UNIVERSITÀ DEGLI STUDI DI BERGAMO
DIPARTIMENTO DI INGEGNERIA

QUADERNI DEL DIPARTIMENTO

Department of Engineering

Working Paper

Series "*Mathematics and Statistics*"

n. 08/MS – 2014

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https://dx.doi.org/10.6092/10446_30860

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ZEROS OF DEDEKIND ZETA FUNCTIONS UNDER GRH

LOÏC GRENIÉ AND GIUSEPPE MOLTENI

Abstract. Assuming GRH, we prove an explicit upper bound for the number of zeros of a Dedekind zeta function having imaginary part in $[T-a, T+a]$. We also prove a bound for the multiplicity of the zeros.

1. Introduction and results

sec:C1

Let $\mathbb{K}$ be a number field of dimension $n_{\mathbb{K}}$ and let $\Delta_{\mathbb{K}}$ be the absolute value of its discriminant. Let $n_{\mathbb{K}}(T;a)$ denote the number of zeros $\rho=\beta+i\gamma$ of the Dedekind zeta function $\zeta_{\mathbb{K}}$ with $|\gamma-T|\le a$ and which are non trivial, i.e. with $0<\beta<1$.
An upper bound can be deduced via the equality $n_{\mathbb{K}}(T;a)=\frac{1}{2}(N_{\mathbb{K}}(T+a)-N_{\mathbb{K}}(T-a^-))$ where $N_{\mathbb{K}}(T)$ counts the nontrivial zeros with imaginary part in $[-T,T]$. In this way from the explicit bound for $N_{\mathbb{K}}(T)$ recently proved by Trudgian [12] it follows that

eq:C1

$$n_{\mathbb{K}}(T;a)\le\Big(\frac{a}{\pi}+0.248\Big)\big(\log\Delta_{\mathbb{K}}+n_{\mathbb{K}}\log T\big)+\text{lower order terms}\qquad\forall T>a+1, \tag{1.1}$$

where the remaining terms are explicit, have lower order as a function of T, and can be estimated independently of the discriminant. The constant 0.248 in (1.1) comes from the remainder term in the formula for $N_{\mathbb{K}}(T)$, which is explicit but has the classical size $O(\log T)$. This changes if one assumes the Generalized Riemann Hypothesis, since the works of Littlewood [7], Selberg [9] and Lang [6] show that in this case the remainder term drops to $O(\frac{\log T}{\log\log T})$, so that now one gets

eq:C2

$$n_{\mathbb{K}}(T;a)=\frac{a}{\pi}\big(\log\Delta_{\mathbb{K}}+n_{\mathbb{K}}\log T\big)+\text{lower order terms.} \tag{1.2}$$

Recent computations of Goldston and Gonek [2] show that for the Riemann zeta function the constant in the remainder term of $N_{\mathbb{Q}}(T)$ is $(\frac{1}{2}+o(1))\frac{\log T}{\log\log T}$, at most. Applying Lang's heuristic [6], the general case should be similar to $(1+o(1))\frac{\log\Delta_{\mathbb{K}}+n_{\mathbb{K}}\log T}{\log\log T}$ and thus the previous formula is probably

$$n_{\mathbb{K}}(T;a)=\Big(\frac{a}{\pi}+\frac{c}{\log\log T}\Big)\big(\log\Delta_{\mathbb{K}}+n_{\mathbb{K}}\log T\big)+\text{lower order terms}$$

with an absolute constant $c\lesssim 1$. Due to the very slow decay of the function $1/\log\log T$, this tentative formula would improve on a result of type (1.1) only for very large T. As a consequence, for numerical applications it is interesting to work out a totally explicit bound for $n_{\mathbb{K}}(T;a)$ under GRH, with an asymptotically non-optimal but small constant in front of the main term $\log\Delta_{\mathbb{K}}+n_{\mathbb{K}}\log T$, and possibly small constants in every other position. With this spirit in this paper we prove the following results, the first one for the zeros in the window $[T-a,T+a]$, the second for the multiplicity of a zero.

Date: July 15, 2014. File name: `paper-CXXIII.tex`.
2010 *Mathematics Subject Classification.* Primary 11R42.

Theorem. *(GRH) One has*

eq:C3 (1.3) $$n_{\mathbb{K}}(T;a) \le \frac{a}{2}\tilde{f}_{\mathbb{K}}\Big(\frac{1}{2}+\frac{a}{4}+iT\Big) \qquad \forall a\in(0,2),\quad T\ge 10+a,$$

and

eq:C4 (1.4) $$n_{\mathbb{K}}(T;0^+) \le \frac{3}{10}(2\sigma-1)\tilde{f}_{\mathbb{K}}(\sigma+iT) \qquad \forall \sigma\in\Big(\frac{1}{2},1\Big),\quad T\ge 10,$$

where

$$\tilde{f}_{\mathbb{K}}(\sigma+iT) := Q+2\Big(\frac{n_{\mathbb{K}}}{1-\sigma}+\frac{\log(\frac{1}{2\sigma-1})}{\pi}+\frac{0.64}{2\sigma-1}+1.37\Big)Q^{2-2\sigma}+\Big(\frac{0.14}{2\sigma-1}-20\Big)n_{\mathbb{K}}$$

and $Q := \log\Delta_{\mathbb{K}}+(\log T+20)n_{\mathbb{K}}+11$.

