UNIVERSITÀ DEGLI STUDI DI BERGAMO
DIPARTIMENTO DI INGEGNERIA

QUADERNI DEL DIPARTIMENTO

Department of Engineering

Working Paper

Series "*Mathematics and Statistics*"

n. 05/MS – 2013

DISCREPANCY AND NUMERICAL INTEGRATION IN SOBOLEV SPACES ON METRIC MEASURE SPACES

by

L. Brandolini, W. W. L. Chen, L. Colzani, G. Gigante, and G. Travaglini

[§] L'accesso alle *Series* è approvato dal Comitato di Redazione. I *Working Papers* della Collana dei Quaderni del Dipartimento di Ingegneria dell'Informazione e Metodi Matematici costituiscono un servizio atto a fornire la tempestiva divulgazione dei risultati dell'attività di ricerca, siano essi in forma provvisoria o definitiva.

Discrepancy and numerical integration in Sobolev spaces on metric measure spaces

L. Brandolini, W. W. L. Chen, L. Colzani, G. Gigante, and G. Travaglini

Abstract. We study the error for quasi Monte Carlo quadrature rules on metric measure spaces adapted to a decomposition of the space into disjoint subsets. We consider both the error for a single given function, and the worst case error for all functions in a given class of potentials. The main tools are the classical Kintchine-Marcinkiewicz-Zygmund inequality and more recent suitable definitions of Sobolev classes on metric measure spaces.

1. Introduction

Consider the integral of a continuous function $f(x)$ over a metric measure space $\mathcal{M}$ with measure dx,

$$\int_{\mathcal{M}} f(x)\,dx,$$

and the associated Riemann sums

$$\sum_{j=1}^{N} \omega_j f(x_j),$$

where $\{x_j\}_{j=1}^N$ are points in $\mathcal{M}$ and $\{\omega_j\}_{j=1}^N$ are given weights. We are interested in the rate of decay of the error

$$\mathcal{E}(x_j,\omega_j) f = \sum_j \omega_j f(x_j) - \int_{\mathcal{M}} f(x)dx$$

as N goes to $+\infty$. It is reasonable to conjecture that this decay depends on some smoothness of the function $f(x)$ and on the distribution of the nodes $\{x_j\}_{j=1}^N$ in $\mathcal{M}$. For references on this problem when the metric space is a torus, a sphere or more generally a compact symmetric space see, for example, [**5**], [**6**], [**7**], [**8**], [**13**], [**17**], [**18**], [**19**], [**20**], [**21**], [**24**]. For some results related to metric measure spaces see also [**26**]. In particular, it has been proved in [**4**, Corollary 2.15] that when $\mathcal{M}$ is a d-dimensional compact Riemannian manifold and $\mathbb{H}_p^\alpha(\mathcal{M})$ is the classical fractional Sobolev space with regularity index $\alpha > d/p$ and $1 \le p \le +\infty$, then for any choice of N nodes $\{x_j\}$ with comparable minimal separation distance and

2010 *Mathematics Subject Classification.* Primary 65D30, 65D32, 31E05; Secondary 11K38, 46E35.

Key words and phrases. Discrepancy, numerical integration, potentials, Besov spaces, Sobolev spaces, metric measure spaces .

mesh norm (roughly speaking this means that the nodes are uniformly spread out in the whole manifold), there is a suitable choice of positive weights $\{\omega_j\}$ such that

$$\|\mathcal{E}(x_j,\omega_j)\|_{\alpha,p} \leq cN^{-\alpha/d}.$$

Here $\|\mathcal{E}(x_j,\omega_j)\|_{\alpha,p}$ denotes the norm of the functional $f \to \mathcal{E}(x_j,\omega_j) f$ on the space $\mathbb{H}_p^{\alpha}(\mathcal{M})$.

The first step in the proof of the above result consists of estimating the decay of the error of a quadrature rule which is exact on the first, say, r eigenfunctions of the Laplace-Beltrami operator on $\mathcal{M}$. Previous results when $\mathcal{M}$ is the Euclidean sphere can be found in [**18**], [**19**], [**21**]. The second ingredient is a result in [**14**], which proves that for any choice of N points in $\mathcal{M}$ satisfying the above mentioned condition on the minimal separation distance and the mesh norm, there is a choice of weights that gives an exact quadrature rule for the first $r = r(N)$ eigenfunctions. A close look at the proof shows furthermore that these N weights can be chosen to satisfy the condition $\omega_j \geq cN^{-1}$.

On the other hand, $N^{-\alpha/d}$ is the highest possible speed of convergence since, again in [**4**, Theorem 2.16], it has been proved that for any choice of N nodes $\{x_j\}$ and weights $\{\omega_j\}$,

$$\|\mathcal{E}(x_j,\omega_j)\|_{\alpha,p} \geq cN^{-\alpha/d}.$$

Again, previous results for the sphere are in [**17**], [**18**], [**20**], while for compact two-point homogeneous spaces can be found in [**24**].

A possible way to find N nodes satisfying the minimal separation distance vs. mesh norm condition is to partition the manifold into N disjoint subsets $\mathcal{M} = \cup_{j=1}^{N}\mathcal{X}_j$ with measure $|\mathcal{X}_j| \approx N^{-1}$ and $\operatorname{diam}(\mathcal{X}_j) \approx N^{-1/d}$, setting the weights $\omega_j = |\mathcal{X}_j|$ and picking a node x_j in each subset $\mathcal{X}_j$. In particular, see [**25**] for a decomposition of the sphere into sets of equal measure and minimal diameter.

The following examples show that the function from the space $\mathcal{X}_1 \times \cdots \times \mathcal{X}_N$ to $\mathbb{R}$,

$$\{x_j\}_{j=1}^{N} \to \|\mathcal{E}(x_j, |\mathcal{X}_j|)\|_{\alpha,p}$$

may present conspicuous oscillations.

EXAMPLE 1. *The d-dimensional torus $\mathbb{T}^d$ can be partitioned into $N = m^d$ congruent cubes with sides $1/m$ and this partition generates a mesh of points $\left(m^{-1}\mathbb{Z}^d\right)$ which gives an exact quadrature rule for all exponentials $\exp(2\pi ikx)$ with k in the hypercube $\left\{\max_{i=1,\ldots,d}|k_i| < m\right\}$. By Theorem 2.12 in* [**4**] *for any $\alpha > d/p$ we have*

$$\left\|\mathcal{E}\left(m^{-1}\mathbb{Z}^d, 1/N\right)\right\|_{\alpha,p} \leq cN^{-\alpha/d}.$$

As mentioned before, this rate of decay is the fastest possible.

EXAMPLE 2. *Fix a smooth function $f(x)$ on the manifold $\mathcal{M}$ with a finite number of stationary points and, as above, divide $\mathcal{M}$ into N subsets $\mathcal{X}_1,\ldots,\mathcal{X}_N$ with measures N^{-1} and diameters $N^{-1/d}$ approximately. Choose points $x_j \in \mathcal{X}_j$ where $f(x_j)$ is close to the supremum in $\mathcal{X}_j$, so that for all indices j,*

$$\omega_j f(x_j) - \int_{\mathcal{X}_j} f(z)\,dz \geq 0$$

and for all $\mathcal{X}_j$'s far away from the stationaty points,

$$\omega_j f(x_j) - \int_{\mathcal{X}_j} f(z)\, dz \geq cN^{-1-1/d}.$$

Since this last inequality holds for the majority of the j's, one has

$$\mathcal{E}(x_j, |\mathcal{X}_j|) f = \sum_{j=1}^{N} \left(\omega_j f(x_j) - \int_{\mathcal{X}_j} f(z)\, dz \right) \geq cN^{-1/d}.$$

Hence

$$\|\mathcal{E}(x_j, |\mathcal{X}_j|)\|_{\alpha,p} \geq cN^{-1/d}.$$

For values of $\alpha > 1$, this gives a rate of decay slower than the best possible $N^{-\alpha/d}$.

Hence, as expected, the norm of the error in the quadrature rule is sensitive to the distribution of the nodes. Anyhow, despite this pointwise irregularity, it has been proved in [**4**] that at least for a certain range of α and for $p = 2$, a random choice of nodes gives the best possible decay. See also [**8**, Theorem 7 and Theorem 24] for the case of the sphere.

Theorem A (Theorem 2.7 in [**4**]). *If a d dimensional compact Riemannian manifold $\mathcal{M}$ is decomposed into disjoint pieces $\mathcal{X}_1 \cup \ldots \cup \mathcal{X}_N$, and if $d/2 < \alpha < d/2 + 1$, then*

$$\left\{ \int_{\mathcal{X}_1} \cdots \int_{\mathcal{X}_N} \|\mathcal{E}(x_j, |\mathcal{X}_j|)\|_{\alpha,2}^2 \frac{dx_1}{|\mathcal{X}_1|} \cdots \frac{dx_N}{|\mathcal{X}_N|} \right\}^{1/2} \leq c \max_{1 \leq j \leq N} \{\operatorname{diam}(\mathcal{X}_j)^\alpha\}.$$

In particular if $\operatorname{diam}(\mathcal{X}_j) \leq cN^{-1/d}$ for all j then, by Chebyshev inequality, for every $0 < \varepsilon < 1$ there exists a constant c such that a random choice of points $\{x_j\}$ in $\{\mathcal{X}_j\}$ gives

$$\|\mathcal{E}(x_j, |\mathcal{X}_j|)\|_{\alpha,2} \leq cN^{-\alpha/d},$$

with probability greater than $1 - \varepsilon$.

The proof of the above theorem relies heavily on Hilbert space techniques and on the Riemannian structure of $\mathcal{M}$. The main purpose of this paper is to generalize this result to the Sobolev spaces $\mathbb{H}_p^\alpha(\mathcal{M})$ with $1 \leq p \leq +\infty$ and with $\mathcal{M}$ an arbitrary metric measure space. The main ingredient in this generalization is an extension of the classical Kintchine inequality due to Marcinkiewicz and Zygmund.

The first issue lies in an appropriate definition of Sobolev space in an arbitrary measure space, and a possible one is in terms of potential spaces. See e.g. [**29**, Chapter V] for the Euclidean case. Thus, if $\mathcal{M}$ is a measure space and $1 \leq p, q \leq +\infty$ are conjugate exponents, $1/p + 1/q = 1$, let $\Phi(x, y)$ be a measurable kernel on $\mathcal{M} \times \mathcal{M}$ such that for every x the q–th power of the kernel $y \to \Phi(x, y)$ is integrable, $\int_{\mathcal{M}} |\Phi(x, y)|^q\, dy < +\infty$. Then every function $g(x)$ in $\mathbb{L}^p(\mathcal{M})$ has a pointwise well defined potential

$$f(x) = \int_{\mathcal{M}} \Phi(x, y)\, g(y)\, dy.$$

The space $\mathbb{H}_p^\Phi(\mathcal{M})$ is the space of all potentials of functions in $\mathbb{L}^p(\mathcal{M})$, with norm

$$\|f\|_{\mathbb{H}_p^\Phi(\mathcal{M})} = \inf_g \left\{ \int_{\mathcal{M}} |g(x)|^p\, dx \right\}^{1/p}.$$

Here, the infimum is over all $g(x)$ which give the potential $f(x)$.

We prove (see Corollary 3 below) the following generalization of Theorem A, under the hypothesis that the kernel $\Phi(x,y)$ has a behaviour similar to the Bessel kernel on a Riemannian manifold. We need the following definition.

