $\beta(s,\cdot)$ is decreasing and $\beta(s,t) \to 0$ as $t \to +\infty$. Consider a couple of $\mathcal{H}$-functions $\sigma_1$ and $\sigma_2$, then $\sigma_1 \circ \sigma_2(s) \triangleq \sigma_1(\sigma_2(s))$, besides $\sigma_1^{(j)}(s)$ denotes the $j$-th composition of $\sigma_1$, i.e. $\sigma_1^{(j+1)}(s) = \sigma_1 \circ \sigma_1^{(j)}(s)$ with $\sigma_1^{(1)}(s) \triangleq \sigma_1(s)$. A function $V : \mathbb{R}^a \to \mathbb{R}_{\geq 0}$ is called positive definite if $V(0) = 0$ and there exists a $\mathcal{H}$-function $\alpha$ such that $V(x) \geq \alpha(|x|)$.

## 2 Problem Statement

Consider that the plant to be controlled is modeled by a discrete-time invariant nonlinear difference equation as follows

$$x(k+1) = f(x(k), u(k), d(k), w(k)), \quad k \geq 0 \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the system state, $u(k) \in \mathbb{R}^m$ is the current controlled variable, $d(k) \in \mathbb{R}^q$ is a signal which models external disturbances and $w(k) \in \mathbb{R}^p$ is a signal which models mismatches between the real plant and the model. The solution of
This system is supposed to fulfil the following standing conditions.

**Assumption 1**

1. System (1) has an equilibrium point at the origin, that is \( f(0,0,0,0) = 0 \).
2. The control and state of the plant must fulfill the following constraints on the state and the input:
   \[
   (x(k),u(k)) \in Z
   \]
   where \( Z \subseteq \mathbb{R}^{n+m} \) is closed and contains the origin in its interior.
3. The uncertainty signal \( w \) is modeled as follows
   \[
   w(k) = w_\eta(k) \eta(x(k),u(k))
   \]
   for all \( k \geq 0 \), where \( \eta \) is a known function \( \eta : Z \subseteq \mathbb{R}^{n+m} \to \mathbb{R}_{\geq 0} \) such that it is continuous and \( \eta(0,0) = 0 \). \( w_\eta \in \mathbb{R}^p \) is an exogenous signal contained in a known compact set \( W_\eta \subset \mathbb{R}^p \). \( W \subseteq \mathbb{R}^p \) denotes the set where the signal \( w \) is confined, i.e. \( W \triangleq \{ w \in \mathbb{R}^p : w = w_\eta \eta(x,u), w_\eta \in W_\eta, (x,u) \in Z \} \)
4. The disturbance signal \( d \) is such that \( d(k) \in D \) for all \( k \geq 0 \), where \( D \subseteq \mathbb{R}^q \) is a known compact set containing the origin.
5. The state of the plant \( x(k) \) can be measured at each sample time.

**Remark 1.** The distinction between a disturbance signal \( d \) and a state and input dependent uncertainty signal \( w \) is aimed to enhance the controller performance by taking advantage of the structure of the uncertainties [30]. However, if merely bounded disturbances are used to model uncertainties, then all the presented results can be applied by merely taking \( w(k) = 0 \) for all \( k \geq 0 \).

The nominal model of the plant (1) denotes the system considering zero-disturbance and it is given by
\[
\hat{x}(k+1) = \hat{f}(\hat{x}(k),u(k)), \ k \geq 0
\]
where \( \hat{f}(x,u) \triangleq f(x,u,0,0) \). The solution to this equation for a given initial state \( x(0) \) is denoted as \( \phi(k,x(0),u) \triangleq \phi(k,x(0),u,0,0) \).

This paper is devoted to the stability analysis of the constrained uncertain system (1) (not necessarily continuous) controlled by a model predictive control law \( u(k) = \kappa_N(x(k)) \) (not necessarily continuous). This requires that there exists a stability region \( X_N \) where, if the initial state is inside this region, i.e. \( x(0) \in X_N \), the evolution of the uncertain system fulfils the constraints (that is \( (x(k),\kappa_N(x(k))) \in Z \) for all \( k \geq 0 \) for any possible evolution of the disturbance signals \( d(k) \) and \( w(k) \)) and the robust stability property of the system is ensured.

In the following section of the paper, we show that the notion of input-to-state stability (ISS) is a suitable framework for the robust stability analysis of systems controlled by predictive controllers. Furthermore, this stability notion unifies in a
single framework other commonly-used robust stability notions, allowing existing results on this topic to be reviewed in a more general approach.

3 Input-to-State Stability

Consider that system (1) is controlled by a certain control law \( u(k) = \kappa(x(k)) \), then the closed loop system can be expressed as follows:

\[
x(k + 1) = f_\kappa(x(k), d(k), w(k)), \quad k \geq 0
\]

where \( f_\kappa(x, d, w) \triangleq f(x, \kappa(x), d, w) \). Consider also that assumption [1] holds for the controlled system. Define the set \( X^\kappa \triangleq \{ x \in \mathbb{R}^n : (x, \kappa(x)) \in \mathbb{Z} \} \) and define the function \( \eta_\kappa(x) \triangleq \eta(x, \kappa(x)) \). The solution of this equation at sampling time \( k \), for the initial state \( x(0) \) and the sequences \( d \) and \( w \) is denoted as \( \phi_\kappa(k, x(0), d, w) \). The nominal model function is denoted as \( \tilde{f}_\kappa(x) \triangleq f(x, \kappa(x)) \) and its solution is denoted as \( \tilde{\phi}_\kappa(k, x(0)) \triangleq \phi_\kappa(k, x(0), 0, 0) \).

In this section, the input-to-state stability property is recalled showing its benefits for the analysis of robustness of a controlled system. Afterwards, a more involved notion, useful for the robustness analysis of predictive controllers, is presented: the regional input-to-state practical stability.

3.1 A Gentle Motivation for the ISS Notion

For the sake of clarity, in this section, state dependent uncertainties \( w \) are not considered (or considered zero) and, with a slight abuse of notation, this argument will be dropped in the previously defined functions. Besides, constraints are not taken into account.

A primary requirement of the controlled system is that, in absence of uncertainties, i.e. \( d(k) = 0 \) for all \( k \in \mathbb{Z}_{\geq 0} \), the controlled system \( \tilde{x}(k + 1) = f_\kappa(\tilde{x}(k), 0) \triangleq \tilde{f}_\kappa(\tilde{x}(k)) \) is asymptotically stable. This property is defined as follows:

**Definition 1.** The system \( \tilde{x}(k + 1) = \tilde{f}_\kappa(\tilde{x}(k)) \) is (globally) asymptotically stable (0-AS) if there exists a \( \mathcal{K} \mathcal{L} \)-function \( \beta \) such that \( |\tilde{\phi}_\kappa(j, x(0))| \leq \beta(|x(0)|, j) \).

This property is usually demonstrated by means of the existence of a (not necessarily continuous) Lyapunov function, which is defined as follows [61] §5.9.

**Definition 2.** A function \( V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) is a Lyapunov function of system \( \tilde{x}(k + 1) = \tilde{f}_\kappa(\tilde{x}(k)) \) if there exist three \( \mathcal{K} \mathcal{L} \)-functions, \( \alpha_1, \alpha_2 \) and \( \alpha_3 \), such that \( \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \) and \( V(\tilde{f}_\kappa(x)) - V(x) \leq -\alpha_3(|x|) \).

On the other hand, the effect of the uncertainty makes the system evolution differs from what expected. Then, it would be desirable that this effect is bounded and depends on the size of the uncertainty. This robustness condition has been expressed in the literature using the following notions [13]:
AG: System (5) has an asymptotic gain (AG) if there exists a $\mathcal{K}$-function $\gamma_a$ such that for each $x(0)$ and $d \in \mathcal{M}_D$, the state of the system satisfies the following property:

$$\limsup_{j \to \infty} |\phi_{\kappa}(j, x(0), d)| \leq \gamma_a \left( \limsup_{j \to \infty} |d(j)| \right)$$

This notion is closely related to the ultimately bounded property of a system [22]: the trajectories of the system converge asymptotically to a set which bound depends uniformly on the ultimate bound of the uncertainty.