The main term of the bound (1.3) is $\frac{a}{2}(\log\Delta_{\mathbb{K}}+n_{\mathbb{K}}\log T)$. It improves on the bound (1.1) for $n_{\mathbb{K}}(T;a)$ whenever $a\le 1.364\ldots$. For example we get

$$n_{\mathbb{K}}\Big(T;\frac{1}{2}\Big) \le \frac{1}{4}Q+(1.4n_{\mathbb{K}}+2.2)Q^{3/4}-4n_{\mathbb{K}} \qquad \forall T\ge 10.5$$

and

$$n_{\mathbb{K}}(T;1) \le \frac{1}{2}Q+(4n_{\mathbb{K}}+2.9)\sqrt{Q}-9n_{\mathbb{K}} \qquad \forall T\ge 11.$$

The bound for the multiplicity (1.4) is stronger than what we can deduce from (1.3) in the limit $a\to 0^+$ (to compare the results take $a=(4\sigma-2)$ in (1.3)). Moreover, every $\sigma<0.872$ in (1.4) improves on what one can deduce from (1.1) in the limit $a\to 0^+$; for example for $\sigma=3/4$ we get

$$n_{\mathbb{K}}(T;0^+) \le \frac{3}{20}Q+(1.2n_{\mathbb{K}}+0.9)\sqrt{Q}-2.9n_{\mathbb{K}} \qquad \forall T\ge 10.$$

However, for a better result the form of (1.4) suggests to try with a σ such that $2\sigma-1\to 0$. In fact, a proper choice of σ proves the following claim.

cor:C1 **Corollary 1.1.** *(GRH) Let Q as in the theorem and let $L:=\log Q$. Suppose $T\ge 10$, then*

$$n_{\mathbb{K}}(T;0^+) \le \Big(0.3\log L+0.4+0.2\frac{\log^2 L}{L}+\frac{\log L}{L}(1.9n_{\mathbb{K}}+0.9)\Big)\frac{Q}{\log Q}.$$

Proof. Let $\epsilon := 2\sigma-1$. From (1.4) we get

$$\frac{10}{3Q}n_{\mathbb{K}}(T;0^+) \le \epsilon+\Big(\frac{4\epsilon}{1-\epsilon}n_{\mathbb{K}}+\frac{2}{\pi}\epsilon|\log\epsilon|+2.76\epsilon+1.28\Big)Q^{-\epsilon}+\frac{0.14n_{\mathbb{K}}}{Q}$$

and setting $\epsilon=\frac{\log L}{\log Q}$ (and using $T\ge 10 \implies Q\ge 33 \implies \epsilon\le 0.36$) we get

$$\frac{10}{3Q}n_{\mathbb{K}}(T;0^+) \le \frac{\log L}{\log Q}+\Big(6.3\frac{\log L}{\log Q}n_{\mathbb{K}}+2\frac{\log^2 L}{\pi\log Q}+2.76\frac{\log L}{\log Q}+1.28\Big)\frac{1}{\log Q}.$$

□

As recalled before, we already know that under GRH the multiplicity of $\frac{1}{2}+iT$ is $O\big(\frac{\log T}{\log\log T}\big)$, thus the previous corollary is weaker than the best known result, but only by the presence of an extra $\log L$, i.e. a triple log in T, in the numerator. Moreover, it is uniform in $\mathbb{K}$, and totally explicit.

Due to the presence of the extra factor $\log L$, every explicit bound of the form $n_{\mathbb{K}}(T;0^+) \le c\frac{\log T}{\log\log T}$ based on Corollary 1.1 holds true only for a bounded range; nevertheless, the following result shows that for the Riemann zeta function this range is extremely large, even for a small value of c.

cor:C2 **Corollary 1.2.** *(RH) Let* $\log T \le 10^{70593}$*, then*

$$n_{\mathbb{Q}}(T;0^+) \le \frac{4\log T}{\log\log T}.$$

Proof. By Corollary 1.1 it is sufficient to prove that

eq:C5 (1.5)
$$\Big(\frac{3\log L}{10}+2.8\frac{\log L}{L}+0.2\frac{\log^2 L}{L}+0.4\Big)\frac{Q}{\log Q} \le 4\frac{\log T}{\log\log T}$$

with $Q=\log T+31$ and $L=\log Q$. Taking account of the fact that the zeros ρ of the Riemann zeta function with $|\mathrm{Im}\rho|<10^{10}$ are simple (actually this has been verified up to 10^{12} [3], but we prefer to base our result on a doubly checked computation), we may assume $\log T \ge 23$. In terms of T Inequality (1.5) is difficult, however we can verify that both

$$\frac{3}{10}\log L+2.8\frac{\log L}{L}+0.2\frac{\log^2 L}{L}+0.4 \le 2$$

and

$$\frac{Q}{\log Q} \le 2\frac{\log T}{\log\log T}$$

hold true for $23 \le \log T \le 10^{55}$ (the first one as a function of L, the second one as a function of $\log T$). Thus we can assume $\log T \ge 10^{55}$. Under this hypothesis (1.5) is implied by

$$\frac{3}{10}\log L+2.8\frac{\log L}{L}+0.2\frac{\log^2 L}{L}+0.4 \le 4(1-10^{-10})$$

which holds true for $L \le 162546.6$. □

The theorem is proved in two steps, following an idea which we have introduced in [5]: let $f_{\mathbb{K}}(s) := \sum_\rho \mathrm{Re}\big(\frac{2}{s-\rho}\big)$, where the sum is on the set of nontrivial zeros of $\zeta_{\mathbb{K}}$. First we exploit the fact that the terms appearing in the sum defining $f_{\mathbb{K}}$ are all positive and depend on the zeros of $\zeta_{\mathbb{K}}$ to find a suitable combination of values of $f_{\mathbb{K}}$ providing an upper bound for $n_{\mathbb{K}}(T;a)$ and $n_{\mathbb{K}}(T;0^+)$; then we bound $f_{\mathbb{K}}(s)$ with $\tilde{f}_{\mathbb{K}}(s)$ in the critical strip. Both steps depend on GRH.

It is interesting to remark that to bound $f_{\mathbb{K}}$ we will use a preliminary explicit upper bound for $n_{\mathbb{K}}(T;1)$ which we deduced from a crude version of (1.1). A virtuous circle appears here because the argument could be iterated producing better and better bounds. However, preliminary considerations suggest that the improvement is quite marginal and affects only the secondary constants.