DEFINITION 1. *We say that a metric measure space $\mathcal{M}$ is Ahlfors d-regular, for some positive constant d, if there exist positive constants a, b such that for every $y \in \mathcal{M}$ and $0 < t < 1$,*

$$at^d \leq |\{x \in \mathcal{M},\ |x-y| < t\}| \leq bt^d.$$

THEOREM B. *Let $\mathcal{M}$ be an Ahlfors d-regular metric measure space of finite measure and assume that $\mathcal{M}$ is decomposed into a finite number of disjoint sets $\mathcal{M} = \cup_{j=1}^N \mathcal{X}_j$ with $|\mathcal{X}_j| \approx N^{-1}$ and $\operatorname{diam}(\mathcal{X}_j) \approx N^{-1/d}$. Assume that for some $0 < \alpha < d$, the kernel $\Phi(x,y)$ satisfies the conditions*

$$|\Phi(x,y)| \leq c\,|x-y|^{\alpha-d} \text{ for every } x \text{ and } y,$$

$$|\Phi(x,y) - \Phi(z,y)| \leq c\,|x-z|\,|x-y|^{\alpha-d-1} \text{ if } |x-y| \geq 2\,|x-z|\,.$$

Finally assume that $1 < p \leq +\infty$, $1/p + 1/q = 1$, and $d/p < \alpha < d$. Then

$$\left\{\int_{\mathcal{X}_1} \cdots \int_{\mathcal{X}_N} \|\mathcal{E}(x_j, |\mathcal{X}_j|)\|_{\Phi,p}^q \frac{dx_1}{|\mathcal{X}_1|} \cdots \frac{dx_N}{|\mathcal{X}_N|}\right\}^{1/q}$$
$$\leq \begin{cases} cN^{-\alpha/d} & \text{if } \alpha < d/2+1, \\ cN^{-1/2-1/d}\log^{1/2}(N) & \text{if } \alpha = d/2+1, \\ cN^{-1/2-1/d} & \text{if } \alpha > d/2+1. \end{cases}$$

Again, $\|\mathcal{E}(x_j, |\mathcal{X}_j|)\|_{\Phi,p}$ is the norm of the functional $f \to \mathcal{E}(x_j, |\mathcal{X}_j|) f$ on the potential space $\mathbb{H}_p^\Phi(\mathcal{M})$.

Under natural assumptions on the kernel we also show that this average decay is sharp in the exponents of N, but notice that in the case $\alpha \geq d/2+1$, $\alpha > d/p$ this average decay is slower than the best possible $N^{-\alpha/d}$, which can only be obtained with a suitable choice of nodes and weights. For the case of the sphere and for $p = 2$ see also [**8**, Theorem 24 and Theorem 25]. As for the pointwise result, once again it follows easily from Theorem B that for every $0 < \varepsilon < 1$ there exists a constant c such that a random choice of points $\{x_j\}$ in $\{\mathcal{X}_j\}$ gives

$$\|\mathcal{E}(x_j, |\mathcal{X}_j|)\|_{\Phi,p} \leq \begin{cases} cN^{-\alpha/d} & \text{if } \alpha < d/2+1, \\ cN^{-1/2-1/d}\log^{1/2}(N) & \text{if } \alpha = d/2+1, \\ cN^{-1/2-1/d} & \text{if } \alpha > d/2+1, \end{cases}$$

with probability greater than $1-\varepsilon$.

The techniques in the proofs of the above results can also be used to study the same problem from a different perspective. Rather than trying to find point distributions that give good quadrature rules for all functions in a given class, we can look for point distributions adapted to a given function in a given class. It turns out that the Hajłasz type Sobolev classes are a natural setting for this type of problems (see [**16**]).

THEOREM C. *Let $\mathcal{M}$ be a metric measure space and $\varphi(t)$ a non negative increasing function in $t \geq 0$. An upper gradient of a measurable function $f(x)$ is a function $g(x)$ such that for almost every x and y,*

$$|f(x) - f(y)| \leq \varphi(\operatorname{dist}(x,y))\,(g(x) + g(y))\,.$$

Assume that $\mathcal{M}$ *has finite measure and that it is decomposed into a finite or infinite number of disjoint sets* $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots$, *with measure* $|\mathcal{X}_j| = \omega_j$ *and* $\operatorname{diam}(\mathcal{X}_j) = \delta_j$. *Then, for every* $1 \leq p \leq +\infty$ *there exists a constant* $B(p)$ *depending only on* p, *such that for every integrable function* $f(x)$ *with an upper gradient* $g(x)$ *in* $\mathbb{L}^p(\mathcal{M})$,

$$\left\{ \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots |\mathcal{E}(x_j, \omega_j) f|^p \frac{dx_1}{\omega_1} \frac{dx_2}{\omega_2} \cdots \right\}^{1/p}$$

$$\leq \begin{cases} 2B(p) \sup \left\{ \omega_j^{1-1/p} \right\} \varphi(2 \sup \{\delta_j\}) \left\{ \int_{\mathcal{M}} |g(x)|^p \, dx \right\}^{1/p} & \textit{if } 1 \leq p \leq 2, \\ 2B(p) |\mathcal{M}|^{1/2-1/p} \sup \left\{ \omega_j^{1/2} \right\} \varphi(2 \sup \{\delta_j\}) \left\{ \int_{\mathcal{M}} |g(x)|^p \, dx \right\}^{1/p} & \textit{if } 2 \leq p < +\infty, \\ 2 |\mathcal{M}|^{1-1/p} \ \varphi(2 \sup \{\delta_j\}) \sup \{|g(x)|\} & \textit{if } 1 \leq p \leq +\infty. \end{cases}$$

See Theorem 3 and the comments that follow. One can easily compare the decay rates given by the above Theorems B and C by taking $\omega_j \approx N^{-1}$, $\varphi(t) = t^\alpha$ with $0 < \alpha < 1$, and $\delta_j \approx N^{-1/d}$. Also observe that in the above theorem, the regularity of the function $f(x)$ is measured in terms of its upper gradient, rather than looking at it as a potential. All these different definitions and their relations are illustrated in the next section.

A final comment. In [**2**], [**10**] and [**11**] there are analogues of the above results for characteristic functions and p an even integer. In [**5**] and in [**13**] there are analogues for $1 \leq p \leq 2$ and arbitrary functions. The proofs in all these references depend heavily on Fourier analytic tecniques. On the contrary the tecniques used in this paper are more elementary and have a wider range of application.

2. Besov, Triebel-Lizorkin and potential spaces

Let $\mathcal{M}$ be a metric measure space, that is a metric space equipped with a Borel regular measure. With a small abuse of notation we denote by $|\mathcal{X}|$ the measure of a measurable set $\mathcal{X}$ and by $|x-y|$ the distance between two points x and y. Simple examples are Riemannian manifolds, or non necessarily smooth surfaces in a Euclidean space with the inherited measure and distance. In [**16**] there is a purely metric definition of a Sobolev space: A measurable function $f(x)$ is in the Sobolev space $\mathbb{W}_p^1(\mathcal{M})$, $1 \leq p \leq +\infty$, if there exists a non negative function $g(x)$ in $\mathbb{L}^p(\mathcal{M})$ such that for almost every $x, y \in \mathcal{M}$,

$$|f(x) - f(y)| \leq |x-y| \left(g(x) + g(y)\right).$$

For example, in a Euclidean space one can choose as an upper gradient $g(x)$ the Hardy-Littlewood maximal function of the gradient $\nabla f(x)$. There are several possible generalizations of the above definition. In particular one can replace the distance $|x-y|$ with more general functions $\varphi(|x-y|)$ and localize the upper gradients.

Definition 2. *Let* $\mathcal{M}$ *be a metric measure space and* $\varphi(t)$ *a non negative increasing function in* $t \geq 0$. *A sequence of non negative measurable functions* $\{g_n(x)\}_{n\in\mathbb{Z}}$ *is an upper gradient of a measurable function* $f(x)$ *if there exists a set* $\mathcal{N}$ *with measure zero such that for all* x *and* y *in* $\mathcal{M} \setminus \mathcal{N}$ *with* $|x-y| \leq 2^{-n}$,

$$|f(x) - f(y)| \leq \varphi\left(2^{-n}\right) \left(g_n(x) + g_n(y)\right).$$

(1) *A measurable function $f(x)$ is in the Besov space $\mathbb{B}^{\varphi}_{p,q}(\mathcal{M})$, $0<p\leq+\infty$ and $0<q\leq+\infty$, if $f(x)$ has an upper gradient $\{g_n(x)\}$ with*

$$\left\{\sum_{n=-\infty}^{+\infty}\left(\int_{\mathcal{M}}|g_n(x)|^p\,dx\right)^{q/p}\right\}^{1/q}<+\infty.$$

The infimum of the above expression over all upper gradients defines the semi norm $\|f\|_{\mathbb{B}^{\varphi}_{p,q}(\mathcal{M})}$.

(2) *A measurable function $f(x)$ is in the Triebel-Lizorkin space $\mathbb{F}^{\varphi}_{p,q}(\mathcal{M})$, $0<p<+\infty$ and $0<q\leq+\infty$, if $f(x)$ has an upper gradient $\{g_n(x)\}$ with*

$$\left\{\int_{\mathcal{M}}\left(\sum_{n=-\infty}^{+\infty}|g_n(x)|^q\right)^{p/q}dx\right\}^{1/p}<+\infty.$$

The infimum of the above expression over all upper gradients defines the semi norm $\|f\|_{\mathbb{F}^{\varphi}_{p,q}(\mathcal{M})}$.

As usual, when $q=+\infty$ the above sum has to be replaced by a supremum, and when $\operatorname{diam}(\mathcal{M})\leq 2^{-N}$ then the index n may be restricted to the range $N\leq n<+\infty$. Moreover, in the definition of $\mathbb{F}^{\varphi}_{p,\infty}(\mathcal{M})$ one can replace the sequence $\{g_n(x)\}$ with the single function $\sup\{g_n(x)\}$. In particular, the simplest generalization of a Hajłasz-Sobolev space is a Triebel-Lizorkin space. The above spaces are homogeneous, and constant functions have semi norm zero. If $\varphi(t)\leq\psi(t)$ then $\mathbb{B}^{\varphi}_{p,q}(\mathcal{M})\subseteq\mathbb{B}^{\psi}_{p,q}(\mathcal{M})$ and $\mathbb{F}^{\varphi}_{p,q}(\mathcal{M})\subseteq\mathbb{F}^{\psi}_{p,q}(\mathcal{M})$. Moreover, for fixed $\varphi(t)$ and p, the largest space in the Besov and Triebel-Lizorkin scales is $\mathbb{B}^{\varphi}_{p,\infty}(\mathcal{M})$. In particular, it is proved in [**23**] that when $\mathcal{M}$ is the Euclidean space $\mathbb{R}^d$ and $\varphi(t)=t^{\alpha}$ with $0<\alpha<1$, then the spaces $\mathbb{B}^{\varphi}_{p,q}(\mathcal{M})$ and $\mathbb{F}^{\varphi}_{p,q}(\mathcal{M})$ coincide with the classical Besov and Triebel-Lizorkin spaces defined via a Littlewood-Paley decomposition. To be precise, the definition of upper gradient in [**23**] requires $|f(x)-f(y)|\leq 2^{-\alpha n}(g_n(x)+g_n(y))$ only for x and y with $2^{-n-1}\leq|x-y|\leq 2^{-n}$, but if one defines $G_n(x)=\sum_{k=0}^{+\infty}2^{-\alpha k}g_{n+k}(x)$, then $|f(x)-f(y)|\leq 2^{-\alpha n}(G_n(x)+G_n(y))$ for x and y with $|x-y|\leq 2^{-n}$, and the seminorms defined via $\{g_n(x)\}$ and $\{G_n(x)\}$ are equivalent.