SM: System (5) has a stability margin (SM) if there exists a $\mathcal{K}_\infty$-function $\sigma$ such that for all $d(k) = \delta(k) \sigma(|x(k)|)$ with $|\delta(k)| \leq 1$, system (5) is asymptotically stable, i.e. there exist a $\mathcal{K}_L$-function $\beta$ such that for all $x(0)$

$$|\phi_{\kappa}(j, x(0), d)| \leq \beta(|x(0)|, j)$$

The function $\sigma$ is called a stability margin. This definition states the existence of a (sufficiently small) state-dependent signal for which asymptotic stability of the uncertain system is maintained. This property is also called robust stability.

The notion of ISS is shown in the following definition [18]:

**Definition 3.** System (5) is ISS if there exist a $\mathcal{K}_L$-function $\beta$ and a $\mathcal{K}$-function $\gamma$ such that for all initial state $x(0)$ and sequence of disturbances $d \in \mathcal{M}_D$,

$$|\phi_{\kappa}(j, x(0), d)| \leq \beta(|x(0)|, j) + \gamma(\|d[j-1]\|)$$

The definition of input-to-state stability of a system comprises both effects (nominal stability and uniformly bounded effect of the uncertainties) in a single condition. In effect, notice that the ISS condition implies asymptotic stability of the undisturbed system (0-AS) (just taking $d = 0$) and that the effect of the disturbance on the evolution of the states is bounded. Furthermore, if the disturbance signal fades, then the disturbed system asymptotically converges to the origin. Then, it is sensible to think that there exists a relation between ISS and the previous robust stability definitions [60]. This assertion is stated in the following theorem.

**Theorem 1.** Consider system (5), then the following statements are equivalent

1. It is input-to-state stable (ISS).
2. The nominal system is asymptotically stable and the disturbed system has an asymptotic gain (0-AS+AG).
3. There exists a stability margin for the disturbed system (SM).

The equivalence between ISS and SM has been proved in [18]. The fact that ISS is equivalent to 0-AS+AG has been proved in [11] as the counterpart for continuous-time systems [60]. If the model is discontinuous, it can be proved that the equivalence properties are valid [33].

**Remark 2.** Stability margin property can be used to demonstrate that a system is ISS even in the case that the uncertainty signal $d(k)$ is not decaying with the norm of the
state. The existence of a SM does not imply that the real signal will be bounded by the SM along the time.

Therefore, the ISS notion generalizes existing classic notions on stability of disturbed system allowing the study of the effect of state dependent, persistent or fading disturbances in a single framework. Moreover, as it will be presented in the following section, there exists Lyapunov-like conditions for the analysis of this property. It is also interesting to study if a nominally asymptotically stable system (0-AS) has a certain degree of robustness. The answer to this question is negative in general, since examples can be found where the 0-AS systems presents zero robustness [20, 14]. Then, additional requirements on the system are necessary. The following theorem gives some sufficient conditions on the nominal system to be ISS, which extends the results of [14, 20] to ISS and generalizes [29]:

**Theorem 2.** Assume that system (5) is such that the function \( f_κ(x, d) \) is uniformly continuous in \( d \) for all \( x \in \mathbb{R}^n \) and \( d \in D \). Assume that system (5) is nominally asymptotically stable (0-AS), then this system is ISS if one of the following conditions holds:

1. There exists a Lyapunov function \( V(x) \) for the nominal system which is uniformly continuous in \( \mathbb{R}^n \).
2. Function \( \tilde{f}_κ(x) \) is uniformly continuous in \( x \in \mathbb{R}^n \).

The proof of this theorem can be found in the appendix.

In some cases, robustness can only be ensured in a neighborhood of the origin and/or for small enough uncertainties. This problem can also be analyzed within the ISS framework by means of the local ISS notion.

**Definition 4.** System (5) is said to be locally ISS if there exist constants \( c_1 \) and \( c_2 \), a \( \mathcal{H} \mathcal{L} \)-function \( \beta \) and a \( \mathcal{H} \)-function \( \gamma \) such that

\[
|\phi_κ(j, x(0), d)| \leq \beta(|x(0)|, j) + \gamma(\|d[j-1]\|)
\]

for all initial state \( |x(0)| \leq c_1 \) and disturbances \( |d(j)| \leq c_2 \).

Under milder conditions, the local ISS property can be ensured.

**Corollary 1 (Local ISS).** The uniform continuity conditions of the latter theorem can be relaxed to achieve local ISS. In effect, assume that the uniform continuity condition of \( f_κ \), \( \tilde{f}_κ \) and \( V \) is replaced by merely continuity at a neighborhood of \( x = 0 \) and \( d = 0 \). Then in virtue of the Heine-Cantor theorem, there exist \( c_1 \) and \( c_2 \) such that for all \( x \) and \( d \) such that \( |x| \leq c_1 \) and \( |d| \leq c_2 \), these functions are uniformly continuous and the theorem can be applied yielding local ISS.

### 3.2 Regional Input-to-State Practical Stability (ISpS)

In this section, a more general definition of the input-to-state stability is recalled: the regional input-to-state practical stability. The term regional refers to the fact that stability property holds in a certain region, which is compulsory for the analysis...
of constrained systems. The term practical means that the input-to-state stability of a neighborhood of the origin could be only ensured. Furthermore, based on \cite{53}, extending the results of \cite{30, 37, 26}, Lyapunov-type sufficient conditions for ISpS are presented.

In the stability analysis of constrained systems, the invariance notion plays an important role. In the following definition, robust invariance for system \cite{5} is presented.

**Definition 5 (Robust positively invariant (RPI) set).** Consider that hypothesis 1 holds for system \cite{5}. A set $\Gamma \subseteq \mathbb{R}^n$ is a robust positively invariant (RPI) set for system \cite{5} if $f_\kappa(x, d, w) \in \Gamma$, for all $x \in \Gamma$, all $w \in W$ and all $d \in D$. Furthermore, if $\Gamma \subseteq X^\kappa$, then $\Gamma$ is called admissible RPI set.

For the robust stability analysis of systems controlled by predictive controllers it is appropriate to use a quite general notion of ISS: regional input-to-state practical stability (ISpS) \cite{60, 18, 37} which is defined as follows:

**Definition 6 (Regional ISpS in $\Gamma$).** Suppose that assumption 1 is satisfied for system \cite{5}. Given a set $\Gamma \subseteq \mathbb{R}^n$, including the origin as an interior point, system \cite{5} is said to be input-to-state practical stable (ISpS) in $\Gamma$ if $\Gamma$ is a robust positively invariant set for \cite{5} and if there exist a $\mathcal{K}_\infty$-function $\beta$, a $\mathcal{K}$-function $\gamma_2$ and a constant $c \geq 0$ such that

$$|\phi_\kappa(j, x(0), d, w)| \leq \beta(|x(0)|, j) + \gamma_2(||d||_{j-1}) + c$$

(7)

for all $x(0) \in \Gamma$, $w \in M_w$, $d \in M_d$ and $k \geq 0$.

**Remark 3.** In the case that $c = 0$ in (7), the system \cite{5} is said to be input-to-state stable (ISS) in $\Gamma$ with respect to $d$.

**Remark 4.** The constant $c$ describes the fact that, in the case of zero disturbances, the controlled system \cite{5} may not evolve to the origin, but to a compact neighborhood of the origin. Thus the ISpS property can also be defined as ISS with respect to a compact nominal invariant set \cite{60, 18}.

Regional ISpS with respect to $d$ will be now associated to the existence of a suitable Lyapunov-like function (not necessarily continuous), which is defined below \cite{30, 37}.