Acknowledgements. A special thank to Alberto Perelli for valuable remarks and comments.

2. Preliminary computations and the upper bound for $n_{\mathbb{K}}(T;a)$

sec:C2

For $\mathrm{Re}(s)>1$ we have

$$-\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s)=\sum_{n=1}^{\infty}\tilde{\Lambda}_{\mathbb{K}}(n)n^{-s}\quad\text{with}\quad\tilde{\Lambda}_{\mathbb{K}}(n)=\begin{cases}\displaystyle\sum_{\mathfrak{p}|p,\,f_{\mathfrak{p}}|k}\log\mathrm{N}\mathfrak{p} & \text{if } n=p^k\\ 0 & \text{otherwise,}\end{cases}$$

where p is a prime number, $\mathfrak{p}$ a prime ideal in $\mathbb{K}$ above p, $\mathrm{N}\mathfrak{p}$ its absolute norm and $f_{\mathfrak{p}}$ its residual degree. The formula for $\tilde{\Lambda}_{\mathbb{K}}$ shows that $\tilde{\Lambda}_{\mathbb{K}}(n)\le n_{\mathbb{K}}\Lambda(n)$ for every integer n, so that

eq:C6

$$\left|\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s)\right|\le -n_{\mathbb{K}}\frac{\zeta'}{\zeta}(\sigma)\qquad\forall\sigma=\mathrm{Re}(s)>1. \tag{2.1}$$

The functional equation for $\zeta_{\mathbb{K}}$ reads

eq:C7

$$\xi_{\mathbb{K}}(1-s)=\xi_{\mathbb{K}}(s) \tag{2.2}$$

where

eq:C8

$$\xi_{\mathbb{K}}(s):=s(s-1)\Delta_{\mathbb{K}}^{s/2}\Gamma_{\mathbb{K}}(s)\zeta_{\mathbb{K}}(s) \tag{2.3}$$

and

eq:C9

$$\Gamma_{\mathbb{K}}(s):=\left[\pi^{-\frac{s+1}{2}}\Gamma\Big(\frac{s+1}{2}\Big)\right]^{r_2}\left[\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\right]^{r_1+r_2}. \tag{2.4}$$

Since $\xi_{\mathbb{K}}(s)$ is an entire function of order 1 and does not vanish at $s=0$, one has

eq:C10

$$\xi_{\mathbb{K}}(s)=e^{A_{\mathbb{K}}+B_{\mathbb{K}}s}\prod_{\rho}\Big(1-\frac{s}{\rho}\Big)e^{s/\rho} \tag{2.5}$$

for some constants $A_{\mathbb{K}}$ and $B_{\mathbb{K}}$, where ρ runs through all the zeros of $\xi_{\mathbb{K}}(s)$, which are precisely those zeros $\rho=\beta+i\gamma$ of $\zeta_{\mathbb{K}}$ for which $0<\beta<1$. We recall that the zeros are symmetric with respect to the real axis, as a consequence of the fact that $\zeta_{\mathbb{K}}(s)$ is real for $s\in\mathbb{R}$.
Differentiating (2.3) and (2.5) logarithmically we obtain the identity

eq:C11

$$\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s)=B_{\mathbb{K}}+\sum_{\rho}\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big)-\frac{1}{2}\log\Delta_{\mathbb{K}}-\Big(\frac{1}{s}+\frac{1}{s-1}\Big)-\frac{\Gamma_{\mathbb{K}}'}{\Gamma_{\mathbb{K}}}(s), \tag{2.6}$$

valid identically in the complex variable s.
Stark [10, Lemma 1] proved that the functional equation (2.2) implies that $B_{\mathbb{K}}=-\sum_{\rho}\mathrm{Re}(\rho^{-1})$, and that once this information is available one can use (2.6) and the definition of the gamma factor in (2.4) to prove that the function $f_{\mathbb{K}}(s):=\sum_{\rho}\mathrm{Re}\big(\frac{2}{s-\rho}\big)$ can be exactly computed via the alternative representation

eq:C12

$$f_{\mathbb{K}}(s)=2\mathrm{Re}\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s)+\log\frac{\Delta_{\mathbb{K}}}{\pi^{n_{\mathbb{K}}}}+\mathrm{Re}\Big(\frac{2}{s}+\frac{2}{s-1}\Big)+(r_1+r_2)\mathrm{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)+r_2\mathrm{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{s+1}{2}\Big). \tag{2.7}$$

The relevance of this function for our problem comes from two facts: it is a sum on zeros each one appearing with the weight $\mathrm{Re}\frac{2}{s-\rho}$ which is *positive* under GRH whenever $\mathrm{Re}(s)>\frac{1}{2}$, and it can be computed via the alternative formula (2.7) which *does not involve* the zeros. For example, assuming GRH we get

$$n_{\mathbb{K}}(T;a)\le\frac{1}{c(\sigma)}\sum_{\rho}\mathrm{Re}\frac{2}{\sigma+iT-\rho}=\frac{1}{c(\sigma)}f_{\mathbb{K}}(\sigma+iT)\qquad T>a$$

with $c(\sigma) := \frac{2\sigma-1}{(\sigma-1/2)^2+a^2}$, which is a lower bound for the weight of the zeros counted by $n_{\mathbb{K}}(T;a)$. By (2.7) the part depending on the discriminant in $c(\sigma)^{-1}f_{\mathbb{K}}(\sigma+iT)$ is simply $c(\sigma)^{-1}\log\Delta_{\mathbb{K}}$, hence to bound the contribution of this parameter to $n_{\mathbb{K}}(T;a)$ we need to choose σ such that $c(\sigma)$ is maximum. This happens when $\sigma=\frac{1}{2}+a$, giving the bound

eq:C13 (2.8)
$$n_{\mathbb{K}}(T;a) \le a f_{\mathbb{K}}\Big(\frac{1}{2}+a+iT\Big) \qquad \forall T>a.$$