EXAMPLE 3. *Given a measurable set $\mathcal{B}$, it is easy to show that the functions*

$$g_n(x)=\begin{cases}\varphi(2^{-n})^{-1} & \text{if } x \text{ is in } \mathcal{B} \text{ and } \operatorname{dist}\{x,\partial\mathcal{B}\}\leq 2^{-n},\\ 0 & \text{otherwise.}\end{cases}$$

are upper gradients of the characteristic function $\chi_{\mathcal{B}}(x)$. Hence,

$$\|\chi_{\mathcal{B}}\|_{\mathbb{B}^{\varphi}_{p,\infty}(\mathcal{M})}\leq\sup_n\left\{\varphi\left(2^{-n}\right)^{-1}\left|\left\{x\in\mathcal{M}:\ \operatorname{dist}\{x,\partial\mathcal{B}\}\leq 2^{-n}\right\}\right|^{1/p}\right\}.$$

In particular, the Besov norm of $\chi_{\mathcal{B}}(x)$ is related to the Minkowski content of the boundary of $\mathcal{B}$, defined by

$$\psi(t)=|\{x\in\mathcal{M}:\ \operatorname{dist}\{x,\partial\mathcal{B}\}\leq t\}|.$$

The higher $\psi(t)$ as $t\to 0$, the higher the fractal dimension of the boundary. For example, if $\mathcal{M}$ is a d dimensional Riemannian manifold and if $\mathcal{B}$ is a bounded open *set with regular boundary then $\psi(t) \approx t$, while if $\psi(t) \approx t^{d-\beta}$ then the boundary has fractal dimension β. In this case, $\chi_{\mathcal{B}}(x)$ is in $\mathbb{B}^{\varphi}_{p,\infty}(\mathcal{M})$ with $\varphi(t) = t^{(d-\beta)/p}$.*

Another possible generalization of Sobolev spaces is via potentials.

Definition 3. *Let $\mathcal{M}$ be a measure space, let $1 \leq p, q \leq +\infty$ with $1/p + 1/q = 1$, and let $\Phi(x,y)$ be a measurable kernel on $\mathcal{M} \times \mathcal{M}$. Assume that for every x the q–th power of the kernel $y \to \Phi(x,y)$ is integrable, $\int_{\mathcal{M}} |\Phi(x,y)|^q \, dy < +\infty$. Then every function $g(x)$ in $\mathbb{L}^p(\mathcal{M})$ has a pointwise well defined potential*

$$f(x) = \int_{\mathcal{M}} \Phi(x,y) \, g(y) \, dy.$$

The space $\mathbb{H}^{\Phi}_p(\mathcal{M})$ is the space of all potentials of functions in $\mathbb{L}^p(\mathcal{M})$, with norm

$$\|f\|_{\mathbb{H}^{\Phi}_p(\mathcal{M})} = \inf_g \left\{ \int_{\mathcal{M}} |g(x)|^p \, dx \right\}^{1/p}.$$

The infimum is over all $g(x)$ which give the potential $f(x)$.

Observe that the above definition does not even require a metric. Potentials can also be defined under weaker assumptions on the kernel, but the above assumptions guarantee that these potentials are defined pointwise everywhere, and this will be necessary in what follows. In particular, when $\mathcal{M}$ is the Euclidean space $\mathbb{R}^d$ and when $\Phi(x,y) = |x-y|^{\alpha-d}$ with $0 < \alpha < d$ is the Riesz kernel, then $\mathbb{H}^{\Phi}_p(\mathcal{M})$ is the fractional Sobolev space $\mathbb{H}^{\alpha}_p(\mathcal{M})$. However the cases $p = 1$ and $p = +\infty$ require some extra care. For interesting examples of generalized potential spaces see e.g. [**22**].

Example 4. *Assume that the finite metric measure space $\mathcal{M}$ is Ahlfors d-regular and assume that for some $0 < \alpha < d$,*

$$|\Phi(x,y)| \leq c\,|x-y|^{\alpha-d} \text{ for every } x \text{ and } y,$$
$$|\Phi(x,y) - \Phi(z,y)| \leq c\,|x-z|\,|x-y|^{\alpha-d-1} \text{ if } |x-y| \geq 2\,|x-z|.$$

Finally, assume that $g(x)$ is in $\mathbb{L}^p(\mathcal{M})$ and that

$$f(x) = \int_{\mathcal{M}} \Phi(x,y) \, g(y) \, dy.$$

Then, if $0 < \alpha < 1$ and if $Mg(x)$ is the Hardy Littlewood maximal function,

$$\begin{aligned}
|f(x) - f(z)| &\leq \int_{\mathcal{M}} |\Phi(x,y) - \Phi(z,y)|\,|g(y)|\,dy \\
\leq c \int_{\{|x-y| \leq 3|x-z|\}} |x-y|^{\alpha-d}\,|g(y)|\,dy &+ c \int_{\{|z-y| \leq 3|x-z|\}} |z-y|^{\alpha-d}\,|g(y)|\,dy \\
&+ c\,|x-z| \int_{\{|x-y| \geq 2|x-z|\}} |x-y|^{\alpha-d-1}\,|g(y)|\,dy \\
&\leq c\,|x-z|^{\alpha}\,(Mg(x) + Mg(z)).
\end{aligned}$$

In particular, if $1 < p < +\infty$ and if $\varphi(t) = t^{\alpha}$ with $0 < \alpha < 1$, then $\mathbb{H}^{\Phi}_p(\mathcal{M}) \subseteq \mathbb{F}^{\varphi}_{p,\infty}(\mathcal{M})$.

3. Discrepancy and numerical integration

The main ingredient in what follows is the Kintchine-Marcinkiewicz-Zygmund inequality for sums of independent random variables.

As it is well known, the variance of the sum of independent random variables is the sum of the variances. More explicitly, for every sequence of probability spaces $\{\mathcal{X}_j, dx_j\}$ and every sequence of measurable functions $f_j(x_j)$ with mean zero,

$$\int_{\mathcal{X}_1}\int_{\mathcal{X}_2}\cdots\left|\sum_{j=1}^{+\infty} f_j(x_j)\right|^2 dx_1 dx_2 \ldots = \sum_{j=1}^{+\infty}\int_{\mathcal{X}_j} |f_j(x_j)|^2\, dx_j.$$

In fact, there is a similar result with the second moment replaced with other moments and the equality replaced with two inequalities.

Theorem 1 (Kintchine-Marcinkiewicz-Zygmund). *For every $1 \le p < +\infty$ there exist two positive constants $A(p)$ and $B(p)$ such that for every sequence of probability spaces $\{\mathcal{X}_j, dx_j\}$ and for every non zero sequence of measurable functions $\{f_j(x_j)\}$ with mean zero, $\int_{\mathcal{X}_j} f_j(x_j)\, dx_j = 0$,*

$$A(p) \le \frac{\left\{\int_{\mathcal{X}_1}\int_{\mathcal{X}_2}\cdots\left|\sum_{j=1}^{+\infty} f_j(x_j)\right|^p dx_1 dx_2 \ldots\right\}^{1/p}}{\left\{\int_{\mathcal{X}_1}\int_{\mathcal{X}_2}\cdots\left(\sum_{j=1}^{+\infty} |f_j(x_j)|^2\right)^{p/2} dx_1 dx_2 \ldots\right\}^{1/p}} \le B(p).$$

For a proof, see [**27**] and [**28**], or [**12**]. In what follows, special attention will be paid to the constants, and $A(p)$ and $B(p)$ will denote the best constants in the Kintchine-Marcinkiewicz-Zygmund inequality. In particular $B(p) \to +\infty$ as $p \to +\infty$. See [**15**] and [**9**].

The following result on the discrepancy of a random set of points extends [**10**, Lemma 5].

Theorem 2. *Let $\mathcal{M}$ be a metric measure space, let $\mathcal{B}$ be a measurable subset of $\mathcal{M}$, and let*

$$\psi(t) = |\{x \in \mathcal{M} :\ \mathrm{dist}\{x, \partial\mathcal{B}\} \le t\}|.$$

Assume that $\mathcal{M}$ is decomposed into a finite or infinite number of disjoint sets $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots$, with measure $0 < |\mathcal{X}_j| = \omega_j < +\infty$ and $0 < \mathrm{diam}(\mathcal{X}_j) = \delta_j < +\infty$. If $\mathcal{J}$ is the set of indices j such that $\mathcal{X}_j$ intersects both $\mathcal{B}$ and its complement, then the following hold:

(i): *For every choice of points $\{x_j\}$ in $\{\mathcal{X}_j\}$,*

$$\left|\sum_j \omega_j \chi_{\mathcal{B}}(x_j) - |\mathcal{B}|\right| \le \psi\left(\sup_{j\in\mathcal{J}}\{\delta_j\}\right).$$

(ii): *For every* $1 \leq p < +\infty$,

$$\left\{ \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots \left| \sum_j \omega_j \chi_{\mathcal{B}}(x_j) - |\mathcal{B}| \right|^p \frac{dx_1}{\omega_1} \frac{dx_2}{\omega_2} \cdots \right\}^{1/p}$$
$$\leq B(p) \sqrt{\sup_{j \in \mathcal{J}} \{\omega_j\} \, \psi\left(\sup_{j \in \mathcal{J}} \{\delta_j\} \right)}.$$

Observe that $\sup_{j \in \mathcal{J}} \{\omega_j\} \leq \psi\left(\sup_{j \in \mathcal{J}} \{\delta_j\}\right)$, and that the estimate (ii) is better than (i) when

$$B(p) \leq \sqrt{\frac{\psi\left(\sup_{j \in \mathcal{J}} \{\delta_j\}\right)}{\sup_{j \in \mathcal{J}} \{\omega_j\}}}.$$

On the other hand, recall that $B(p) \to +\infty$ as $p \to +\infty$, hence eventually estimate (i) wins.

Proof. The proof of (i) is elementary. For every choice of $x_j \in \mathcal{X}_j$ one has

$$\sum_j \omega_j \chi_{\mathcal{B}}(x_j) - |\mathcal{B}| = \sum_j \left(\omega_j \chi_{\mathcal{B} \cap \mathcal{X}_j}(x_j) - |\mathcal{B} \cap \mathcal{X}_j| \right).$$

If $\mathcal{X}_j \subseteq \mathcal{B}$ or if $\mathcal{B} \cap \mathcal{X}_j = \emptyset$ then $\omega_j \chi_{\mathcal{B} \cap \mathcal{X}_j}(x_j) - |\mathcal{B} \cap \mathcal{X}_j| = 0$. In particular, this happens if $\operatorname{dist}\{x_j, \partial\mathcal{B}\} > \operatorname{diam}(\mathcal{X}_j)$. Moreover,

$$\left| \omega_j \chi_{\mathcal{B} \cap \mathcal{X}_j}(x_j) - |\mathcal{B} \cap \mathcal{X}_j| \right| \leq \omega_j$$

for every x_j. For every j the function $\omega_j \chi_{\mathcal{B} \cap \mathcal{X}_j}(x_j) - |\mathcal{B} \cap \mathcal{X}_j|$ has mean zero on $\mathcal{X}_j$. Then, by the triangle inequality,

$$\left| \sum_j \left(\omega_j \chi_{\mathcal{B} \cap \mathcal{X}_j}(x_j) - |\mathcal{B} \cap \mathcal{X}_j| \right) \right| \leq \sum_{j \in \mathcal{J}} \omega_j$$
$$\leq \left| \left\{ x \in \mathcal{M} : \operatorname{dist}\{x, \partial\mathcal{B}\} \leq \sup_{j \in \mathcal{J}} \{\delta_j\} \right\} \right|.$$

The proof of (ii) is similar, as long as one replaces the triangle inequality with the Kintchine-Marcinkiewicz-Zygmund inequality,

$$\left\{ \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots \left| \sum_j \left(\omega_j \chi_{\mathcal{B} \cap \mathcal{X}_j}(x_j) - |\mathcal{B} \cap \mathcal{X}_j| \right) \right|^p \frac{dx_1}{\omega_1} \frac{dx_2}{\omega_2} \cdots \right\}^{1/p}$$
$$\leq B(p) \left\{ \sum_{j \in \mathcal{J}} \omega_j^2 \right\}^{1/2} \leq B(p) \sup_{j \in \mathcal{J}} \left\{ \omega_j^{1/2} \right\} \left\{ \sum_{j \in \mathcal{J}} \omega_j \right\}^{1/2}$$
$$\leq B(p) \sup_{j \in \mathcal{J}} \left\{ \omega_j^{1/2} \right\} \left| \left\{ x \in \mathcal{M} : \operatorname{dist}\{x, \partial\mathcal{B}\} \leq \sup_{j \in \mathcal{J}} \{\delta_j\} \right\} \right|^{1/2}.$$

□

The following is just a restatement of the above theorem in the relevant case of an Ahlfors regular space.