**Definition 7.** (ISpS-Lyapunov function in $\Gamma$) Suppose that assumption 1 is satisfied for system \cite{5}. Consider that $\Gamma$ is a RPI set containing the origin in its interior. A function $V: \mathbb{R}^n \rightarrow \mathbb{R} \geq 0$ is called an ISpS-Lyapunov function in $\Gamma$ for system \cite{5} with respect to $d$, if there exists a compact set $\Omega \subseteq \Gamma$ (including the origin as an interior point), suitable $\mathcal{K}_\infty$-functions $\alpha_1, \alpha_2, \alpha_3$, a $\mathcal{K}$-function $\lambda_2$ and a couple of constants $c_1, c_2 \geq 0$ such that:

$$V(x) \geq \alpha_1(|x|), \quad \forall x \in \Gamma$$

(8)

$$V(x) \leq \alpha_2(|x|) + c_1, \quad \forall x \in \Omega$$

(9)

and for all $x \in \Gamma$, $w \in W$, and $d \in D$. 
\[ V(f_k(x,d,w)) - V(x) \leq -\alpha_3(|x|) + \lambda_2(|d|) + c_2 \] (10)

**Remark 5.** A function \( V: \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) is called an ISS-Lyapunov function in \( \Gamma \) if it is an ISpS-Lyapunov function in \( \Gamma \) with \( c_1 = c_2 = 0 \).

A sufficient condition, that extends the ISS results of [37] and [30] by means of lemma 4 (see the appendix), is stated in the following theorem.

**Theorem 3.** Consider system (5) which fulfils assumption 1. If this system admits an ISpS-Lyapunov function in \( \Gamma \) w.r.t. \( d \), then it is ISpS in \( \Gamma \) w.r.t. \( d \).

The proof can be derived from [53] taking into account lemma 4.

**Remark 6.** Notice also that ISpS w.r.t. \( d \) implicitly states that the state-dependent uncertainty \( w \) is bounded by a stability margin. Under this assumption, an ISpS-Lyapunov function w.r.t. \( w \) and \( d \) is also an ISpS-Lyapunov function w.r.t. \( d \) with a suitable redefinition of the supply functions [53].

### 4 Input-to-State Stability of Nominal MPC

In this section, robust stability of a nominal model predictive controller for system (1) fulfilling assumption 1 is analyzed. This predictive controller is called nominal because it has been designed using the nominal model of the system (4), that is, neglecting the disturbances. We focus on the standard model predictive control formulation presented in [42], which control law is derived from the solution of the following mathematical programming problem \( P_N(x) \) parameterized in the current state \( x \).

\[
\min_u V_N(x,u) \triangleq \sum_{j=0}^{N-1} L(\bar{x}(j),u(j)) + V_f(\bar{x}(N))
\] (11)

\[
s.t. \quad \bar{x}(j) = \bar{\phi}(j,x,u), \ j \in \mathbb{Z}_{[0,N]}
\] (12)

\[
(\bar{x}(j),u(j)) \in Z, \ j \in \mathbb{Z}_{[0,N-1]}
\] (13)

\[
\bar{x}(N) \in X_f
\] (14)

where \( L: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0} \) is the stage cost function, \( V_f: \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) is the terminal cost function and \( X_f \subseteq \mathbb{R}^n \) is the terminal region. It is assumed that \( P_N(x) \) is feasible in a non-empty region denoted \( X_N \). For each \( x \in X_N \), the optimal decision variable of \( P_N(x) \) is denoted \( u^*(x) \) and the optimal cost is \( V_N^*(x) \). The MPC control law derives from the application of the solution in a receding horizon manner \( \kappa_N(x) \triangleq u^*(0,x) \) and it is defined for all \( x \in X_N \).

The stage cost \( L \), the terminal cost \( V_f \) and the terminal set \( X_f \) of the MPC controller are considered to fulfil the following conditions.

**Assumption 2.** The stage cost function \( L(x,u) \) is such that \( L(0,0) = 0 \) and there exists a \( \mathcal{K}_\infty \) function \( \alpha(\cdot) \) such that \( L(x,u) \geq \alpha(|x|) \) for all \( (x,u) \in Z \). \( X_f \) is an
admissible control invariant set for system \( (4) \), i.e. for all \( x \in X_f \) there exists \( u \in \mathbb{R}^m \) such that \((x, u) \in Z \) and \( \tilde{f}(x, u) \in X_f \). \( V_f \) is a control Lyapunov function (CLF) for system \( (4) \) such that for all \( x \in X_f \) there exist two \( \mathcal{K}_\infty \)-functions \( \alpha_{V_f} \) and \( \beta_{V_f} \) satisfying \( \alpha_{V_f}(|x|) \leq V_f(x) \leq \beta_{V_f}(|x|) \) and

\[
\min_{u} \{ V_f(\tilde{f}(x, u)) - V_f(x) + L(x, u) : (x, u) \in Z, \tilde{f}(x, u) \in X_f \} \leq 0, \quad \forall x \in X_f
\]

It is well known that this assumption suffices to prove that the optimal cost \( V^*_N(x) \) is a Lyapunov function of the closed-loop nominal system and the feasible region \( X_N \) is an admissible positively invariant set. Then the MPC control law asymptotically stabilizes system \( (4) \) with a domain of attraction \( X_N \) \([42]\). This property holds for any prediction horizon \( N \), but a larger \( N \) may yield to a better closed-loop performance and a larger domain of attraction, but at expense of a larger computational burden of the calculation of the optimal solution.

The obtained control law stabilizes the nominal system but, is the stability maintained when it is applied to the uncertain system \( (1) \)? This important question has been recently studied. In \( [12, 8, 40, 17] \) robustness is analyzed for nonlinear systems based on the optimality of the control problem for unconstrained systems. In \( [59, 14, 29, 9, 56] \) more general results are obtained using continuity conditions on the solution of the optimization problem. In the following theorem, some of these results are generalized and extended to the ISS notion.

**Theorem 4.** Consider a system given by \( (1) \) fulfilling assumption \( 7 \). Let \( \kappa_N(x) \) be the predictive controller derived from the solution of PN \( (x) \) satisfying assumption \( 2 \) and let \( X_N \) be its feasibility region. Let the model function of system, \( f(x, \kappa_N(x), d, w) \), be uniformly continuous in \( d \) and \( w \) for all \( x \in X_N \), \( d \in D \) and \( w \in W \). If one of the following conditions holds:

1. Function \( \tilde{f}(x, \kappa_N(x)) \) is uniformly continuous in \( x \) for all \( x \in X_N \).
2. The optimal cost \( V^*_N(x) \) is uniformly continuous in \( X_N \).

then system \( (1) \) controlled by the nominal model predictive controller \( u(k) = \kappa_N(x(k)) \) fulfils the ISS property in a robust invariant set \( \Omega_r \subseteq X_N \) for a sufficiently small bound of the uncertainties.

The proof of this theorem can be found in the appendix.

Between the two conditions for ISS stated in the latter theorem, uniform continuity of the optimal cost function results more interesting from a practical point of view since this can be ensured under certain conditions on the MPC problem. Some of these conditions are shown in the following proposition.

**Proposition 1.** Assume that hypotheses of theorem \( 2 \) hold.

**C1:** If the plant has only constraints on the inputs (i.e. \( Z = \mathbb{R}^n \times U \) where \( U \subseteq \mathbb{R}^m \)), the terminal region is \( X_f = \mathbb{R}^n \) and \( V_N(x, u) \) is uniformly continuous in \( x \) for all \( x \in \mathbb{R}^n \) and \( u \in \mathcal{M}(U) \), then the optimal cost \( V^*_N(x) \) is uniformly continuous in \( \mathbb{R}^n \).
Besides, if \( f(x,u,d,w) \) is uniformly continuous in \( x \), \( L(x,u) \) is uniformly continuous in \( x \) and \( V_f(x) \) is uniformly continuous for all \( x \in \mathbb{R}^n \), \( u \in U \), \( d \in D \) and \( w \in W \), then uniform continuity condition of \( V_N(x,u) \) holds.

**C2:** If the plant has constraints on the state and on the inputs (i.e. \( Z \triangleq X \times U \) with \( X \subseteq \mathbb{R}^n \) and \( U \subseteq \mathbb{R}^m \)), \( V_f(x) \) is given by \( V_f(x) \triangleq \lambda V(x) \) where \( V(x) \) is a uniformly continuous CLF in \( X_f \) such that

\[
\min_u \{ V(f(x,u)) - V(x) + L(x,u) : (x,u) \in Z, f(x,u) \in X_f \} \leq 0
\]

for all \( x \in X_f \) with \( \lambda \geq 1 \) a given weighting factor, and \( V_N(x,u) \) is uniformly continuous in \( x \) for all \( x \in X_N \) and \( u \in M \) then there exists a region \( \Omega_r \triangleq \{ x \in \mathbb{R}^n : V^*_N(x) \leq r \} \subseteq X_N \) in which the optimal cost is uniformly continuous. Besides larger \( \lambda \) yields to larger region \( \Omega_r \).