Let $\frac{1}{2}+iT$ be a zero for $\zeta_{\mathbb{K}}$, and let ν be its multiplicity. Then $f_{\mathbb{K}}\big(\frac{1}{2}+a+iT\big) \sim 2\nu a^{-1}$ as $a\to 0$. Thus the previous formula overestimates the multiplicity of zeros by a factor two. The following argument improves (2.8) adding greater flexibility to the choice of the weight. Let $g\colon \mathbb{R}\to\mathbb{R}^+$ be any map and $\mathrm{d}\mu$ be a measure in $\mathbb{R}$ such that

eq:C14 (2.9)
$$g(\gamma) \le \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} \qquad \forall\gamma\in\mathbb{R}.$$

Then summing on all zeros and assuming GRH we have

$$\sum_\rho g(\gamma) \le \sum_\rho \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} = \frac{1}{2\sigma-1}\int_{\mathbb{R}}\sum_\rho \frac{2\sigma-1}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}\,\mathrm{d}\mu(t),$$

producing the bound

eq:C15 (2.10)
$$\sum_\rho g(\gamma) \le \frac{1}{2\sigma-1}\int_{\mathbb{R}} f_{\mathbb{K}}(\sigma+it)\,\mathrm{d}\mu(t).$$

Moreover, suppose that μ is symmetric with respect to a point t_0 and such that for some real c the measure $\mathrm{d}\mu+c\delta_{t_0}$ is positive, where δ_{t_0} is the Dirac measure at t_0. Furthermore, let $\tilde{f}_{\mathbb{K}}(\sigma+it)$ be a concave upper bound for $f_{\mathbb{K}}(\sigma+it)$ in the support of μ, then we immediately deduce that

eq:C16 (2.11)
$$\sum_\rho g(\gamma) \le \frac{\mu(\mathbb{R})}{2\sigma-1}\,\tilde{f}_{\mathbb{K}}(\sigma+it_0).$$

The argument has a variational flavor: finding the minimum for $\frac{\mu(\mathbb{R})}{2\sigma-1}$ in the set of (symmetric around t_0 and) positive (outside t_0) measures μ satisfying (2.9).
We apply the argument with $g(\gamma) := \chi_{[T-a,T+a]}(\gamma)$ and $T>a$, so that

$$\sum_\rho g(\gamma) = n_{\mathbb{K}}(T;a).$$

It is interesting to remark that integrating (2.9) in γ we get the lower bound $\frac{\mu(\mathbb{R})}{2\sigma-1} \ge \frac{a}{\pi}$, which happens to be exactly the bound given by the general principle (1.2): this shows that there is no motivation to exclude a priori that the simple upper bound (2.11) may be able to produce the asymptotically correct bound for $n_{\mathbb{K}}(T;a)$. The best result we have been able to prove is not trivial but is only $a/2$.

We have experimented with several possible measures, but actually our best results come from a very simple choice. In fact, we set

eq:C17 (2.12)
$$\mathrm{d}\mu(t) := \sum_{j=-2}^{2} c_j\delta_{T-b_j}(t),$$

the sum of five Dirac's deltas, with $c_{-j}=c_j$ and $b_{-j}=-b_j$ for every j, in order to make $\mathrm{d}\mu$ symmetric around T. With this choice (2.9) becomes

eq:C18 (2.13)
$$\chi_{[-a,a]}(\gamma)\le\sum_{j=-2}^{2}\frac{c_j}{\alpha^2+(\gamma-b_j)^2}\qquad\forall\gamma\in\mathbb{R}$$

where $\alpha:=\sigma-\frac12$; we have also removed the parameter T via the shift $T+\gamma\to\gamma$. We are interested into a combination of parameters producing a small value for

$$\frac{\mu(\mathbb{R})}{2\sigma-1}=\frac{c_0+2c_1+2c_2}{2\alpha}.$$

For the moment we do not have yet determined any set of values for the parameters, however we can make a simple test proving that our strategy has a good chance to produce something interesting. Suppose that $1/2+iT$ is a zero and let ν be its multiplicity, then $f_{\mathbb{K}}(s)\sim 2\mathrm{Re}(\nu(s-1/2-iT)^{-1})$ as s goes to $1/2+iT$. By (2.10) with the measure (2.12) and letting $\sigma\to1/2$ we get

eq:C19 (2.14)
$$n_{\mathbb{K}}(T;0^+)\le\frac{1}{2\sigma-1}\sum_{j=-2}^{2}c_jf_{\mathbb{K}}(\sigma+iT-ib_j)=\nu\sum_{j=-2}^{2}\frac{c_j}{\alpha^2+b_j^2}+R(\sigma),$$

where the remainder $R(\sigma)$ is $O(\sum|b_j|)$. Hence the function to the right hand side of (2.14) is substantially ν if we require that the constants c_j's produce an equality in (2.13) when $\gamma=0$ and the b_j's are small. In this way we improve on what comes from the elementary argument (2.8).