Corollary 1. *Let $\mathcal{M}$ be an Ahlfors d-regular metric measure space of finite measure and assume that $\mathcal{M}$ is decomposed into a finite number of disjoint sets $\mathcal{M} = \cup_{j=1}^{N} \mathcal{X}_j$ with $\omega_j = |\mathcal{X}_j| \approx N^{-1}$ and $\operatorname{diam}(\mathcal{X}_j) \approx N^{-1/d}$. Finally, assume that a measurable set $\mathcal{B}$ has boundary of Minkowski dimension $\alpha > 0$,*

$$|\{x \in \mathcal{M} : \ \operatorname{dist}\{x, \partial\mathcal{B}\} \leq t\}| \leq ct^{\alpha}.$$

Then for every $0 < \varepsilon < 1$ there exists a constant c such that a random choice of points $\{x_j\}$ in $\{\mathcal{X}_j\}$ gives

$$\left| \sum_{j=1}^{N} \omega_j \chi_{\mathcal{B}}(x_j) - |\mathcal{B}| \right| \leq cN^{-1/2-\alpha/2d}$$

with probability greater than $1 - \varepsilon$.

Proof. This follows from the above theorem via Chebyshev inequality. □

The above theorem is a particular case of a more general estimate of the discrepancy between integrals and random Riemann sums.

Theorem 3. *Assume that a metric measure space $\mathcal{M}$ of finite measure is decomposed into a finite or infinite number of disjoint sets $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots$, with measure $0 < |\mathcal{X}_j| = \omega_j < +\infty$ and $0 < \operatorname{diam}(\mathcal{X}_j) = \delta_j < +\infty$. Also let $\varphi(t)$ be a non negative increasing function in $t \geq 0$, and let $\mathbb{B}^{\varphi}_{p,\infty}(\mathcal{M})$ be the associated Besov space. Then the following hold:*

(i): *For every $1 \leq p \leq +\infty$,*

$$\begin{aligned}&\left\{ \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots |\mathcal{E}(x_j, \omega_j) f|^p \frac{dx_1}{\omega_1} \frac{dx_2}{\omega_2} \cdots \right\}^{1/p} \\ &\leq 2 |\mathcal{M}|^{1-1/p} \varphi(2 \sup\{\delta_j\}) \|f\|_{\mathbb{B}^{\varphi}_{p,\infty}(\mathcal{M})}.\end{aligned}$$

(ii): *If $1 \leq p \leq 2$,*

$$\begin{aligned}&\left\{ \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots |\mathcal{E}(x_j, \omega_j) f|^p \frac{dx_1}{\omega_1} \frac{dx_2}{\omega_2} \cdots \right\}^{1/p} \\ &\leq 2B(p) \sup\left\{\omega_j^{1-1/p}\right\} \varphi(2 \sup\{\delta_j\}) \|f\|_{\mathbb{B}^{\varphi}_{p,\infty}(\mathcal{M})}.\end{aligned}$$

(iii): *If $2 \leq p < +\infty$,*

$$\begin{aligned}&\left\{ \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots |\mathcal{E}(x_j, \omega_j) f|^p \frac{dx_1}{\omega_1} \frac{dx_2}{\omega_2} \cdots \right\}^{1/p} \\ &\leq 2B(p) |\mathcal{M}|^{1/2-1/p} \sup\left\{\omega_j^{1/2}\right\} \varphi(2 \sup\{\delta_j\}) \|f\|_{\mathbb{B}^{\varphi}_{p,\infty}(\mathcal{M})}.\end{aligned}$$

Observe that the estimate (i) is of some interest only for p large. Indeed if $1 \leq p \leq 2$ then (ii) is better than (i), and if $2 \leq p < +\infty$ and $B(p) \leq |\mathcal{M}|^{1/2} / \sup\left\{\omega_j^{1/2}\right\}$ then (iii) is better than (i).

Proof. The proof of (i) is elementary. Functions in $\mathbb{B}^{\varphi}_{p,\infty}(\mathcal{M})$ satisfy a Poincaré inequality. Indeed, assume that $\{g_n(x)\}$ is an upper gradient of $f(x)$, that is for almost every x and y with $|x - y| \leq 2^{-n}$,

$$|f(x) - f(y)| \leq \varphi\left(2^{-n}\right)(g_n(x) + g_n(y)).$$

If $\mathcal{X}$ is a measurable subset with $0 < |\mathcal{X}| = \omega$ and $\operatorname{diam}(\mathcal{X}) = \delta \le 2^{-n}$, then for every $1 \le q < +\infty$,

$$\begin{aligned}\left\{\int_{\mathcal{X}} \left| f(x) - \int_{\mathcal{X}} f(y) \frac{dy}{\omega} \right|^q \frac{dx}{\omega}\right\}^{1/q} &\le \left\{\int_{\mathcal{X}}\int_{\mathcal{X}} |f(x) - f(y)|^q \frac{dx}{\omega}\frac{dy}{\omega}\right\}^{1/q} \\ &\le \varphi\left(2^{-n}\right)\left\{\int_{\mathcal{X}}\int_{\mathcal{X}} |g_n(x) + g_n(y)|^q \frac{dx}{\omega}\frac{dy}{\omega}\right\}^{1/q} \\ &\le 2\varphi\left(2^{-n}\right)\left\{\int_{\mathcal{X}} |g_n(x)|^q \frac{dx}{\omega}\right\}^{1/q}.\end{aligned}$$

Hence if $2^{-n-1} < \sup\{\delta_j\} \le 2^{-n}$, by the Poincaré and the triangle inequalities we have

$$\begin{aligned}&\left\{\int_{\mathcal{X}_1}\int_{\mathcal{X}_2}\cdots\left|\sum_j \omega_j f(x_j) - \int_{\mathcal{M}} f(x)dx\right|^p \frac{dx_1}{\omega_1}\frac{dx_2}{\omega_2}\cdots\right\}^{1/p} \\ &\le \sum_j \omega_j \left\{\int_{\mathcal{X}_j}\left| f(x_j) - \int_{\mathcal{X}_j} f(y_j)\frac{dy_j}{\omega_j}\right|^p \frac{dx_j}{\omega_j}\right\}^{1/p} \\ &\le 2\varphi\left(2^{-n}\right)\sum_j \omega_j\left\{\int_{\mathcal{X}_j} |g_n(x_j)|^p \frac{dx_j}{\omega_j}\right\}^{1/p} \\ &\le 2\varphi\left(2^{-n}\right)\left\{\sum_j \omega_j\right\}^{1-1/p}\left\{\sum_j\int_{\mathcal{X}_j} |g_n(x_j)|^p dx_j\right\}^{1/p} \\ &\le 2\varphi\left(2\sup\{\delta_j\}\right)|\mathcal{M}|^{1-1/p}\left\{\int_{\mathcal{M}} |g_n(x)|^p dx\right\}^{1/p}.\end{aligned}$$

The proofs of (ii) and (iii) are similar. By the Kintchine-Marcinkiewicz-Zygmund inequality,

$$\begin{aligned}&\left\{\int_{\mathcal{X}_1}\int_{\mathcal{X}_2}\cdots\left|\sum_j \omega_j f(x_j) - \int_{\mathcal{M}} f(x)dx\right|^p \frac{dx_1}{\omega_1}\frac{dx_2}{\omega_2}\cdots\right\}^{1/p} \\ &= \left\{\int_{\mathcal{X}_1}\int_{\mathcal{X}_2}\cdots\left|\sum_j \omega_j\left(f(x_j) - \int_{\mathcal{X}_j} f(y_j)\frac{dy_j}{\omega_j}\right)\right|^p \frac{dx_1}{\omega_1}\frac{dx_2}{\omega_2}\cdots\right\}^{1/p} \\ &\le B(p)\left\{\int_{\mathcal{X}_1}\int_{\mathcal{X}_2}\cdots\left(\sum_j \omega_j^2\left|f(x_j) - \int_{\mathcal{X}_j} f(y_j)\frac{dy_j}{\omega_j}\right|^2\right)^{p/2}\frac{dx_1}{\omega_1}\frac{dx_2}{\omega_2}\cdots\right\}^{1/p}.\end{aligned}$$

If $2^{-n-1} < \sup\{\delta_j\} \leq 2^{-n}$ and $1 \leq p \leq 2$, the Poincaré inequality gives

$$\left\{ \int_{\mathcal{X}_1}\int_{\mathcal{X}_2} \cdots \left(\sum_j \omega_j^2 \left| f(x_j) - \int_{\mathcal{X}_j} f(y_j) \frac{dy_j}{\omega_j} \right|^2 \right)^{p/2} \frac{dx_1}{\omega_1}\frac{dx_2}{\omega_2}\cdots \right\}^{1/p}$$

$$\leq \left\{ \int_{\mathcal{X}_1}\int_{\mathcal{X}_2} \cdots \left(\sum_j \omega_j^p \left| f(x_j) - \int_{\mathcal{X}_j} f(y_j) \frac{dy_j}{\omega_j} \right|^p \right) \frac{dx_1}{\omega_1}\frac{dx_2}{\omega_2}\cdots \right\}^{1/p}$$

$$= \left\{ \sum_j \omega_j^p \int_{\mathcal{X}_j} \left| f(x_j) - \int_{\mathcal{X}_j} f(y_j) \frac{dy_j}{\omega_j} \right|^p \frac{dx_j}{\omega_j} \right\}^{1/p}$$

$$\leq 2\varphi\left(2^{-n}\right) \left\{ \sum_j \omega_j^p \int_{\mathcal{X}_j} |g_n(x_j)|^p \frac{dx_j}{\omega_j} \right\}^{1/p}$$

$$\leq 2\sup\left\{\omega_j^{1-1/p}\right\} \varphi\left(2\sup\{\delta_j\}\right) \left\{ \int_{\mathcal{M}} |g_n(x)|^p \, dx \right\}^{1/p}.$$

Similarly, if $2 \leq p < +\infty$ then

$$\left\{ \int_{\mathcal{X}_1}\int_{\mathcal{X}_2} \cdots \left(\sum_j \omega_j^2 \left| f(x_j) - \int_{\mathcal{X}_j} f(y_j) \frac{dy_j}{\omega_j} \right|^2 \right)^{p/2} \frac{dx_1}{\omega_1}\frac{dx_2}{\omega_2}\cdots \right\}^{1/p}$$

$$\leq \left\{ \sum_j \omega_j^{(2p-2)/(p-2)} \right\}^{(p-2)/2p} \left\{ \sum_j \omega_j \int_{\mathcal{X}_j} \left| f(x_j) - \int_{\mathcal{X}_j} f(y_j) \frac{dy_j}{\omega_j} \right|^p \frac{dx_j}{\omega_j} \right\}^{1/p}$$

$$\leq \sup\left\{\omega_j^{1/2}\right\} \left\{ \sum_j \omega_j \right\}^{(p-2)/2p} 2\varphi\left(2^{-n}\right) \left\{ \sum_j \omega_j \int_{\mathcal{X}_j} |g_n(x_j)|^p \frac{dx_j}{\omega_j} \right\}^{1/p}$$

$$\leq 2\,|\mathcal{M}|^{1/2-1/p} \sup\left\{\omega_j^{1/2}\right\} \varphi\left(2\sup\{\delta_j\}\right) \left\{ \int_{\mathcal{M}} |g_n(x)|^p \, dx \right\}^{1/p}.$$

□

COROLLARY 2. *Let $\mathcal{M}$ be an Ahlfors d-regular metric measure space of finite measure and assume that $\mathcal{M}$ is decomposed into a finite number of disjoint sets $\mathcal{M} = \cup_{j=1}^N \mathcal{X}_j$ with $\omega_j = |\mathcal{X}_j| \approx N^{-1}$ and $\operatorname{diam}(\mathcal{X}_j) \approx N^{-1/d}$. Then for every $1 \leq p < +\infty$ and for every $0 < \varepsilon < 1$ there exists a constant c with the following property. For every function $f(x)$ in the Besov space $\mathbb{B}^{\varphi}_{p,\infty}(\mathcal{M})$, $\varphi(t) = t^{\alpha}$ and $\alpha > 0$, a random choice of points $\{x_j\}$ in $\{\mathcal{X}_j\}$ gives*

$$\left| \sum_{j=1}^N \omega_j f(x_j) - \int_{\mathcal{M}} f(x)dx \right| \leq \begin{cases} c\,\|f\|_{\mathbb{B}^{\varphi}_{p,\infty}(\mathcal{M})} N^{1/p-1-\alpha/d} & \text{if } 1 \leq p \leq 2, \\ c\,\|f\|_{\mathbb{B}^{\varphi}_{p,\infty}(\mathcal{M})} N^{-1/2-\alpha/d} & \text{if } 2 \leq p < +\infty, \end{cases}$$

with probability greater than $1 - \varepsilon$.