**C3:** If the nominal model is linear, the cost function \( V_N(x,u) \) is linear and the set of constraints \( Z \) is a convex closed polyhedron, then optimal cost \( V^*_N(x) \) is uniformly continuous in \( X_N \).

**C4:** If the nominal model is linear, the cost function \( V_N(x,u) \) is continuous and the set of constraints \( Z \) is a compact convex polyhedron, then optimal cost \( V^*_N(x) \) is uniformly continuous in \( X_N \).

The proof can be found in the appendix.

As studied in [14], MPC controllers may exhibit zero-robustness and hence may not be ISS. From the previously presented results, it can be seen that ISS can be ensured for some special classes of models and ingredients of the MPC. Thus, linear systems with bounded inputs controlled with a predictive control law are ISS if the cost function is continuous. For the case of nonlinear systems uniformly continuous in the uncertainty signals, uniform continuity of the cost function is assumed to derive ISS property. In the uncommon case that the set of constraints of the optimization problem does not depend on the state, ISS is proved. In the case that there exist some constraints depending on the state, such as constraints on the predicted states of the system or terminal constraints, these may cause discontinuity [45] and zero-robustness [14 Section 5]. Fortunately, in this case the closed-loop system will be ISS by merely weighting the terminal cost function. This is a simple and practical method to ensure robustness of the nominal MPC for continuous functions.

Uniform continuity of some functions involved in theorem 4 and property 1 play an important role in the demonstration of the ISS property. Taking into account the Heine-Cantor theorem, the conditions on the functions can be relaxed obtaining simpler results, as stated in the following corollary.

**Corollary 2.** Under the hypotheses of theorem 4 and proposition 7, if the uniform continuity condition is replaced by continuity of the corresponding functions at a neighborhood of \( x = 0 \) and \( d = 0 \), then there exist \( c_1 \) and \( c_2 \) such that for all \( x \) and \( d \) such that \( |x| \leq c_1 \) and \( |d| \leq c_2 \), these functions are uniformly continuous and theorem 4 and proposition 7 can be applied yielding to local ISS.

**Remark 7 (Inherent robustness of suboptimal nominal MPC).** Notice that in the previously presented results on local robustness of the nominal MPC it is implicitly
assumed that the optimal solution of $P_N(x)$ is achieved. However, it is well-known that this requirement is difficult to address when the system is non-linear. In [58], a stabilizing control algorithm based on suboptimal solutions to the optimization problem is presented. Nominal stability of this suboptimal practical procedure has been proved [58, 31].

The question that arises from this fact is if the local ISS property of the MPC based on optimal solutions still holds in case of suboptimality. Observe that suboptimal nominal predictive controller makes sense in presence of uncertainties since, in absence of uncertainties in the system, the feasible solution computed from the last optimal solution suffices for stability. Extending the results from [27, Thm 5.16], it can be proved that continuity of the optimal cost suffices to prove local ISS of the suboptimal controller, but a larger degree of suboptimality implies a lower stability margin.

5 Input-to-State Stability of Robust MPC

Earlier approaches of robust MPC formulations derive the control law from the solution of an optimization problem based on open-loop predictions of the uncertain system evolution. This open-loop scheme results to be very conservative from both a performance and domain of attraction points of view (see [42, Section 4]). In order to reduce this conservativeness, a closed-loop (or feedback) formulation of the MPC has been proposed [57]. In this case, control policies instead of control actions are taken as decision variables, yielding to an infinite dimensional optimization problem that is in general very difficult to solve and for which there exist few efficient algorithms in the literature in the case of linear systems [48, 13]. A practical formulation between these two approaches is the so-called semi-feedback formulation, where a family of parameterized control laws is used [23, 10]. Thus the decision variables are the sequence of the parameters of the control laws, and hence the optimization problem is a finite-dimensional mathematical programming problem.

Consider that the control actions are derived from a given family of controllers parameterized by $v \in \mathbb{R}^s$, $u(k) = \pi(x(k), v(k))$. Thus, system (1) is transformed in

\[ x(k+1) = f_\pi(x(k), v(k), d(k), w(k)), \quad k \geq 0 \]  

where $f_\pi(x, v, d, w) \triangleq f(x, \pi(x, v), d, w)$ and $v$ plays the role of the input of the modified system. The family of control laws is typically chosen as an affine function of the state. Notice that the open-loop formulation is included in the proposed semi-feedback approach.

Consider assumption 1 holds for system (1). Then it is convenient to pose constraint (2) in terms of $v$ as follows

\[ (x(k), v(k)) \in Z_\pi \]  

where $Z_\pi$ is such that $(x, \pi(x, v)) \in Z$ for all $(x, v) \in Z_\pi$. State and input dependent uncertainty signal $w$ can also be written as follows
\[ w(k) = w_\eta(k) \eta_\pi(x(k), v(k)) \] (17)

where \( \eta_\pi(x, v) \triangleq \eta(x, \pi(x, v)) \). It will be useful to define the model function in terms of \( w_\eta \) as follows
\[ f_{\pi\eta}(x, v, d, w_\eta) \triangleq f_\pi(x, v, d, \eta_\pi(x, v)) \] (18)

The nominal model of system (15) is denoted by \( \tilde{f}_\pi(x, v) \triangleq f_\pi(x, v, 0, 0) \). The solution to the difference equation (15) at \( k \) sampling times, starting from \( x \) and for inputs \( v, d \) and \( w \), is denoted by \( \phi_\pi(k, x, v, d, w) \) and the solution to the nominal model \( \tilde{\phi}_\pi(k, x, v) \triangleq \phi_\pi(k, x, v, 0, 0) \).

In the following sections, robust stability of robust predictive controllers based on the nominal cost function and based on the worst-case cost function are analyzed by means of input to state stability framework. In both cases, robust constraint satisfaction will be guaranteed by means of the calculation of a sequence of reachable sets, commonly known as tube.

### 5.1 Tube Based Methods for Robust Constraint Satisfaction

The notion of tube (of trajectories), or sequence of reachable sets, was firstly introduced in [4] as a sequence of sets such that each set can be reached from the previous one. Recently, this idea has emerged again as a tool for robust constraint satisfaction [7, 28, 32, 5, 15] or for design robust predictive controllers for linear [24, 44] and for nonlinear systems [54, 55, 43]. In this section, this notion is presented as a practically attractive method for solving the robust constraint satisfaction.

**Definition 8.** A sequence of sets \( \{X_0, X_1, \cdots, X_N\} \), with \( X_i \subset \mathbb{R}^n \), is called a tube (or a sequence of reachable sets) for system (15) and a given sequence of control inputs \( v \), if \( f_{\pi\eta}(X_i, v(i), D, W_\eta) \subseteq X_{i+1} \), for all \( i \in \mathbb{Z}_{[0,N-1]} \).

A tube can be calculated by means of a suitable procedure to estimate the range of a function.

**Definition 9.** Let \( C^b \) be a class of compact sets contained in \( \mathbb{R}^b \) and let \( F : \mathbb{R}^a \to \mathbb{R}^b \). Then a procedure is called a (guaranteed) range estimator of the function \( F \), denoted by \( \diamond F \), if for every compact set \( X \subset \mathbb{R}^a \), \( \diamond F(X) \) returns a set in \( C^b \) such that \( F(x) \in \diamond F(X) \) for all \( x \in X \).