We have six parameters: α, b_1 and b_2 and the three $c_j's$. Equation (2.13) shows an homogeneity in a: once we have found a set of parameters α, b_j and c_j for $a=1$, the parameters $a\alpha$, ab_j and a^2c_j can be used for any a. We thus suppose $a=1$. Moreover, we set $\alpha=\frac14$ and to fix the value of the other parameters we impose the equality in (2.13) for $\gamma=0$ (ensuring the good asymptotic estimate for the multiplicity), $\gamma=\pm\frac35$ and $\gamma=\pm1$: values $\frac14$ for α and $\pm\frac35$ have been chosen by trial and error and produce our best result, but are essentially arbitrary. The equalities form a linear system in the c_j's which may be explicitly solved in terms of b_1 and b_2.
Then we impose two extra conditions: the first one requiring that the contact in $\gamma=\pm\frac35$ has at least double order, the second one requiring that the contact in $\gamma=0$ has at least fourth order. Due to the symmetry, these conditions correspond to the two equations

eq:C20 (2.15)
$$\begin{cases}\displaystyle\sum_{j=-2}^{2}\frac{c_j(\frac35-b_j)}{(\frac{1}{16}+(\frac35-b_j)^2)^2}=0\\ \displaystyle\sum_{j=-2}^{2}\frac{c_j(\frac{1}{16}-3b_j^2)}{(\frac{1}{16}+b_j^2)^3}=0.\end{cases}$$

Observe that the equation

eq:C21 (2.16)
$$\sum_{j=-2}^{2}\frac{c_j}{\frac{1}{16}+(\gamma-b_j)^2}=1$$

(which is an algebraic equation in γ of degree 10) has at least the roots 0 (multiplicity ≥4), $\pm\frac35$ (multiplicity ≥2, each) and ±1 (multiplicity ≥1, each). Hence there are no other roots,

and comparing the two sides of (2.16) as $\gamma \to \infty$ we conclude that (2.13) holds true in $[-1,1]$. Moreover, (2.13) is also true for $|\gamma| > 1$ if the c_j's are not negative.
Equation (2.15) can be seen as an algebraic system in the variables b_1^2 and b_2^2. Using the resultant of the polynomials, we can check that there is a unique solution such that $0 \le b_1 \le b_2$ and it is

$$b_1 = 0.355\ldots, \quad b_2 = 0.875\ldots$$

giving

$$c_1 = 0.0200\ldots, \quad c_2 = 0.0491\ldots, \quad c_3 = 0.0651\ldots$$

For this combination we get

$$\frac{\mu(\mathbb{R})}{2\sigma-1} \le \frac{1}{2}$$

that, coming back to generic a, via (2.11) yields

eq:C22

$$n_{\mathbb{K}}(T;a) \le \frac{a}{2}\tilde{f}_{\mathbb{K}}\Big(\frac{1}{2}+\frac{a}{4}+iT\Big) \qquad \forall T > a \tag{2.17}$$

where for the moment $\tilde{f}_{\mathbb{K}}$ denotes any concave upper bound of $f_{\mathbb{K}}$.
Our second result (1.4) is proved with a slightly different argument; it needs the formulas for the c_j's to be made explicit in terms of the other parameters, thus we further simplify our definition of the measure imposing $c_2 = 0$ in (2.12), i.e. setting

$$\mathrm{d}\mu(t) := c_1\delta_{T+b}(t)+c_0\delta_T(t)+c_1\delta_{T-b}(t).$$

We fix the c_j's in such a way as to get an equality in (2.13) for $\gamma = 0$ and $\gamma = \pm a$; this happens for

$$\begin{cases} c_0 = \dfrac{-\alpha^6+(3b^2-2a^2)\alpha^4+(3b^2-a^2)a^2\alpha^2}{b^2(5\alpha^2+a^2+b^2)} \\ c_1 = \dfrac{\alpha^6+(2a^2+3b^2)\alpha^4+(a^4+3b^4)\alpha^2+(a^2-b^2)^2b^2}{2b^2(5\alpha^2+a^2+b^2)} \end{cases}$$

which produces

$$\frac{c_0+2c_1}{2\alpha} = \frac{6\alpha^4+3(a^2+b^2)\alpha^2+(a^2-b^2)^2}{2(5\alpha^2+a^2+b^2)\alpha}.$$

Formulas show that c_1 is always positive, but c_0 may be negative for some values of the parameters. However, for these c_j's the function appearing to the right-hand side of (2.13) can be written as sum of squares and thus is always positive. As a consequence only the range $[-a,a]$ has to be considered for (2.13). Previously we have taken advantage of the homogeneity of the problem in the a parameter, but for the present application it is useful to workout a formula allowing the limit $a \to 0$ without contemporarily sending α to 0. As a consequence we set $b^2 = \frac{a^2}{2}$, giving

$$\begin{cases} c_0 = -2\dfrac{(2\alpha^2-a^2)(\alpha^2+a^2)\alpha^2}{a^2(10\alpha^2+3a^2)} \\ c_1 = \dfrac{(2\alpha^2+a^2)(4\alpha^4+12a^2\alpha^2+a^4)}{4a^2(10\alpha^2+3a^2)} \end{cases}$$

which produces

$$\frac{c_0+2c_1}{2\alpha}=\frac{24\alpha^4+18a^2\alpha^2+a^4}{4(10\alpha^2+3a^2)\alpha}$$

but we do not assume any proportionality between α and a. Once again the choice $b^2=\frac{a^2}{2}$ is the result of a trial and error procedure. Inequality (2.13) holds as an equality

$$\frac{c_1}{\alpha^2+(\gamma-\frac{a}{\sqrt{2}})^2}+\frac{c_0}{\alpha^2+\gamma^2}+\frac{c_1}{\alpha^2+(\gamma+\frac{a}{\sqrt{2}})^2}=1$$

in six complex points (multiplicity included). The definitions of c_j's gives the roots $\gamma=0$ (multiplicity ≥ 2) and $\gamma=\pm a$; the remaining solutions solve

$$\gamma^2=\frac{a^4-36\alpha^4}{20\alpha^2+6a^2}.$$

Inequality (2.13) is satisfied in the range $\gamma\in[-a,a]$ if and only these two extra solutions are either 0 or non real. This is what happens as long as $a^2<6\alpha^2$, and since $n_{\mathbb{K}}(T;0^+)\leq n_{\mathbb{K}}(T;a)$ for every a, we deduce that

$$n_{\mathbb{K}}(T;0^+)\leq\frac{24\alpha^4+18a^2\alpha^2+a^4}{4(10\alpha^2+3a^2)\alpha}\tilde{f}_{\mathbb{K}}(\sigma+iT),$$

where again $\tilde{f}_{\mathbb{K}}$ denotes any concave upper bound of $f_{\mathbb{K}}$. Setting $a\to 0$ to the right-hand side we conclude that

eq:C23

$$n_{\mathbb{K}}(T;0^+)\leq\frac{3}{10}(2\sigma-1)\tilde{f}_{\mathbb{K}}(\sigma+iT)\qquad\forall T>0,\qquad\forall\sigma>\frac{1}{2}. \tag{2.18}$$