PROOF. This follows from the above theorem via Chebyshev inequality. □

The following example shows that the above theorem is essentially sharp.

EXAMPLE 5. *Decompose the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d = [0,1)^d$ into $m^d = N$ equal cubes $\{\mathcal{X}_j\}$ with centers $\{z_j\}$ and sides of length $m^{-1} = N^{-1/d}$. Choose a smooth function $g(x)$ on $\mathbb{R}^d$ with support in the cube $(-1/2,1/2)^d$ and mean zero,*

$$\int_{\mathbb{R}^d} g(x)dx = 0.$$

Shrink and copy $g(x)$ into a single $\mathcal{X}_j$ and define

$$f(x) = g\left(N^{1/d}(x - z_j)\right).$$

Then for every $1 \le p \le +\infty$,

$$\left\{\int_{\mathbb{T}^d} |f(x)|^p\, dx\right\}^{1/p} = N^{-1/p}\left\{\int_{\mathbb{R}^d} |g(x)|^p\, dx\right\}^{1/p},$$

$$\left\{\int_{\mathbb{T}^d} |\nabla f(x)|^p\, dx\right\}^{1/p} = N^{1/d-1/p}\left\{\int_{\mathbb{R}^d} |\nabla g(x)|^p\, dx\right\}^{1/p}.$$

It then follows by interpolation that if $\varphi(t) = t^\alpha$ with $0 < \alpha < 1$, then

$$\|f\|_{\mathbb{B}^{\varphi}_{p,\infty}(\mathbb{T}^d)} \le cN^{\alpha/d-1/p}.$$

Moreover, since $f(x)$ has mean zero,

$$\left\{\int_{\mathcal{X}_1}\int_{\mathcal{X}_2}\cdots\left|\sum_{j=1}^{N}\omega_j f(x_j) - \int_{\mathbb{T}^d} f(x)dx\right|^p \frac{dx_1}{\omega_1}\frac{dx_2}{\omega_2}\cdots\right\}^{1/p}$$

$$= \left\{\int_{\mathcal{X}_j} |\omega_j f(x_j)|^p \frac{dx_j}{\omega_j}\right\}^{1/p} = N^{-1}\left\{\int_{\mathbb{R}^d} |g(x)|^p\, dx\right\}^{1/p}.$$

Finally,

$$N^{-1} = N^{1/p-1}N^{-\alpha/d}N^{\alpha/d-1/p} \ge c\sup\left\{\omega_j^{1-1/p}\right\}\varphi\left(2\sup\{\delta_j\}\right)\|f\|_{\mathbb{B}^{\varphi}_{p,\infty}(\mathbb{T}^d)}.$$

In particular, this estimate shows that Theorem 3 with $1 \le p \le 2$ is sharp.

Similarly, shrink and copy $g(x)$ on each $\mathcal{X}_j$ and define

$$f(x) = \sum_{j=1}^{N} g\left(N^{1/d}(x - z_j)\right).$$

Then for every $1 \le p \le +\infty$,

$$\left\{\int_{\mathbb{T}^d} |f(x)|^p\, dx\right\}^{1/p} = \left\{\int_{\mathbb{R}^d} |g(x)|^p\, dx\right\}^{1/p},$$

$$\left\{\int_{\mathbb{T}^d} |\nabla f(x)|^p\, dx\right\}^{1/p} = N^{1/d}\left\{\int_{\mathbb{R}^d} |\nabla g(x)|^p\, dx\right\}^{1/p}.$$

It then follows by interpolation that if $\varphi(t) = t^\alpha$ with $0 < \alpha < 1$, then

$$\|f\|_{\mathbb{B}^{\varphi}_{p,\infty}(\mathbb{T}^d)} \le cN^{\alpha/d}.$$

See [**1**]. *Moreover, the Kintchine-Marcinkiewicz-Zygmund inequality gives, for* $2 \leq p < +\infty$,

$$\left\{ \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots \left| \sum_{j=1}^{N} \omega_j f(x_j) - \int_{\mathbb{T}^d} f(x) dx \right|^p \frac{dx_1}{\omega_1} \frac{dx_2}{\omega_2} \cdots \right\}^{1/p}$$

$$\geq A(p) \left\{ \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots \left(\sum_{j=1}^{N} \left| \omega_j f(x_j) \right|^2 \right)^{p/2} \frac{dx_1}{\omega_1} \frac{dx_2}{\omega_2} \cdots \right\}^{1/p}$$

$$\geq A(p) \left\{ \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots \left(\sum_{j=1}^{N} \left| \omega_j f(x_j) \right|^2 \right) \frac{dx_1}{\omega_1} \frac{dx_2}{\omega_2} \cdots \right\}^{1/2}$$

$$= A(p) N^{-1/2} \left\{ \int_{\mathbb{R}^d} |g(x)|^2 dx \right\}^{1/2}.$$

Finally,

$$N^{-1/2} = N^{-1/2} N^{-\alpha/d} N^{\alpha/d} \geq c \sup \left\{ \omega_j^{1/2} \right\} \varphi(2 \sup \{\delta_j\}) \|f\|_{\mathbb{B}^{\varphi}_{p,\infty}(\mathbb{T}^d)}.$$

In particular, this estimate shows that Theorem 3 with $2 \leq p < +\infty$ *is sharp.*

Again, define $f(x) = \sum_{j=1}^{N} g\left(N^{1/d}(x - z_j)\right)$, *fix a point* z *with* $g(z) \neq 0$, *and choose* $x_j = z_j + N^{-1/d} z$. *Then*

$$\left| \sum_{j=1}^{N} \omega_j f(x_j) - \int_{\mathbb{T}^d} f(x) dx \right| = |g(z)| \geq c \varphi(2 \sup \{\delta_j\}) \|f\|_{\mathbb{B}^{\varphi}_{p,\infty}(\mathcal{M})}.$$

In particular, this estimate shows that Theorem 3 with $p = +\infty$ *is sharp.*

Finally, similar counterexamples work in any compact Riemannian manifold.

The above theorems state that for every function in a suitable class, a random distribution of nodes gives a good quadrature rule, with the nodes that depend on the function. But one can also search for a distribution of nodes which gives good quadrature rules for all functions in a suitable class. In what follows this suitable class is a class of potentials.

For every choice of nodes $\{x_j\}$ and weights $\{\omega_j\}$ denote by $\mathcal{E}(x_j, \omega_j) f$ the linear functional

$$\mathcal{E}(x_j, \omega_j) f = \sum_j \omega_j f(x_j) - \int_{\mathcal{M}} f(x) dx.$$

Also denote by $\|\mathcal{E}(x_j, \omega_j)\|_{\Phi,p}$ the norm of this functional on $\mathbb{H}^{\Phi}_p(\mathcal{M})$,

$$\|\mathcal{E}(x_j, \omega_j)\|_{\Phi,p} = \sup \left\{ \frac{|\mathcal{E}(x_j, \omega_j) f|}{\|f\|_{\mathbb{H}^{\Phi}_p(\mathcal{M})}} \right\}.$$

LEMMA 1. *Assume that a measure space* $\mathcal{M}$ *is decomposed into a finite or infinite number of disjoint sets* $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots$, *with measure* $|\mathcal{X}_j| = \omega_j > 0$. *Assume that* $1 \leq p \leq +\infty$, $1 \leq q \leq +\infty$, $1/p + 1/q = 1$, *and that for every* x

$$\int_{\mathcal{M}} |\Phi(x, y)|^q dy < +\infty.$$

Finally assume that

$$\int_{\mathcal{M}}\left(\int_{\mathcal{M}}\left|\Phi\left(x,y\right)\right|dx\right)^{q}dy<+\infty.$$

Then the functional $\mathcal{E}\left(x_j,\omega_j\right)$ *is well defined and continuous on* $\mathbb{H}_p^{\Phi}\left(\mathcal{M}\right)$ *and its norm is*

$$\left\|\mathcal{E}\left(x_j,\omega_j\right)\right\|_{\Phi,p}=\left\{\int_{\mathcal{M}}\left|\sum_j\int_{\mathcal{X}_j}\left(\Phi(x_j,y)-\Phi(x,y)\right)dx\right|^q dy\right\}^{1/q}.$$

Proof. Assume that $f(x)$ is the potential of a function $g(x)$ in $\mathbb{L}^p\left(\mathcal{M}\right)$,

$$f\left(x\right)=\int_{\mathcal{M}}\Phi\left(x,y\right)g\left(y\right)dy.$$

Since

$$\int_{\mathcal{M}}\left|\Phi\left(x,y\right)\right|^q dy<+\infty,$$

$f\left(x\right)$ is pointwise well defined, and since

$$\int_{\mathcal{M}}\left(\int_{\mathcal{M}}\left|\Phi\left(x,y\right)\right|dx\right)^{q}dy<+\infty,$$

one can apply Fubini's theorem and obtain

$$\int_{\mathcal{M}}f\left(x\right)dx=\int_{\mathcal{M}}\left(\int_{\mathcal{M}}\Phi\left(x,y\right)g\left(y\right)dy\right)dx=\int_{\mathcal{M}}g\left(y\right)\left(\int_{\mathcal{M}}\Phi\left(x,y\right)dx\right)dy.$$

Then the functional $\mathcal{E}\left(x_j,\omega_j\right)f$ is well defined. Moreover

$$\begin{aligned}\left|\mathcal{E}\left(x_j,\omega_j\right)f\right|&=\left|\sum_j\omega_j f\left(x_j\right)-\int_{\mathcal{M}}f(x)dx\right|\\
&=\left|\sum_j\omega_j\int_{\mathcal{M}}\Phi\left(x_j,y\right)g\left(y\right)dy-\int_{\mathcal{M}}g\left(y\right)\left(\int_{\mathcal{M}}\Phi\left(x,y\right)dx\right)dy\right|\\
&=\left|\int_{\mathcal{M}}\left(\sum_j\int_{\mathcal{X}_j}\left(\Phi\left(x_j,y\right)-\Phi\left(x,y\right)\right)dx\right)g\left(y\right)dy\right|\\
&\leq\left\{\int_{\mathcal{M}}|g(y)|^p dy\right\}^{1/p}\left\{\int_{\mathcal{M}}\left|\sum_j\int_{\mathcal{X}_j}\left(\Phi\left(x_j,y\right)-\Phi\left(x,y\right)\right)dx\right|^q dy\right\}^{1/q}.\end{aligned}$$