This procedure is assumed to be computationally tractable and uses a specialized algorithm or property to calculate a compact set of a certain class (as for instance, balls, boxes, intervals, zonotopes, polytopes) which is an outer bound of the exact range [32, 5, 1, 54]. Using this range estimator procedure of the function model \( f_\pi(\cdot, \cdot, \cdot, \cdot) \), a tube can be computed by means of the following recursion:
\[ X_{i+1} = \diamond f_{\pi\eta}(X_i, v(i), D, W_\eta) \] (19)
for \( i \in \mathbb{Z}_{\{0,N-1\}} \) and a given \( X_0 \). Notice that the procedure computes the range for the four arguments. This method has been used to compute a tube by several authors. In [47, 28], Lipschitz continuity of the model function is used in the estimation range procedure providing a ball \( B_r \equiv \{ x \in \mathbb{R}^n : |x| \leq r \} \) as an estimation set (i.e. \( \mathbb{C}^n \) is the set of balls in \( \mathbb{R}^n \)). In [32], the tube is calculated by using a range estimation procedure based on the interval extension of a function, returning interval sets as estimation. In [5] the procedure used is based on zonotope inclusion and an extension of the mean value theorem, and returns a zonotope (i.e. an affine mapping of a hypercube) as estimation region. In [1] a procedure based on DC-programming is used resulting in parallelepipeds. In the case of linear systems, a tube centered in a nominal trajectory and a robust invariant set as cross section is used in [44] and in [55] a tube is computed for a class of nonlinear systems by means of a suitable transformation and using linear-case methods. Notice the latter procedures return convex compact sets, typically polytopes. For the computation of the tubes, the procedure is assumed to fulfil the following hypotheses.

**Assumption 3.** The procedure \( \diamond f_{\pi\eta}(X,v,D,W_\eta) \) is such that

1. For every \( A,B \subset \mathbb{R}^n \) such that \( A \subseteq B \), we have that \( \diamond f_{\pi}(A,v,D,W_\eta) \subseteq \diamond f_{\pi}(B,v,D,W_\eta) \) for every \( v, D \) and \( W_\eta \).
2. Let \( A \) be a robust invariant set for system (1) controlled by \( u = \pi(x,v_f) \), for all \( d(k) \in D \) and \( w_\eta(k) \in W_\eta \). Then \( \diamond f_{\pi\eta}(A,v_f,D,W_\eta) \subseteq A \).

These conditions are slightly restrictive but provide useful properties to the tube when used to design a predictive controller. These properties can be addressed (or relaxed) by a suitable procedure.

In the following sections, the tube-based method will be used in the predictive control formulations. The idea consists in replacing the constraints on the sequence of predicted trajectory with the constraints on the sequence of predicted reachable sets derived from the current state [32, 5]. In [44], the authors propose a tube with an invariant set as constant section and centered in a nominal predicted trajectory which initial state is considered as decision variable. Based on this ingredients and the superposition principle, a nice robust controller is derived. This formulation will not be used in this paper.

### 5.2 Predictive Controllers Based on Nominal Predictions

The robust nominal MPC is a natural extension of the nominal MPC to the case of uncertain systems [47, 28, 15] where the cost function is calculated for nominal predictions while the decision variables must be such that the constraints are fulfilled for any possible realization of the uncertainties.

The proposed predictive controller based on tubes is derived from the following optimization problem \( P_{\text{nt}}^N(x) \):
\[ \min_{\nu} V_N(x, \nu) \triangleq \sum_{j=0}^{N-1} L_\pi(\tilde{x}(j), \nu(j)) + V_f(\tilde{x}(N)) \]  
\[ s.t. \quad \tilde{x}(j) = \tilde{\phi}_\pi(j, x, \nu) \]  
\[ X_0 = \{x\} \]  
\[ X_{j+1} = \circ f_{\pi\eta}(X_j, \nu(j), D, W_\eta), j \in \mathbb{Z}_{[0,N-1]} \]  
\[ X_j \times \nu(j) \subseteq Z_\pi, j \in \mathbb{Z}_{[0,N-1]} \]  
\[ X_N \subseteq X_f \]  
where \( L_\pi(x, \nu) \triangleq L(x, \pi(x, \nu)) \). The feasibility region of this optimization problem is denoted by \( X_{nt}^N \) and \( \nu^*(x) \) denotes the optimal solution. The predictive control law is given by \( u(k) = \kappa_{nt}^N(x(k)) \triangleq \pi(x(k), \nu^*(0; x(k))) \). The ingredients of the optimization problem and the system function must fulfil the following assumption.

**Assumption 4.** The stage cost function \( L(x, u) \) is such that \( L(0, 0) = 0 \) and there exists a \( \mathcal{K}_\infty \) function \( \alpha(\cdot) \) such that \( L(x, u) \geq \alpha(|x|) \) for all \( (x, u) \in \mathbb{Z} \). \( X_f \) is an admissible robust invariant set for system (15) for a suitable parameter \( v_f \), i.e. for all \( x \in X_f, \ (x, v_f) \in Z_\pi \) and \( f_{\pi\eta}(x, v_f, d, w_\eta) \in X_f \) for all \( d \in D, w_\eta \in W_\eta \). \( V_f \) is a Lyapunov function for the nominal system such that for all \( x \in X_f \) there exist \( \alpha_{v_f} \) and \( \beta_{v_f} \), both \( \mathcal{K}_\infty \)-functions, satisfying \( \alpha_{v_f}(|x|) \leq V_f(x) \leq \beta_{v_f}(|x|) \) and

\[ V_f(f_\pi(x, v_f)) - V_f(x) \leq -L_\pi(x, v_f) \]

Function \( f_\pi(x, v, d, w) \) is uniformly continuous in \( x, d \) and \( w \), \( V_f(x) \) is uniformly continuous and \( L_\pi(x, v) \) is uniformly continuous in \( x \) for all \( (x, v) \in Z_\pi, d \in D, \ w \in W \).

In the following theorem, stability of this controller is stated.

**Theorem 5.** Assume that the feasible set \( X_{nt}^N \) is not empty and consider that assumptions 3 and 4 hold. Then for all \( x(0) \in X_{nt}^N \), system (1) controlled by \( u = \kappa_{nt}^N(x) \) robustly fulfils constraint (2) and it is ISS in \( X_{nt}^N \).

The proof of this theorem can be found in the appendix.

**Remark 8.** From the proof of theorem 5 it can be derived that at any sample \( k > 0 \), a feasible solution ensuring the ISS property can be constructed from the solution of the last sample. Then, optimality of the solution of \( P_{nt}^N(x) \) is not required, but merely an enhanced solution (if possible) to the constructed feasible solution.

### 5.3 Min-Max Model Predictive Controllers

Robust predictive controllers based on nominal predictions have demonstrated to robustly stabilize the uncertain system. From a closed-loop performance index point
of view, there may exist uncertainty scenarios where the cost function based on nominal predictions is not a suitable measure of the closed-loop performance. This leads us to the well-known min-max formulation of the robust predictive controller \[57, 42, 41, 10, 37, 30\] and the \(\mathcal{H}_\infty\) control \[36, 16, 35, 38\]. In this section, the ISS property of these controllers is studied. As in the previous section, a semi-feedback formulation of the controller is considered due to practical reasons, but the presented results can be extended to the feedback case. A tube-based formulation for the robust constraint satisfaction is also used leading to a novel min-max formulation. Notice that if it is assumed that the range estimation procedure returns the exact set, results can be extended to the feedback case. A tube-based formulation for the robust control \[36, 16, 35, 38\]. In this section, the ISS property of these controllers is studied. As in the previous section, a semi-feedback formulation of the controller is considered due to practical reasons, but the presented results can be extended to the feedback case. A tube-based formulation for the robust constraint satisfaction is also used leading to a novel min-max formulation. Notice that if it is assumed that the range estimation procedure returns the exact set, results can be extended to the feedback case. A tube-based formulation for the robust control law \[36, 16, 35, 38\]. In this section, the ISS property of these controllers is studied. As in the previous section, a semi-feedback formulation of the controller is considered due to practical reasons, but the presented results can be extended to the feedback case. A tube-based formulation for the robust constraint satisfaction is also used leading to a novel min-max formulation. Notice that if it is assumed that the range estimation procedure returns the exact set, results can be extended to the feedback case.