In the next section we will prove that $f_{\mathbb{K}}(\sigma+it)\leq\tilde{f}_{\mathbb{K}}(\sigma+it)$ where $\tilde{f}_{\mathbb{K}}$ is the function given in the theorem. With (2.11) this suffices to prove (1.3) and (1.4) from (2.17) and (2.18) respectively, because $\tilde{f}_{\mathbb{K}}(\sigma+it)$ is a concave map in the t variable.

3. Bounds in the critical strip

sec:C3

The following two lemmas collect some elementary inequalities involving the gamma function which we will need later.

lem:C1

Lemma 3.1. *Let $s=\sigma+it$ with $\sigma\geq 0$ and $|t|\geq\sigma+2$. Then*

$$\mathrm{Re}\Big(\frac{\Gamma'}{\Gamma}(s)\Big)\leq\log|s-1/2|.$$

Proof. Using the explicit formula [1, Th. 1.4.2] for $\log\Gamma(s)$ coming from the Euler–Maclaurin summation formula one gets:

$$\begin{aligned}\frac{\Gamma'}{\Gamma}(s)-\log\Big(s-\frac{1}{2}\Big)&=-\log\Big(1-\frac{1}{2s}\Big)-\frac{1}{2s}-\frac{1}{12s^2}+\frac{1}{120s^4}+\int_0^{+\infty}\frac{B_4(\{u\})}{(s+u)^5}\,\mathrm{d}u\\&=\frac{1}{24s^2}+\sum_{n=3}^{+\infty}\frac{1}{n2^ns^n}+\frac{1}{120s^4}+\int_0^{+\infty}\frac{B_4(\{u\})}{(s+u)^5}\,\mathrm{d}u\end{aligned}$$

where $B_4(x)=x^4-2x^3+x^2-\frac{1}{30}$. Thus, if t is positive we get

$$\mathrm{Re}\Big(\frac{\Gamma'}{\Gamma}(s)-\log\Big(s-\frac{1}{2}\Big)\Big)\leq\frac{1}{24}\mathrm{Re}\Big(\frac{1}{s^2}\Big)+\sum_{n=3}^{+\infty}\frac{1}{n2^n|s|^n}+\frac{1}{120|s|^4}+\int_0^{+\infty}\frac{|B_4(\{u\})|}{|s+u|^5}\,\mathrm{d}u$$

$$\leq \frac{1}{24}\mathrm{Re}\Big(\frac{1}{s^2}\Big)+\frac{1}{12|s|^2(2|s|-1)}+\frac{1}{120|s|^4}+\frac{1}{30}\int_0^{+\infty}\frac{\mathrm{d}u}{(u^2+t^2)^{5/2}}$$
$$\leq \frac{1}{24}\mathrm{Re}\Big(\frac{1}{s^2}\Big)+\frac{1}{12|s|^2(2t-1)}+\frac{1}{120|s|^4}+\frac{1}{45t^4}.$$

This is a rational function of σ and t, and with elementary arguments one proves that it is negative for $\sigma \geq 0$ and $t \geq \sigma+2$. □

lem:C2 eq:C24 **Lemma 3.2.** *Let $u \in [-3/4, -1/4]$, then*

eq:C24a (3.1a)
$$\int_{\mathbb{R}} |\Gamma(u+iy)|\, dy \leq 4.73,$$

eq:C24b (3.1b)
$$\int_{\mathbb{R}} |\Gamma(u+i(t-y))| \log(1+|y|)\, \mathrm{d}y \leq 4.73 \log(1+|t|) \qquad \forall t.$$

Furthermore suppose $|t| \geq 10$, then

eq:C24c (3.1c)
$$\int_{\mathbb{R}} \frac{|\Gamma(u+i(t-y))|}{|1+4iy|}\, dy \leq 0.171$$

eq:C24d (3.1d)
$$\int_{\mathbb{R}} \frac{|\Gamma(u+i(t-y))|}{1+\alpha y^2}\, dy \leq \begin{cases} 0.013 & \alpha = 1 \\ 0.007 & \alpha = 2. \end{cases}$$

Proof. The map $u \to \int_{\mathbb{R}} |\Gamma(u+iv)|\, \mathrm{d}v$ is log-convex for $u \in (-1, 0)$ as a consequence of a general inequality of Hardy, Ingham and Pólya (see [8, Ch. 11, Prop. 4]), and the uniform exponential decay of $\Gamma(u+iv)$ for $v \to \infty$. Hence,

$$\int_{\mathbb{R}} |\Gamma(u+iy)|\, \mathrm{d}y \leq \max\Big\{ \int_{\mathbb{R}} \Big|\Gamma\Big(-\frac{3}{4}+iy\Big)\Big|\, \mathrm{d}y, \int_{\mathbb{R}} \Big|\Gamma\Big(-\frac{1}{4}+iy\Big)\Big|\, \mathrm{d}y \Big\}$$

for any $u \in [-3/4, -1/4]$. The last two integrals are bounded respectively by 4.43 and 4.73 (any effective version of the Stirling bound [1, Cor. 1.4.4] may be used to prove that the contribution of the range $y \in \mathbb{R}\backslash[-10, 10]$ to the integral is smaller than 10^{-6}, and then, using the monotonicity of $y \mapsto |\Gamma(u+iy)|$, a Riemann sum with 10000 points produces the result). This proves (3.1a).