Taking the infimum as $g\left(y\right)$ varies among all possible functions $g\left(y\right)$ in $\mathbb{L}^p\left(\mathcal{M}\right)$ with potential $f\left(x\right)$, one obtains

$$\left\|\mathcal{E}\left(x_j,\omega_j\right)\right\|_{\Phi,p}\leq\left\{\int_{\mathcal{M}}\left|\sum_j\int_{\mathcal{X}_j}\left(\Phi\left(x_j,y\right)-\Phi\left(x,y\right)\right)dx\right|^q dy\right\}^{1/q}.$$

Conversely, set

$$F\left(y\right)=\sum_{j}\int_{\mathcal{X}_{j}}\left(\Phi\left(x_{j},y\right)-\Phi\left(x,y\right)\right)dx,$$

$$g\left(y\right)=\begin{cases}\overline{F\left(y\right)}\left|F\left(y\right)\right|^{q/p-1} & \text{if } F\left(y\right)\neq 0,\\ 0 & \text{if } F\left(y\right)=0,\end{cases}$$

$$f\left(x\right)=\int_{\mathcal{M}}\Phi\left(x,y\right)g\left(y\right)dy.$$

With this choice one has

$$\begin{aligned}\left|\mathcal{E}\left(x_{j},\omega_{j}\right)f\right| &= \left|\sum_{j}\omega_{j}f\left(x_{j}\right)-\int_{\mathcal{M}}f(x)dx\right|\\ &= \left|\int_{\mathcal{M}}F\left(y\right)g\left(y\right)dy\right| = \int_{\mathcal{M}}\left|F\left(y\right)\right|^{1+q/p}dy\\ &= \left\{\int_{\mathcal{M}}\left|F\left(y\right)\right|^{q}dy\right\}^{1/q}\left\{\int_{\mathcal{M}}\left|F\left(y\right)\right|^{q}dy\right\}^{1/p}\\ &= \left\{\int_{\mathcal{M}}\left|F\left(y\right)\right|^{q}dy\right\}^{1/q}\left\{\int_{\mathcal{M}}\left|g\left(y\right)\right|^{p}dy\right\}^{1/p}\\ &\geq \left\{\int_{\mathcal{M}}\left|F\left(y\right)\right|^{q}dy\right\}^{1/q}\left\|f\right\|_{\mathbb{H}_{p}^{\Phi}}.\end{aligned}$$

This implies that

$$\left\|\mathcal{E}\left(x_{j},\omega_{j}\right)\right\|_{\Phi,p}\geq\left\{\int_{\mathcal{M}}\left|F\left(y\right)\right|^{q}dy\right\}^{1/q}.$$

□

THEOREM 4. *Assume that a measure space $\mathcal{M}$ is decomposed into a finite or infinite number of disjoint sets $\mathcal{X}_1\cup\mathcal{X}_2\cup\cdots$, with measure $|\mathcal{X}_j|=\omega_j>0$. Assume that $1\leq p\leq+\infty$, $1\leq q\leq+\infty$, $1/p+1/q=1$, and that for every x*

$$\int_{\mathcal{M}}\left|\Phi\left(x,y\right)\right|^{q}dy<+\infty.$$

Finally assume that

$$\int_{\mathcal{M}}\left(\int_{\mathcal{M}}\left|\Phi\left(x,y\right)\right|dx\right)^{q}dy<+\infty.$$

Define

$$\Gamma\left(\Phi\right)=\sum_{j}\left\{\int_{\mathcal{M}}\int_{\mathcal{X}_{j}}\left|\int_{\mathcal{X}_{j}}\left(\Phi(x_{j},y)-\Phi(z_{j},y)\right)dz_{j}\right|^{q}dy\frac{dx_{j}}{\omega_{j}}\right\}^{1/q},$$

$$\Delta\left(\Phi\right)=\left\{\int_{\mathcal{M}}\int_{\mathcal{X}_{1}}\int_{\mathcal{X}_{2}}\cdots\left(\sum_{j}\left|\int_{\mathcal{X}_{j}}\left(\Phi(x_{j},y)-\Phi(z_{j},y)\right)dz_{j}\right|^{2}\right)^{q/2}dy\frac{dx_{1}}{\omega_{1}}\frac{dx_{2}}{\omega_{2}}\cdots\right\}^{1/q}.$$

Then

(i): *For every* $1 \leq p \leq +\infty$,

$$\left\{ \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots \left\| \mathcal{E}\left(x_j, \omega_j\right) \right\|_{\Phi,p}^{q} \frac{dx_1}{\omega_1} \frac{dx_2}{\omega_2} \cdots \right\}^{1/q} \leq \Gamma\left(\Phi\right).$$

(ii): *For every* $1 < p \leq +\infty$,

$$A\left(q\right) \Delta\left(\Phi\right) \leq \left\{ \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots \left\| \mathcal{E}\left(x_j, \omega_j\right) \right\|_{\Phi,p}^{q} \frac{dx_1}{\omega_1} \frac{dx_2}{\omega_2} \cdots \right\}^{1/q} \leq B\left(q\right) \Delta\left(\Phi\right).$$

In particular, there exist choices of nodes $\{x_j\}$ *in* $\{\mathcal{X}_j\}$ *with the property that for every function* $f(x)$ *in the potential space* $\mathbb{H}_p^{\Phi}(\mathcal{M})$,

$$\left| \sum_j \omega_j f\left(x_j\right) - \int_{\mathcal{M}} f(x)dx \right| \leq \begin{cases} \Gamma\left(\Phi\right) \left\|f\right\|_{\mathbb{H}_p^{\Phi}(\mathcal{M})} & \text{for every } 1 \leq p \leq +\infty, \\ \\ B\left(q\right) \Delta\left(\Phi\right) \left\|f\right\|_{\mathbb{H}_p^{\Phi}(\mathcal{M})} & \text{for every } 1 < p \leq +\infty. \end{cases}$$

The constants $\Gamma(\Phi)$ and $\Delta(\Phi)$ are related to the smoothness of the kernel $\Phi(x,y)$ and can be estimated in terms of Sobolev norms, as in the proof of Theorem 3. However in the applications the estimates in terms of Sobolev norms are not always optimal and it is more convenient to keep the above complicated expressions. As before, the estimate (i) is better than (ii) only when p is close to 1.

Proof. By Lemma 1 and the triangle inequality, the mean value of $\left\| \mathcal{E}\left(x_j, \omega_j\right) \right\|_{\Phi,p}$ is controlled by

$$\left\{ \int_{\mathcal{M}} \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots \left| \sum_j \int_{\mathcal{X}_j} \left(\Phi(x_j, y) - \Phi(z_j, y)\right) dz_j \right|^q dy \frac{dx_1}{\omega_1} \frac{dx_2}{\omega_2} \cdots \right\}^{1/q}$$

$$\leq \sum_j \left\{ \int_{\mathcal{M}} \int_{\mathcal{X}_j} \left| \int_{\mathcal{X}_j} \left(\Phi(x_j, y) - \Phi(z_j, y)\right) dz_j \right|^q dy \frac{dx_j}{\omega_j} \right\}^{1/q}.$$

This gives the proof with $\Gamma(\Phi)$. The proof with $\Delta(\Phi)$ is similar. By the Kintchine-Marcinkiewicz-Zygmund inequality,

$$\left\{ \int_{\mathcal{M}} \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots \left| \sum_j \int_{\mathcal{X}_j} \left(\Phi(x_j, y) - \Phi(z_j, y)\right) dz_j \right|^q dy \frac{dx_1}{\omega_1} \frac{dx_2}{\omega_2} \cdots \right\}^{1/q}$$

$$\leq B\left(q\right) \left\{ \int_{\mathcal{M}} \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots \left(\sum_j \left| \int_{\mathcal{X}_j} \left(\Phi(x_j, y) - \Phi(z_j, y)\right) dz_j \right|^2 \right)^{q/2} dy \frac{dx_1}{\omega_1} \frac{dx_2}{\omega_2} \cdots \right\}^{1/q}.$$

The proof for the lower bound is similar. □

As we said in the Introduction, it has been proved in [**4**, Corollary 2.15] that when $\mathcal{M}$ is a d-dimensional compact Riemannian manifold and $\mathbb{H}_p^{\alpha}(\mathcal{M})$ is the classical fractional Sobolev space with $\alpha > d/p$, then for any choice of N nodes $\{x_j\}$ and weights $\{\omega_j\}$,

$$\left\| \mathcal{E}\left(x_j, \omega_j\right) \right\|_{\alpha,p} \geq cN^{-\alpha/d}.$$

Moreover, one can achieve this optimal speed of convergence with any choice of N nodes $\{x_j\}$ with comparable minimal separation distance and mesh norm, and a suitable choice of positive weights $\{\omega_j\}$,

$$\left\|\mathcal{E}\left(x_j,\omega_j\right)\right\|_{\alpha,p} \leq cN^{-\alpha/d}.$$

The following corollary shows that if $\mathcal{M}$ is decomposed into a finite number of disjoint sets $\mathcal{M}=\cup_{j=1}^{N}\mathcal{X}_j$ with $\omega_j=|\mathcal{X}_j|\approx N^{-1}$ and $\operatorname{diam}(\mathcal{X}_j)\approx N^{-1/d}$ then, at least for certain values of α, a random choice of nodes $\{x_j\}$ in $\{\mathcal{X}_j\}$ gives again this best possible exponent.

Corollary 3. *Let $\mathcal{M}$ be an Ahlfors d-regular metric measure space of finite measure and assume that $\mathcal{M}$ is decomposed into a finite number of disjoint sets $\mathcal{M}=\cup_{j=1}^{N}\mathcal{X}_j$ with $\omega_j=|\mathcal{X}_j|\approx N^{-1}$ and $\delta_j=\operatorname{diam}(\mathcal{X}_j)\approx N^{-1/d}$. Assume that for some $0<\alpha<d$,*

$$\begin{gathered}\left|\Phi(x,y)\right| \leq c\left|x-y\right|^{\alpha-d} \text{ for every } x \text{ and } y,\\ \left|\Phi(x,y)-\Phi(z,y)\right| \leq c\left|x-z\right|\left|x-y\right|^{\alpha-d-1} \text{ if } \left|x-y\right|\geq 2\left|x-z\right|.\end{gathered}$$

Finally assume that $1<p\leq+\infty$, $1/p+1/q=1$, and $d/p<\alpha<d$. Then

$$\begin{aligned}&\left\{\int_{\mathcal{X}_1}\cdots\int_{\mathcal{X}_N}\left\|\mathcal{E}\left(x_j,\omega_j\right)\right\|_{\Phi,p}^{q}\frac{dx_1}{\omega_1}\cdots\frac{dx_N}{\omega_N}\right\}^{1/q}\\ &\leq\begin{cases} cN^{-\alpha/d} & \text{if } \alpha<d/2+1,\\ cN^{-1/2-1/d}\log^{1/2}(N) & \text{if } \alpha=d/2+1,\\ cN^{-1/2-1/d} & \text{if } \alpha>d/2+1.\end{cases}\end{aligned}$$

Proof. By Theorem 4 it suffices to estimate

$$\left\{\int_{\mathcal{M}}\int_{\mathcal{X}_1}\cdots\int_{\mathcal{X}_N}\left(\sum_j\left|\int_{\mathcal{X}_j}\left(\Phi(x_j,y)-\Phi(z_j,y)\right)dz_j\right|^2\right)^{q/2}dy\frac{dx_1}{\omega_1}\cdots\frac{dx_2}{\omega_N}\right\}^{1/q}.$$