Assumption 5. The stage cost function \(L(x, u, d, w)\) is such that \(L(0, 0, 0, 0) = 0\) and there exists a couple of \(\mathcal{H}_\infty\) functions \(\alpha_{Lx}, \alpha_{Li}\) such that \(L(x, u, d, w) \geq \alpha_{Lx}(|x|) - \alpha_{Li}(|d|)\) for all \((x, u) \in Z, w \in W\) and \(d \in D\). \(X_f\) is an admissible robust invariant set for system \[15\] for a suitable parameter \(v_f\), i.e., for all \(x, X_f\) such that \(\alpha_{\nu_f}(|x|) \leq \beta_{\nu_f}(|x|)\) and \(X_f\) is ISS-Lyapunov function w.r.t. signal \(d\) in \(X_f\) for all \(w \in W, d \in D, V_f\) is an ISS-Lyapunov function w.r.t. signal \(d\) in \(X_f\) for all \(w \in W, d \in D, V_f\) is an ISS-Lyapunov function w.r.t. signal \(d\) in \(X_f\) for all \(w \in W, d \in D\). Notice that this assumption implies that the system controlled by the terminal control law \(u(k) = \pi(x(k), v_f)\) is ISS w.r.t the disturbance \(d\) in \(X_f\) and hence \(w\) is bounded by a stability margin for all \(x(k) \in X_f\). Input-to-state stability of the min-max controller is proved in the following theorem which is a generalization of \[30, 37\].
Theorem 6. [53] Assume that the feasibility region $X^r_N$ is a non-empty compact set and assumptions 3 and 5 hold, then system (1) controlled by $u(k) = \kappa^r_N(x(k))$ is ISpS with respect to $d$ in the robust invariant region $X^r_N$. The proof of this theorem is derived demonstrating that the optimal cost function of $P_N^r(x)$ is an ISpS-Lyapunov function in $X^r_N$. Moreover, this satisfies the following inequality

$$V^r_N(f(x, \kappa^r_N(x), d, w)) - V^r_N(x) \leq -\alpha_{Lx}(|x|) + \alpha_{Ld}(|d|) + \rho(D^{sup})$$

This highlights two interesting properties of min-max controllers [30, 26]: if the uncertainty $w(k)$ is locally bounded by a stability margin for the system controlled by the terminal control law, then the min-max controller inherits the stability margin and extends it to its domain of attraction $X^r_N$ (see remark 6). The second interesting property stems from the term $\rho(D^{sup})$. This term is the responsible for the practical stability of the closed-loop system and it is derived from the worst-case nature of the min-max optimization problem. Thus, despite the signal $d(k)$ fades, the state of the closed loop system may not converge to the origin.

Input-to-state stability of the min-max controllers can be achieved by two methods: the most simple one is based on the application of the controller in a dual-mode manner, switching to the terminal control law once $x(k) \in X_f$ [26]. The second method is based on the $H_{\infty}$ strategy. This is derived from the optimization problem $P_N^r(x)$ taking a suitable choice of the stage cost [35], as stated in the following theorem.

Theorem 7. [53] Let $L_d(d)$ be a definite positive function and let the stage cost function $L(x, u, d, w) \triangleq L_x(x, u, w) - L_d(d)$ be such that assumption 5 holds for $\rho \equiv 0$. Assume that the feasibility region $X^r_N$ is non-empty then, under assumptions 3 and 5 the closed-loop system $x(k+1) = f(x(k), \kappa^r_N(x(k)), d(k), w(k))$ is ISS with respect to $d$ in the robust invariant region $X^r_N$.

The stability proof of min-max predictive controllers can be obtained by means of dynamic programming [41, 30] or by means of monotonicity of the cost function [35, 37, 2].

Remark 9. Notice that no assumption on the continuity of the model function, stage cost function or terminal cost function are considered in the stability conditions of the min-max controller. This means that min-max is suitable for robust stability of discontinuous systems and/or discontinuous cost functions.

Remark 10. From standard arguments, it can be proved that suboptimal solutions in the minimization of $P_N^r(x)$ can be tolerated retaining stability, but sub-optimality of the maximization stage requires further analysis. This problem is rather interesting since the maximization stage may be computationally more demanding than the minimization one. In [2] it is demonstrated for linear systems how maximization can be relaxed yielding to input-to-state practical stability. This problem has been extended to nonlinear systems in [52].
6 Conclusions

In this paper, existing results on robust model predictive control are reviewed demonstrating how an unifying framework for robust stability analysis can be used: input-to-state stability. Sufficient conditions for local ISS of nominal MPC are stated. In the case of robust MPC, a semi-feedback tube-based formulation is proposed and sufficient conditions for ISS are given in the case of nominal predictions. This framework has also been used for the min-max MPC and its stability has also been studied. Moreover, the robust stability results are valid in the suboptimal case.

However, there are many interesting and open topics on stability and robustness analysis of MPC, as the following: stability in presence of discontinuous model functions, robust constraint satisfaction and enhanced estimation of the tubes, suboptimal approaches of predictive controllers, mainly in the min-max approach, decentralized and distributed formulations for large-scale systems applications or tracking of changing operating points and trajectories.

A Proof of the Theorems

A.1 Technical Lemmas

The following lemmas play an important role in the theoretical development of the results presented in this paper. Their proofs are omitted by lack of space.

Lemma 1. Let \( f \) be a function \( f(x,y) : \mathbb{R}^a \times \mathbb{R}^b \rightarrow \mathbb{R}^c \). Then \( f \) is a uniformly continuous function in \( x \) for all \( x \in A \) and \( y \in B \) iff there exists a \( K_\infty \)-function \( \sigma \) such that

\[
|f(x_1,y) - f(x_2,y)| \leq \sigma(|x_1 - x_2|), \quad \forall x_1, x_2 \in A, \forall y \in B
\]

Lemma 2. Consider a system defined by the difference equation \( x(k+1) = f(x(k),u(k)) \) with \( (x(k),u(k)) \in \mathbb{Z} \). Denote \( \phi(j,x,u) \) the solution to this equation. Assume that \( f \) is assumed to be uniformly continuous in \( x \) for all \( (x,u) \in \mathbb{Z} \) and \( \sigma_x \) is a suitable function such that \( |f(x,u) - f(y,u)| \leq \sigma_x(|x - y|) \). Then

\[
|\phi(j,x,u) - \phi(j,y,u)| \leq \sigma_x(|x - y|).
\]

Lemma 3. Let \( \Gamma \subseteq \mathbb{R}^n \) be a set with the origin in its interior and let \( V(x) : \Gamma \rightarrow \mathbb{R}_{\geq 0} \) be a positive definite function continuous at a neighborhood of the origin, then there exists a \( \mathscr{K}_\infty \)-function \( \alpha \) such that \( V(x) \leq \alpha(|x|) \) for all \( x \in \Gamma \).

Lemma 4. Consider a couple of sets \( \Gamma, \Omega \subseteq \mathbb{R}^n \) both of them with the origin in their interior and such that \( \Omega \subseteq \Gamma \). Let \( V(x) : \Gamma \subseteq \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) be a function such that: (i) \( V(x) < \infty \) for all \( x \in \Gamma \), (ii) there exists a \( \mathscr{K} \)-function \( \alpha_1 \) such that \( V(x) \geq \alpha_1(|x|) \) for all \( x \in \Gamma \), and (iii) there exists a \( \mathscr{K} \)-function \( \alpha_2 \) and a positive constant \( c \geq 0 \) such that \( V(x) \leq \alpha_2(|x|) + c, \forall x \in \Omega \). Then there exists a \( \mathscr{K}_\infty \)-function \( \beta \) such that

\[
V(x) \leq \beta(|x|) + c, \forall x \in \Gamma.
\]
A.2 Proof of Theorem 2

1. Let $V(x)$ be the uniform continuous Lyapunov function according to the definition [2]. In virtue of lemma [1] there exists a $\mathcal{K}$-function $\sigma_V$ such that $|V(y) - V(x)| \leq \sigma_V(y - x)$. Moreover, from the uniform continuity of $f_k$ w.r.t. $d$, there exists a $\mathcal{K}$-function $\sigma_d$ such that $|f_k(x, d_1) - f_k(x, d_2)| \leq \sigma_d(|d_2 - d_1|)$. From these facts, it is inferred that

$$V(f_k(x, d)) - V(x) = V(f_k(x, d)) - V(f_k(x, 0)) + V(f_k(x, 0)) - V(x)$$

$$\leq |V(f_k(x, d)) - V(f_k(x, 0))| - \alpha_3(|x|)$$

$$\leq \sigma_V(|f_k(x, d) - f_k(x, 0)|) - \alpha_3(|x|)$$

$$\leq \sigma_V \circ \sigma_d(|d|) - \alpha_3(|x|)$$

Then $V(x)$ is a ISS-Lyapunov function and in virtue of [18] the system is ISS.