Without loss of generality we can assume $t > 0$. By (3.1a), in order to prove (3.1b) it is sufficient to show that

eq:C25 (3.2)
$$\int_{\mathbb{R}} |\Gamma(u+i(t-y))| \log\Big(\frac{1+|y|}{1+t}\Big)\, \mathrm{d}y$$

is negative. Let

$$F_1(u, v) := -\int_v^{+\infty} |\Gamma(u+iw)|\, \mathrm{d}w$$
$$F_2(u, v) := -\int_v^{+\infty} F_1(u, w)\, \mathrm{d}w = \int_v^{+\infty} (w-v)|\Gamma(u+iw)|\, \mathrm{d}w,$$

so that $\partial_v F_1(u, v) = |\Gamma(u+iv)|$ and $\partial_v F_2(u, v) = F_1(u, v)$. We split (3.2) into three ranges $(-\infty, 0]\cup[0, t]\cup[t, +\infty)$, getting

$$\int_{\mathbb{R}} |\Gamma(u+i(t-y))| \log\Big(\frac{1+|y|}{1+t}\Big)\, \mathrm{d}y = \int_{-\infty}^{0} |\Gamma(u+i(t-y))| \log\Big(\frac{1-y}{1+t}\Big)\, \mathrm{d}y$$

$$+\int_0^t |\Gamma(u+i(t-y))| \log\left(\frac{1+y}{1+t}\right) \mathrm{d}y + \int_t^{+\infty} |\Gamma(u+i(y-t))| \log\left(\frac{1+y}{1+t}\right) \mathrm{d}y$$

where in the last term we have used the equality $|\Gamma(u-iw)| = |\Gamma(u+iw)|$ to ensure the positivity of the imaginary part of the argument of the gamma. Then an integration by parts shows that it is

$$= -\int_{-\infty}^0 \frac{F_1(u,t-y)}{1-y}\,\mathrm{d}y + \int_0^t \frac{F_1(u,t-y)}{1+y}\,\mathrm{d}y - \int_t^{+\infty} \frac{F_1(u,y-t)}{1+y}\,\mathrm{d}y.$$

A second integration by parts produces

$$= 2F_2(u,t) - \int_{-\infty}^0 \frac{F_2(u,t-y)}{(1-y)^2}\,\mathrm{d}y - \int_0^t \frac{F_2(u,t-y)}{(1+y)^2}\,\mathrm{d}y - \int_t^{+\infty} \frac{F_2(u,y-t)}{(1+y)^2}\,\mathrm{d}y.$$

Function F_2 being positive, this is

$$\le 2F_2(u,t) - \frac{1}{(1+t)^2}\int_0^t F_2(u,t-y)\,\mathrm{d}y = 2F_2(u,t) - \frac{1}{(1+t)^2}\int_0^t F_2(u,y)\,\mathrm{d}y$$

which is negative if and only if

eq:C26

$$2(1+t)^2 F_2(u,t) + \int_t^{+\infty} F_2(u,y)\,\mathrm{d}y \le \int_0^{+\infty} F_2(u,y)\,\mathrm{d}y. \tag{3.3}$$

This inequality holds true when t is large enough because the left-hand side decreases to 0 as a function of t. In order to prove that this happens already for $t \ge 10$ we use an effective version of the Stirling bound (see [1, Cor. 1.4.4]) on the gamma, giving, when $u < 0$ and $v > 0$,

$$|\Gamma(u+iv)| = \sqrt{2\pi}|u+iv|^{u-1/2} e^{-\frac{\pi}{2}v} e^{-u+v\,\mathrm{atan}(u/v)} e^{R(u,v)}$$

with $|R(u,v)| \le -\frac{1}{8v}\,\mathrm{atan}(v/u)$. Thus, if furthermore $u \in [-3/4, -1/4]$, we get

$$|F_2(u+iv)| \le \frac{4}{\pi^2}\sqrt{2\pi} e^{3/4} v^{-3/4} e^{\frac{\pi}{16v}} e^{-\frac{\pi}{2}v}$$

and

$$\int_t^{+\infty} F_2(u+iw)\,\mathrm{d}w \le \frac{8}{\pi^3}\sqrt{2\pi} e^{3/4} t^{-3/4} e^{\frac{\pi}{16t}} e^{-\frac{\pi}{2}t} \qquad \forall t > 0.$$

These bounds show that the left-hand side in (3.3) is smaller than 10^{-4} for every $t \ge 10$. On the contrary,

$$\int_0^{+\infty} F_2(u,y)\,\mathrm{d}y = \frac{1}{2}\int_0^{+\infty} w^2 |\Gamma(u+iw)|\,\mathrm{d}w \ge \frac{1}{2}\int_0^1 w^2 |\Gamma(u+iw)|\,\mathrm{d}w$$
$$\ge \frac{1}{6} \min_{\substack{u\in[-3/4,-1/4]\\ w\in\times[0,1]}} |\Gamma(u+iw)|.$$

The maximum modulus principle for holomorphic functions (applied to $1/\Gamma(z)$) shows that the minimum is reached at the boundary of the region $[-3/4,-1/4]\times[0,1]$, and it is easy to verify that here $|\Gamma(z)|$ is always larger than 0.4.