When $\operatorname{dist}(y,\mathcal{X}_j)\leq 2\delta_j$, then for every x_j in $\mathcal{X}_j$,

$$\begin{aligned}\left|\int_{\mathcal{X}_j}\left(\Phi(x_j,y)-\Phi(z_j,y)\right)dz_j\right| &\leq c\int_{\mathcal{X}_j}\left(\left|x_j-y\right|^{\alpha-d}+\left|z_j-y\right|^{\alpha-d}\right)dz_j\\ &\leq c\omega_j\left|x_j-y\right|^{\alpha-d}+c\delta_j^{\alpha}\leq cN^{-1}\left|x_j-y\right|^{\alpha-d}.\end{aligned}$$

When $\operatorname{dist}(y,\mathcal{X}_j)\geq 2\delta_j$, then

$$\begin{aligned}\left|\int_{\mathcal{X}_j}\left(\Phi(x_j,y)-\Phi(z_j,y)\right)dz_j\right| &\leq c\int_{\mathcal{X}_j}\left|x_j-z_j\right|\left|x_j-y\right|^{\alpha-d-1}dz_j\\ &\leq c\delta_j\omega_j\left|x_j-y\right|^{\alpha-d-1}\leq cN^{-1-1/d}\left|x_j-y\right|^{\alpha-d-1}.\end{aligned}$$

Hence

$$\left\{\int_{\mathcal{M}}\int_{\mathcal{X}_1}\cdots\int_{\mathcal{X}_N}\left(\sum_j\left|\int_{\mathcal{X}_j}\left(\Phi(x_j,y)-\Phi(z_j,y)\right)dz_j\right|^2\right)^{q/2}dy\frac{dx_1}{\omega_1}\cdots\frac{dx_N}{\omega_N}\right\}^{1/q}$$

$$\leq c\left\{\int_{\mathcal{M}}\int_{\mathcal{X}_1}\cdots\int_{\mathcal{X}_N}\left(N^{-2}\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)\leq 2\delta_j}|x_j-y|^{2\alpha-2d}\right)^{q/2}dy\frac{dx_1}{\omega_1}\cdots\frac{dx_N}{\omega_N}\right\}^{1/q}$$

$$+c\left\{\int_{\mathcal{M}}\int_{\mathcal{X}_1}\cdots\int_{\mathcal{X}_N}\left(N^{-2-2/d}\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)>2\delta_j}|x_j-y|^{2\alpha-2d-2}\right)^{q/2}dy\frac{dx_1}{\omega_1}\cdots\frac{dx_N}{\omega_N}\right\}^{1/q}.$$

Under the assumption that $\delta_j\approx N^{-1/d}$ there is only a bounded number of $\mathcal{X}_j$ with $\mathrm{dist}\,(y,\mathcal{X}_j)\leq 2\delta_j$. Hence, since $\alpha>d/p$,

$$\left\{\int_{\mathcal{M}}\int_{\mathcal{X}_1}\cdots\int_{\mathcal{X}_N}\left(N^{-2}\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)\leq 2\delta_j}|x_j-y|^{2\alpha-2d}\right)^{q/2}dy\frac{dx_1}{\omega_1}\cdots\frac{dx_N}{\omega_N}\right\}^{1/q}$$

$$\leq c\left\{N^{-q}\sum_j\int_{\mathcal{X}_j}\left(\int_{\left\{|y-x_j|\leq cN^{-1/d}\right\}}|x_j-y|^{\alpha q-dq}\,dy\right)\frac{dx_j}{\omega_j}\right\}^{1/q}\leq cN^{-\alpha/d}.$$

Moreover,

$$\begin{aligned}
&N^{-2-2/d}\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)>2\delta_j}|x_j-y|^{2\alpha-2d-2}\\
&\leq cN^{-1-2/d}\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)>2\delta_j}\omega_j\,|x_j-y|^{2\alpha-2d-2}\\
&\leq cN^{-1-2/d}\int_{\left\{|x-y|>cN^{-1/d}\right\}}|x-y|^{2\alpha-2d-2}\,dx\\
&\leq\begin{cases} cN^{-2\alpha/d} & \text{if } \alpha<1+d/2,\\ cN^{-1-2/d}\log(N) & \text{if } \alpha=1+d/2,\\ cN^{-1-2/d} & \text{if } \alpha>1+d/2.\end{cases}
\end{aligned}$$

Hence,

$$\left\{\int_{\mathcal{M}}\int_{\mathcal{X}_1}\cdots\int_{\mathcal{X}_N}\left(N^{-2-2/d}\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)>2\delta_j}|x_j-y|^{2\alpha-2d-2}\right)^{q/2}dy\frac{dx_1}{\omega_1}\cdots\frac{dx_N}{\omega_N}\right\}^{1/q}$$

$$\leq\begin{cases} cN^{-\alpha/d} & \text{if } \alpha<1+d/2,\\ cN^{-1/2-1/d}\log^{1/2}(N) & \text{if } \alpha=1+d/2,\\ cN^{-1/2-1/d} & \text{if } \alpha>1+d/2.\end{cases}$$

□

The following result shows that, under some natural assumptions on the kernel, the mean value estimate in the above corollary is essentially sharp.

Corollary 4. *Let $\mathcal{M}$ be an Ahlfors d-regular metric measure space of finite measure and assume that $\mathcal{M}$ is decomposed into a finite number of disjoint sets $\mathcal{M} = \cup_{j=1}^{N} \mathcal{X}_j$ with $\omega_j = |\mathcal{X}_j| \approx N^{-1}$ and $\delta_j = \operatorname{diam}(\mathcal{X}_j) \approx N^{-1/d}$. Assume that there exists an α with $0 < \alpha < d$, such that for any $j = 1, \ldots, N$, for any $z \in \mathcal{X}_j$, and for any y such that* $\operatorname{dist}(y, \mathcal{X}_j) \geq 2\delta_j$,

$$\int_{\mathcal{X}_j} |\Phi(x,y) - \Phi(z,y)|\, dx \geq cN^{-1-1/d} \left(\operatorname{dist}(y, \mathcal{X}_j)\right)^{\alpha-d-1}.$$

Also assume that for any $y \in \mathcal{M}$, the function $x \to \Phi(x,y)$ is continuous in $x \neq y$. Finally assume that $1 < p \leq +\infty$, $1/p + 1/q = 1$, and $d/p < \alpha < d$. Then

$$\left\{ \int_{\mathcal{X}_1} \cdots \int_{\mathcal{X}_N} \|\mathcal{E}(x_j, \omega_j)\|_{\Phi,p}^{q} \frac{dx_1}{\omega_1} \cdots \frac{dx_N}{\omega_N} \right\}^{1/q}$$
$$\geq \begin{cases} cN^{-\alpha/d} & \text{if } \alpha < d/2+1, \\ cN^{-1/2-1/d} \log^{1/2}(N) & \text{if } \alpha = d/2+1, \\ cN^{-1/2-1/d} & \text{if } \alpha > d/2+1. \end{cases}$$

Proof. It follows from Lemma 1 that

$$\left\{ \int_{\mathcal{X}_1} \cdots \int_{\mathcal{X}_N} \|\mathcal{E}(x_j, \omega_j)\|_{\Phi,p}^{q} \frac{dx_1}{\omega_1} \cdots \frac{dx_N}{\omega_N} \right\}^{1/q}$$
$$= \left\{ \int_{\mathcal{M}} \int_{\mathcal{X}_1} \cdots \int_{\mathcal{X}_N} \left| \sum_j \int_{\mathcal{X}_j} (\Phi(x_j,y) - \Phi(z_j,y))\, dz_j \right|^q dy \frac{dx_1}{\omega_1} \cdots \frac{dx_N}{\omega_N} \right\}^{1/q}$$
$$\geq |\mathcal{M}|^{-1/p} \int_{\mathcal{M}} \int_{\mathcal{X}_1} \cdots \int_{\mathcal{X}_N} \left| \sum_j \int_{\mathcal{X}_j} (\Phi(x_j,y) - \Phi(z_j,y))\, dz_j \right| dy \frac{dx_1}{\omega_1} \cdots \frac{dx_N}{\omega_N}.$$

By the Kintchine-Marcinkiewicz-Zygmund inequality, this is bounded from below by

$$|\mathcal{M}|^{-1/p} A(1) \int_{\mathcal{M}} \int_{\mathcal{X}_1} \cdots \int_{\mathcal{X}_N} \left(\sum_j \left| \int_{\mathcal{X}_j} (\Phi(x_j,y) - \Phi(z_j,y))\, dz_j \right|^2 \right)^{1/2} dy \frac{dx_1}{\omega_1} \cdots \frac{dx_N}{\omega_N}$$
$$\geq |\mathcal{M}|^{-1/p} A(1) \int_{\mathcal{M}} \int_{\mathcal{X}_1} \cdots \int_{\mathcal{X}_N} \left(\sum_{j:\operatorname{dist}(y,\mathcal{X}_j) \geq 2\delta_j} \left| \int_{\mathcal{X}_j} (\Phi(x_j,y) - \Phi(z_j,y))\, dz_j \right|^2 \right)^{1/2} dy \frac{dx_1}{\omega_1} \cdots \frac{dx_N}{\omega_N}.$$

By the continuity of $z_j \to \Phi(z_j, y)$, there exists a point x_j^* depending on y such that $\int_{\mathcal{X}_j} \Phi(z_j,y) dz_j = \omega_j \Phi(x_j^*, y)$. Thus we have

$$\int_{\mathcal{X}_1} \cdots \int_{\mathcal{X}_N} \left(\sum_{j:\operatorname{dist}(y,\mathcal{X}_j) \geq 2\delta_j} \left| \int_{\mathcal{X}_j} (\Phi(x_j,y) - \Phi(z_j,y))\, dz_j \right|^2 \right)^{1/2} \frac{dx_1}{\omega_1} \cdots \frac{dx_N}{\omega_N}$$
$$= \int_{\mathcal{X}_1} \cdots \int_{\mathcal{X}_N} \left(\sum_{j:\operatorname{dist}(y,\mathcal{X}_j) \geq 2\delta_j} \omega_j^2 \left| \Phi(x_j,y) - \Phi(x_j^*,y) \right|^2 \right)^{1/2} \frac{dx_1}{\omega_1} \cdots \frac{dx_N}{\omega_N}.$$

Now let $\{\alpha_j\}$ and $\{\beta_j\}$ be two sequences, and let $A = \sum \alpha_j^2 < +\infty$. Then

$$\left(\sum_j \alpha_j^2 \beta_j^2\right)^{1/2} = A^{1/2}\left(\sum_j \frac{\alpha_j^2}{A}\beta_j^2\right)^{1/2} \geq A^{1/2}\sum_j \frac{\alpha_j^2}{A}\left|\beta_j\right| = A^{-1/2}\sum_j \alpha_j^2\left|\beta_j\right|.$$

Thus, if v_j is a point of the closure of $\mathcal{X}_j$ that minimizes the distance from y,

$$\int_{\mathcal{X}_1}\cdots\int_{\mathcal{X}_N}\left(\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)\geq 2\delta_j}\omega_j^2\left|v_j-y\right|^{2\alpha-2d-2}\frac{\left|\Phi(x_j,y)-\Phi(x_j^*,y)\right|^2}{\left|v_j-y\right|^{2\alpha-2d-2}}\right)^{1/2}\frac{dx_1}{\omega_1}\cdots\frac{dx_N}{\omega_N}$$

$$\geq \int_{\mathcal{X}_1}\cdots\int_{\mathcal{X}_N}\left[\left(\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)\geq 2\delta_j}\omega_j^2\left|v_j-y\right|^{2\alpha-2d-2}\right)^{-1/2}\times\right.$$

$$\left.\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)\geq 2\delta_j}\omega_j^2\left|v_j-y\right|^{2\alpha-2d-2}\frac{\left|\Phi(x_j,y)-\Phi(x_j^*,y)\right|}{\left|v_j-y\right|^{\alpha-d-1}}\right]\frac{dx_1}{\omega_1}\cdots\frac{dx_N}{\omega_N}$$

$$=\left(\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)\geq 2\delta_j}\omega_j^2\left|v_j-y\right|^{2\alpha-2d-2}\right)^{-1/2}\times$$

$$\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)\geq 2\delta_j}\omega_j^2\left|v_j-y\right|^{\alpha-d-1}\left(\int_{\mathcal{X}_1}\cdots\int_{\mathcal{X}_N}\left|\Phi(x_j,y)-\Phi(x_j^*,y)\right|\frac{dx_1}{\omega_1}\cdots\frac{dx_N}{\omega_N}\right)$$

$$\geq c\left(\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)\geq 2\delta_j}\omega_j^2\left|v_j-y\right|^{2\alpha-2d-2}\right)^{-1/2}\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)\geq 2\delta_j}\omega_j^2\left|v_j-y\right|^{2\alpha-2d-2}N^{-1/d}$$

$$= c\left(\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)\geq 2\delta_j}\omega_j^2\left|v_j-y\right|^{2\alpha-2d-2}\right)^{1/2}N^{-1/d}.$$