2. The asymptotic stability of the nominal system implies that there exists a $\mathcal{K}$-$\mathcal{L}$-function $\beta$ such that $|\phi_k(j, x(0), 0)| \leq \beta(|x(0)|, j)$.

From this property, it will be proved that there exists a uniform continuous Lyapunov function for the nominal system, following a similar procedure to [50]. For every $\mathcal{K}$-$\mathcal{L}$-function $\beta$, there exists a couple of $\mathcal{K}$-functions $\alpha_1$ and $\alpha_2$ and a constant $a > 0$ such that $\beta(s, t) \leq \alpha_1^{-1}(\alpha_2(s)e^{-at})$ [50].

Define the function

$$V(x) \triangleq \sup_{j \geq 0} \left( \alpha_1(\phi_k(j, x, 0))e^{aj} \right)$$

Since $f_k(x, 0)$ is uniformly continuous, then $\phi_k(j, x, 0)$ is uniformly continuous in $x$, and hence $V(x)$ inherits this property.

See that $\alpha_1(\phi_k(j, x, 0))e^{aj}$ is upper bounded by $\alpha_2(|x|)$, and hence the function is well-defined. Besides, this function satisfies $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$. Then, it suffices to proof the decreasing property:

$$V(f_k(x, 0)) = \sup_{j \geq 0} \left( \alpha_1(\phi_k(j, f_k(x, 0), 0))e^{aj} \right)$$

$$= \sup_{j \geq 1} \left( \alpha_1(\phi_k(j, x, 0))e^{a(j-1)} \right)$$

$$\leq \sup_{j \geq 0} \left( \alpha_1(\phi_k(j, x, 0))e^{aj} \right) e^{-a}$$

$$= V(x)e^{-a} = V(x) - (1 - e^{-a})V(x)$$

$$\leq V(x) - (1 - e^{-a})\alpha_1(|x|)$$

Hence there exists a uniformly continuous Lyapunov function $V(x)$ for the nominal system and then in virtue of the first statement of theorem [2] the disturbed system is ISS.
\(2.\) ISS property directly stems from theorem 2.

1. Consider a given sequence of control actions \(f \in \Omega_r \subseteq X_N\) for all \(x \in \Omega_r\) and hence the control law is defined along the time.

2. ISS property directly stems from theorem 2.

Once the ISS property is derived, the existence of a compact robust invariant set \(\Omega_r\) is proved for a sufficiently small bound of the uncertainties. From the first part of the proof, it can be said that in all the cases there exists a uniformly continuous Lyapunov function \(\tilde{V}(x)\) in \(X_N\). Define \(\Omega_r\) as \(\Omega_r \triangleq \{x : \tilde{V}(x) \leq r\}\) such that \(\Omega_r \subseteq X_N\). Then there exist a couple of \(\mathcal{K}\)-functions \(\gamma_d, \gamma_w\)

\[
\tilde{V}(f(x, \kappa_N(x), d, w)) \leq \tilde{V}(x) - L(x, \kappa_N(x)) + \gamma_d(|d|) + \gamma_w(|w|)
\]

for all \(x \in X_N\). From standard Lyapunov theory arguments (see [18] for instance) there exists a \(\mathcal{K}_\infty\) function \(\alpha_1\) such that \(\tilde{V}(x) \geq \alpha_1(|x|)\). Furthermore, there exists a \(\mathcal{K}_\infty\) function \(\rho\) such that \(\theta(s) = s - \rho(s)\) is a \(\mathcal{K}\)-function and ensures that \(\tilde{V}(f(x, \kappa_N(x), d, w)) \leq \rho \circ \tilde{V}(x) + \gamma_d(|d|) + \gamma_w(|w|)\) [18 Lemma 3.5]. Taking a couple of constants \(c_1\) and \(c_2\) such that \(\gamma_d(c_1) + \gamma_w(c_2) \leq \theta(r)\), then we have that for all \(|d| \leq c_1\) and \(|w| \leq c_2\),

\[
\tilde{V}(f(x, \kappa_N(x), d, w)) \leq \rho(r) + \gamma_d(|d|) + \gamma_w(|w|)
\]

\[
\leq \rho(r) + \theta(r) = r
\]

Then, robust invariance of \(\Omega_r\) is derived.

\(A.4\) Proof of Proposition 7.

1. Consider a given sequence of control actions \(u\) and \(x, z \in \mathbb{R}^n\), then in virtue of the uniform continuity of the cost function \(V_N(x, u)\) there exists a \(\mathcal{K}_\infty\)-function \(\theta\) such that \(|V_N(x, u) - V_N(z, u)| \leq \theta(|x - z|)\).

Let \(u^*(x)\) be the optimal solution of \(P_N(x)\). Since the constraints of \(P_N(x)\) do not depend on the state \(x\), \(u^*(x)\) is feasible for any \(x \in \Omega_r\). Assume (without loss of generality) that \(V_N^*(z) \geq V_N^*(x)\), then

\[
|V_N^*(z) - V_N^*(x)| \leq |V_N(z, u^*(x)) - V_N(x, u^*(x))| \leq \theta(|x - z|)
\]

Therefore, the optimal cost is uniformly continuous in \(\mathbb{R}^n\).

In order to prove the second statement, denote \(\tilde{x}(i) = \tilde{\phi}(i, \tilde{x}, u)\) and \(\tilde{z}(i) = \tilde{\phi}(i, \tilde{z}, u)\). Uniform continuity of the model function implies the existence of a \(\mathcal{K}_\infty\)-function \(\sigma_x\) such that \(|f(x, u, d, w) - f(z, u, d, w)| \leq \sigma_x(|x - z|)\) for all \(x, z\) in \(\mathbb{R}^n\). Then from lemma 2 it is derived that \(|\tilde{x}(i) - \tilde{z}(i)| \leq \sigma^2_x(|\tilde{x}(0) - \tilde{z}(0)|)\).
Analogously, there also exists a couple of $\mathcal{H}_\infty$-functions $\sigma_L, \sigma_{V_f}$ such that $|L(x,u) - L(z,u)| \leq \sigma_L(|x-z|)$ and $|V_f(x) - V_f(z)| \leq \sigma_{V_f}(|x-z|)$ for all $x, z \in \mathbb{R}^n$ and $u \in U$. Considering these results we have that

$$
|V_N(x,u) - V_N(z,u)| \leq \sum_{j=0}^{N-1} |L(\bar{\chi}(j), u(j)) - L(\bar{\chi}(j), u(j))| + |V_f(\bar{\chi}(N)) - V_f(\bar{\chi}(N))|
$$

$$
\leq \sum_{j=0}^{N-1} \sigma_L \circ \sigma^L_x(|x-z|) + \sigma_{V_f} \circ \sigma^N_x(|x-z|)
$$

Hence the cost function is uniformly continuous.

2. Consider the compact set $\Omega_r \triangleq \{ x \in \mathbb{R}^n : V^*_N(x) \leq r \}$ contained in the interior of $X_N \subseteq X$. Then this set is an admissible invariant set for the nominal system controlled by the MPC. Take a $x \in \Omega_r$ and denote $x^*(j)$ the optimal trajectory of the solution of $P_N(x)$, then from the Bellman’s optimality principle, $V^*_N(x) \geq V^*_{N-j}(x^*(j))$ for all $j \in \mathbb{Z}_{[0,N]}$. Given that for all $z \in X_N$, $V^*_N(z) \leq V^*_{N-j}(z)$, we have that $V^*_N(x^*(j)) \leq V^*_N(x)$. Then for all $x \in \Omega_r$, the predicted optimal trajectory remains in $\Omega_r$ and hence the constraint on the states is not active. Therefore, this can be removed from $P_N(x)$.