The desired result now follows from the simple estimates

$$\sum_{j:\mathrm{dist}(y,\mathcal{X}_j)\geq 2\delta_j}\omega_j^2\left|v_j-y\right|^{2\alpha-2d-2}\geq cN^{-1}\int_{N^{-1/d}}^1\rho^{2\alpha-2d-2}\rho^{d-1}d\rho$$

$$\geq\begin{cases} cN^{-1} & \text{if } \alpha>1+d/2,\\ cN^{-1}\log(N) & \text{if } \alpha=1+d/2,\\ cN^{-2\alpha/d+2/d} & \text{if } \alpha<1+d/2.\end{cases}$$

□

EXAMPLE 6. *Let $\mathcal{M}$ be a d dimensional compact Riemannian manifold with total measure one. Let $\left\{\lambda^2\right\}$ and $\{\varphi_\lambda(x)\}$ be the eigenvalues and an orthonormal complete system of eigenfunctions of the Laplace Beltrami operator Δ. Every tempered distribution on $\mathcal{M}$ has Fourier transform and series,*

$$\mathcal{F}f(\lambda)=\int_{\mathcal{M}}f(y)\overline{\varphi_\lambda(y)}dy,\quad f(x)=\sum_\lambda \mathcal{F}f(\lambda)\varphi_\lambda(x).$$

The Bessel kernel $B^{\alpha}(x,y)$, $-\infty<\alpha<+\infty$, *is a distribution defined by the expansion*

$$B^{\alpha}(x,y)=\sum_{\lambda}\left(1+\lambda^{2}\right)^{-\alpha/2}\varphi_{\lambda}(x)\overline{\varphi_{\lambda}(y)}.$$

A distribution $f(x)$ *is the Bessel potential of a distribution* $g(x)$ *if*

$$f(x)=\int_{\mathcal{M}}B^{\alpha}(x,y)g(y)dy=\sum_{\lambda}\left(1+\lambda^{2}\right)^{-\alpha/2}\mathcal{F}g(\lambda)\varphi_{\lambda}(x).$$

Bessel potentials of functions in $\mathbb{L}^{p}(\mathcal{M})$ *define the fractional Sobolev space* $\mathbb{H}_{p}^{\alpha}(\mathcal{M})$. *If* $0<\alpha<d$, *then the Bessel kernel satisfies the estimate*

$$|B^{\alpha}(x,y)|\leq c\,|x-y|^{\alpha-d},\quad |\nabla B^{\alpha}(x,y)|\leq c\,|x-y|^{\alpha-d-1}.$$

See [**4**]. *In particular, Corollary 3 applies. Indeed, an application of the technique of the Hadamard parametrix (see e.g.* [**3**]*) gives a more precise result: there is a smooth positive function* $C(y)$ *and positive constants* ε *and* c *such that*

$$B^{\alpha}(x,y)=C(y)\,|x-y|^{\alpha-d}+E(x,y),$$

with

$$|E(x,y)|\leq c\,|x-y|^{\alpha-d+\varepsilon}$$
$$|\nabla E(x,y)|\leq c\,|x-y|^{\alpha-d-1+\varepsilon}.$$

It then follows from this lemma that Corollary 4 applies to the Bessel kernel (see also [**8**, Theorems 24 and 25] *for the case of the Euclidean sphere with* $p=2$*).*

Finally, the classical Besov spaces $\mathbb{B}_{p,q}^{\alpha}(\mathcal{M})$ *defined via a Littlewood Paley decomposition are interpolation spaces between Bessel potential spaces. If* $1<p<+\infty$, $1\leq q\leq+\infty$, $0<\vartheta<1$, $\alpha=(1-\vartheta)\alpha_{0}+\vartheta\alpha_{1}$, *then in the real method of interpolation*

$$\left(\mathbb{H}_{p}^{\alpha_{0}}(\mathcal{M}),\mathbb{H}_{p}^{\alpha_{1}}(\mathcal{M})\right)_{\vartheta,q}=\mathbb{B}_{p,q}^{\alpha}(\mathcal{M}).$$

See [**1**]. *In particular, interpolating the results in Corollary 3, one proves that for every* $0<\varepsilon<1$ *there is a positive constant* c *such that a random choice of points* $\{x_{j}\}$ *in* $\{\mathcal{X}_{j}\}$ *has the property that for every function* $f(x)$ *in the Besov space* $\mathbb{B}_{p,\infty}^{\alpha}(\mathcal{M})$,

$$|\mathcal{E}(x_{j},\omega_{j})f|\leq\begin{cases}c\,\|f\|_{\mathbb{B}_{p,\infty}^{\alpha}(\mathcal{M})}N^{-\alpha/d} & \text{if } \alpha<d/2+1,\\ c\,\|f\|_{\mathbb{B}_{p,\infty}^{\alpha}(\mathcal{M})}N^{-1/2-1/d+\gamma} & \text{if } \alpha=d/2+1 \text{ and } \gamma>0,\\ c\,\|f\|_{\mathbb{B}_{p,\infty}^{\alpha}(\mathcal{M})}N^{-1/2-1/d} & \text{if } \alpha>d/2+1,\end{cases}$$

with probability greater than $1-\varepsilon$.

References

[1] Bergh, J., Löfström, J.: Interpolation spaces. Springer Verlag (1976)
[2] Beck, J., Chen, W.W.L.: Note on irregularities of distribution II, Proc. London Math. Soc. 61, 251–272 (1990)
[3] Brandolini, L., Colzani, L.: Decay of Fourier transforms and summability of eigenfunction expansions. Ann. Sc. Norm. Sup. Pisa Cl. Sci. 29, 611–638 (2000)
[4] Brandolini, L., Choirat, C., Colzani, L., Gigante G., Seri R., Travaglini, G.: Quadrature rules and distribution of points on manifolds. To appear in Ann. Sc. Norm. Super. Pisa Cl. Sci.
[5] Brandolini, L., Chen, W.W.L., Gigante, G., Travaglini, G.: Discrepancy for randomized Riemann sums. Proc. Amer. Math. Soc. 137, 3187–3196 (2009)

[6] Brandolini, L., Colzani, L., Gigante, G., Travaglini, G.: On the Koksma–Hlawka inequality, J. Complexity 29, 158–172 (2013)
[7] Brauchart, J.S., Hesse, K.: Numerical integration over spheres of arbitrary dimension. Constr. Approx. 25, 41-71(2007)
[8] Brauchart, J.S., Saff, E.B., Sloan, I.H., Womersley, R.S.: QMC design: optimal order quasi Monte Carlo integration schemes on the sphere. arXiv:1208.3267 (2012)
[9] Burkholder, D.: Sharp inequalities for martingales and stochastic integrals. Astérisque 157-158, 75–94 (1988)
[10] Chen, W.W.L.: On irregularities of distribution. III. J. Austral. Math. Soc. 60, 228–244 (1996)
[11] Chen, W.W.L., Travaglini, G.: Deterministic and probabilistic discrepancies, Ark. Mat. 47, 273–293 (2009)
[12] Chow, Y.S. , Teicher, Y.S.: Probability theory. Independence, interchangeability, martingales. Springer Verlag (1988)
[13] Colzani, L.: Approximation of Lebesgue integrals by Riemann sums and lattice points in domains with fractal boundary. Monatsh. Math. 123, 299–308 (1997)
[14] Filbir, F., Mhaskar, H.N.: A quadrature formula for diffusion polynomials corresponding to a generalized heat kernel. J. Fourier Anal. Appl. 16, 629–657 (2010)
[15] Haagerup, U.: The best constant in the Kintchine inequality. Studia Math. 70, 231–283 (1981)
[16] Hajłasz, P.: Sobolev spaces on an arbitrary metric space. Potential Anal. 5, 403–415 (1996)
[17] Hesse, K.: A lower bound for the worst-case cubature error on spheres of arbitrary dimension. Numer. Math. 103, 413–433 (2006)
[18] Hesse, K., Mhaskar, H.M., Sloan, I.H.: Quadrature in Besov spaces on the Euclidean sphere. J. Complexity 23, 528–552 (2007)
[19] Hesse, K., Sloan, I.H.: Worst-case errors in a Sobolev space setting for cubature over the sphere S^2. Bull. Austral. Math. Soc. 71, 81–105 (2005)
[20] Hesse, K., Sloan, I.H.: Optimal lower bounds for cubature error on the sphere S^2. J. Complexity 21, 790–803 (2005)
[21] Hesse, K., Sloan, I.H.: Cubature over the sphere S^2 in Sobolev spaces of arbitrary order. J. Approx. Theory 141, 118–133 (2006)
[22] Hu, J., Zähle, M.: Generalized Bessel and Riesz Potentials on Metric Measure Spaces. Potential Anal. 30, 315–340 (2009)
[23] Koskela, P., Yang, D., Zhou, Y.: Pointwise characterization of Besov and Triebel Lizorkin spaces and quasiconformal mappings. Adv. Math. 226, 3579–3621 (2011)
[24] Kushpel, A.: Optimal cubature formulas on compact homogeneous manifolds. J. Funct. Anal. 257, 1621–1629 (2009)
[25] Leopardi, P.: Diameter bounds for equal area partitions of the unit sphere. Electron. Trans. Numer. Anal. 35, 1–16 (2009)
[26] Maggioni, M., Mhaskar, H.N.: Diffusion polinomial frames on metric measure spaces. Appl. Comput. Harmon. Anal. 24, 329–353 (2008)
[27] Marcinkiewicz, J., Zygmund, A.: Sur les fonctions indépendantes. Fund. Math. 29, 60–90 (1937)
[28] Marcinkiewicz, J., Zygmund, A.: Quelques théorèmes sur les fonctions indépendantes. Studia Math. 7, 104–120 (1938)
[29] Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press (1970)

(Luca Brandolini) Dipartimento di Ingegneria, Università di Bergamo, Viale Marconi 5, 24044 Dalmine (BG), Italia.

E-mail address: luca.brandolini@unibg.it

(William W. L. Chen) Department of Mathematics, Macquarie University, Sydney, NSW 2109, Australia.

E-mail address: william.chen@mq.edu.au

(Leonardo Colzani) Dipartimento di Matematica e Applicazioni, Edificio U5, Università di Milano-Bicocca, Via R. Cozzi 53, 20125 Milano, Italia.

E-mail address: leonardo.colzani@unimib.it

(Giacomo Gigante) Dipartimento di Ingegneria, Università di Bergamo, Viale Marconi 5, 24044 Dalmine (BG), Italia.

E-mail address: giacomo.gigante@unibg.it

(Giancarlo Travaglini) Dipartimento di Statistica e Metodi Quantitativi, Edificio U7, Università di Milano-Bicocca, Via Bicocca degli Arcimboldi 8, 20126 Milano, Italia.

E-mail address: giancarlo.travaglini@unimib.it