On the other hand, taking into account the results reported in [31], for any $\lambda$ such that $V_f(x) = \lambda V'(x)$, there exists a value of $r(\lambda)$ such that for all $x \in \Omega_r$ the terminal constraint is not active throughout the state evolution. Besides, larger $\lambda$, larger the region $\Omega_r$. Then, the terminal constraint could also be removed from the optimization problem yielding to a set of constraints of $P_N(x)$ which does not depend on the state $x$. From the arguments of the latter case, the optimal cost is uniformly continuous in $\Omega_r$.

3. In this case, the optimal cost function is a piece-wise affine continuous function [3]. Then $V^*_N(x)$ is Lipschitz and hence uniformly continuous in its domain.

4. This proof can be derived from [14, Proposition 12].

### A.5 Proof of Theorem 5

First, recursive feasibility will be proved. Thus, consider a state $x_k \in X_N^d$ and denote $v^*(x)$ the optimal solution of $P_N^d(x)$ and denote $\{X^*_j(x)\}$ the sequence of reachable sets for the optimal solution.

Let $x^+$ be the actual state at next sampling time, i.e. $x^+ = f(x, \kappa^d_N(x), d, w)$. Define the following sequence of control inputs to be applied at this state $\bar{\nu}(x^+) \triangleq \{v^*(1,x), \cdots, v^*(N-1,x), v_f\}$ and define the sequence of sets

$$
\bar{X}_{j+1}(x^+) \triangleq \sigma_{P \eta} (\bar{X}_{j}(x^+), \bar{\nu}(j,x^+), D, W_{\eta})
$$

with $\bar{X}_0(x^+) = \{x^+\}$. Since $x^+ \in X^*_1(x)$, then it can be proved from the property 1 of assumption [3] $\bar{X}_j(x^+) \subseteq X^*_j(x)$. Therefore
\[\tilde{X}_f(x^+) \times \tilde{v}(j,x^+) \subseteq X_{f+1}^*(x) \times v^*(j+1, x) \subseteq Z_\pi, \quad j \in \mathbb{Z}_{[0,N-2]}\]

Moreover, since \(\tilde{X}_{N-1}(x^+) \subseteq X_N^*(x) \subseteq X_f\) we have that
\[\tilde{X}_{N-1}(x^+) \times \tilde{v}(N-1, x^+) \subseteq X_f \times v_f \subseteq Z_\pi\]

Finally, from assumption 3 we have that
\[\tilde{X}_N(x^+) \triangleq \circ \pi_\eta(\tilde{X}_{N-1}(x^+), v_f, D, W_\eta) \subseteq \circ \pi_\eta(X_f, v_f, D, W_\eta) \subseteq X_f\]

Therefore, the sequence \(\tilde{v}(x^+)\) is a feasible solution of \(P_N^{\text{nl}}(x^+)\) and hence recursive feasibility and robust invariance of \(X_N^{\text{nl}}\) are proved.

To demonstrate the ISS property, let define the following sequences \(\tilde{z}(i) \triangleq \tilde{\phi}_\pi(i,x^+, \tilde{v}(x^+))\) and \(\tilde{x}^*(i) \triangleq \tilde{\phi}_\pi(i, x, v^*(x))\) for \(i \in \mathbb{Z}_{[0,N-1]}\). From the assumption 4 there exist some \(\mathcal{K}_\infty\)-functions \(\sigma_\pi, \sigma_d, \sigma_w, \sigma_L, \sigma_{Vf}, \) such that
\[
|f_\pi(x_1, v, d_1, w_1) - f_\pi(x_2, v, d_2, w_2)| \leq \sigma_\pi(|x_1-x_2|) + \sigma_d(|d_1-d_2|)
+ \sigma_w(|w_1-w_2|)

|L_\pi(x, v) - L_\pi(z, v)| \leq \sigma_L(|x-z|)

|V_f(x) - V_f(z)| \leq \sigma_{Vf}(|x-z|)
\]

for all \((x, u)\) in \(Z_\pi\) \(d \in D\) and \(w \in W\).

Taking into account the fact and considering that \(\tilde{v}(i,x^+) \triangleq v^*(i+1, x)\) for all \(i \in \mathbb{Z}_{[0,N-1]}\), it can be inferred that \(|\tilde{z}(i) - \tilde{x}^*(i+1)| \leq \sigma_\pi'(|x^+-\tilde{x}^*(1)|), \quad i \in \mathbb{Z}_{[0,N-1]}\). Denote \(\tilde{V}_N(x^+) \triangleq V_N(x^+, \tilde{v}(x^+))\), then, we can state
\[
\tilde{V}_N(x^+) - V_N^*(x) = -L(x, \kappa_N^{\text{nl}}(x))
+ \sum_{j=0}^{N-2} [L_\pi(\tilde{z}(j), \tilde{v}(j,x^+)) - L_\pi(\tilde{x}^*(j+1), v^*(j+1, x))]
+ L_\pi(\tilde{z}(N-1), v_f) + V_f(\tilde{z}(N)) - V_f(\tilde{x}^*(N-1))
+ V_f(\tilde{z}(N-1)) - V_f(\tilde{x}^*(N))
\]

Taking into account that \(\tilde{v}(i,x^+) \triangleq v^*(i+1, x)\), the following bound can be obtained
\[
L_\pi(\tilde{z}(j), \tilde{v}(j,x^+)) - L_\pi(\tilde{x}^*(j+1), v^*(j+1, x)) \leq \sigma_L \circ \sigma_\pi'(|x^+-\tilde{x}^*(1)|)
\]
for all \(j \in \mathbb{Z}_{[0,N-2]}\) and similarly,
\[
V_f(\tilde{z}(N-1)) - V_f(\tilde{x}^*(N)) \leq \sigma_{Vf} \circ \sigma_\pi'(|x^+-\tilde{x}^*(1)|)
\]

Considering that \(\tilde{z}(N-1) \in \tilde{X}_{N-1}(x^+)\), and \(\tilde{X}_{N-1}(x^+) \subseteq X_N^*(x) \subseteq X_f\), we have that \(\tilde{z}(N-1) \in X_f\). From assumption 4 we have that
\[
L_\pi(\tilde{z}(N-1), v_f) + V_f(\tilde{z}(N)) - V_f(\tilde{z}(N-1)) \leq 0
\]
This yields to $\tilde{V}_N(x^+) - V_N^*(x) \leq -L(x, \kappa_N^u(x)) + \gamma(|x^+ - x^*(1)|)$, where $\gamma(s) \triangleq \sum_{j=0}^{N-2} \sigma_L \circ \sigma_L^j(s) + \sigma_V \circ \sigma_N^N(s)$. On the other hand, since $x^+ \triangleq f(x, \kappa_N^u(x), d, w)$ and $x^*(1) \triangleq f(x, \kappa_N^u(x), 0, 0)$, we have that

$$|x^+ - \bar{x}^*(1)| = |f(x, \kappa_N^u(x), d, w) - f(x, \kappa_N^u(x), 0, 0)| \leq \sigma_d(|d|) + \sigma_w(|w|)$$

Then there exists a couple of $\mathcal{H}$-functions $\theta_d$ and $\theta_w$ such that

$$\tilde{V}_N(x^+) - V_N^*(x) \leq -\alpha(|x|) + \theta_d(|d|) + \theta_w(|w|)$$

Since $\tilde{v}(x^+)$ is a feasible solution of $P_N^l(x^+)$ we have that $V_N^*(x^+) \leq \tilde{V}_N(x^+)$, and then $V_N^*(x^+) - V_N^*(x) \leq -\alpha(|x|) + \theta_d(|d|) + \theta_w(|w|)$. Taking into account that $V_N^*(x)$ is a definite positive function continuous in a neighborhood of the origin, in virtue of lemma 3 there exists a couple of $\mathcal{H}$-functions $\alpha_1$ and $\alpha_2$ such that $\alpha_1(|x|) \leq V_N^*(x) \leq \alpha_2(|x|)$ for all $x \in X_N^u$. Hence the closed-loop system is ISS.

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References