For (3.1c) we note that $|\Gamma(u+iv)| \le 5.3e^{-\pi|v|/2}$ for every $u \in [-3/4,-1/4]$ and every v, thus it is sufficient to bound

$$F(t) := \int_{\mathbb{R}} \frac{e^{-\frac{\pi}{2}|t-y|}}{|1+4iy|}\,\mathrm{d}y.$$

It is the unique bounded solution of the differential equation $F''(t)-\frac{\pi^2}{4}F(t) = -\frac{\pi}{|1+4it|}$. A numerical check shows that $F(10) = 0.032\ldots$ is larger than $\frac{4/\pi}{|1+40i|} = 0.031\ldots$. Suppose that there exists $t_0 > 10$ with $F(t_0) > F(10)$. Then there exists also a point $t_1 > t_0$ with $F(t_1) = \max_{t\in[10,+\infty)} F(t)$, because $F(+\infty) = 0$. Then $F''(t_1) \le 0$, because F is a C^2 map in $(10,+\infty)$. Thus from the differential equation we get

$$\frac{\pi^2}{4}F(t_1) \le \frac{\pi}{|1+4it_1|} < \frac{\pi}{|1+40i|} < \frac{\pi^2}{4}F(10)$$

which violates the definition of t_1. This proves that $F(10) = \max_{t\in[10,+\infty)} F(t)$, so that

$$\int_{\mathbb{R}} \frac{|\Gamma(u+i(t-y))|}{|1+4iy|}\,\mathrm{d}y \le 5.3F(t) \le 5.3F(10) \le 0.171.$$

The same argument may be applied to prove (3.1d), since $F_\alpha(t) := \int_{\mathbb{R}} \frac{e^{-\frac{\pi}{2}|t-y|}}{1+\alpha y^2}\,\mathrm{d}y$ is the unique solution of the differential equation $F''_\alpha(t)-\frac{\pi^2}{4}F_\alpha(t) = -\frac{\pi}{1+\alpha y^2}$ which is bounded for $t \to \pm\infty$, and $F_1(10) = 0.0129\ldots > \frac{4/\pi}{1+10^2} = 0.0126\ldots$, and $F_2(10) = 0.0065\ldots > \frac{4/\pi}{1+2\cdot 10^2} = 0.0063\ldots$. □

Let

eq:C27 (3.4)
$$N_{\mathbb{K}}(T) =: \frac{T}{\pi}\log\Big(\Big(\frac{T}{2\pi e}\Big)^{n_{\mathbb{K}}}\Delta_{\mathbb{K}}\Big)+R_{\mathbb{K}}(T).$$

Trudgian [12, Th. 2] proved unconditionally that for every $T \ge 1$ one has

eq:C28 (3.5)
$$|R_{\mathbb{K}}(T)| \le \tilde{R}_{\mathbb{K}}(T) := d_1W_{\mathbb{K}}(T)+d_2n_{\mathbb{K}}+d_3$$

with

$$W_{\mathbb{K}}(T) := \log\Delta_{\mathbb{K}}+n_{\mathbb{K}}\log\Big(\frac{T}{2\pi}\Big)$$

and $d_1 = 0.317$, $d_2 = 6.333+0.317\log(2\pi) \le 6.9157$ and $d_3 = 3.482$. We use this result first to bound $n_{\mathbb{K}}(t;1)$, and later to bound certain finite sums over zeros (see Lemma (3.4) below).

lemc63 eq:C29 **Lemma 3.3.** *For $t \in \mathbb{R}$,*

eq:C29a (3.6a)
$$n_{\mathbb{K}}(t;1) \le 0.636W_{\mathbb{K}}(t)+6.92n_{\mathbb{K}}+3.49 \qquad \text{if } |t| > 1,$$

eq:C29b (3.6b)
$$n_{\mathbb{K}}(t;1) \le 0.954\log\Delta_{\mathbb{K}}+5.19n_{\mathbb{K}}+3.49 \qquad \text{if } |t| \le 1.$$

Proof. The symmetry of roots allows us to assume $t \ge 0$. For $t \ge 2$, using (3.5):

$$\begin{aligned} n_{\mathbb{K}}(t;1) &= \frac{1}{2}(N_{\mathbb{K}}(t+1^+)-N_{\mathbb{K}}(t-1^-)) \\ &\le \frac{1}{2\pi}\Big((t+1)\log\Big(\Big(\frac{t+1}{2\pi e}\Big)^{n_{\mathbb{K}}}\Delta_{\mathbb{K}}\Big)-(t-1)\log\Big(\Big(\frac{t-1}{2\pi e}\Big)^{n_{\mathbb{K}}}\Delta_{\mathbb{K}}\Big)\Big) \\ &\quad+\frac{d_1}{2}(W_{\mathbb{K}}(t+1)+W_{\mathbb{K}}(t-1))+d_2n_{\mathbb{K}}+d_3. \end{aligned}$$

Since $(t+1)\log(t+1)-(t-1)\log(t-1) \le 2(1+\log t)$ and $W_{\mathbb{K}}(t)$ is a concave map, we get

$$\begin{aligned}&\le \frac{1}{2\pi}(2n_{\mathbb{K}}(1+\log t)+2\log\Delta_{\mathbb{K}}-2n_{\mathbb{K}}\log(2\pi e))+d_1W_{\mathbb{K}}(t)+d_2n_{\mathbb{K}}+d_3\\&=\Big(\frac{1}{\pi}+d_1\Big)W_{\mathbb{K}}(t)+d_2n_{\mathbb{K}}+d_3.\end{aligned}$$

Introducing the values for d_j's we get (3.6a) for $t \ge 2$.
For $1 < t \le 2$, we have

$$\begin{aligned}n_{\mathbb{K}}(t;1) &\le \frac{1}{2}N_{\mathbb{K}}(t+1^+) \le \frac{1}{2}N_{\mathbb{K}}(3)\\&\le \Big(\frac{3}{2\pi}+\frac{d_1}{2}\Big)\log\Delta_{\mathbb{K}}+\Big(\frac{3}{2\pi}\log\Big(\frac{3}{2\pi e}\Big)+\frac{d_1}{2}\log\Big(\frac{3}{2\pi}\Big)+\frac{d_2}{2}\Big)n_{\mathbb{K}}+\frac{d_3}{2}.\end{aligned}$$

This bound is a bit smaller than what we get extrapolating to $t \in (1,2]$ the formula we have found for $t > 2$. Not needing its full strength, we estimate $n_{\mathbb{K}}(t;1)$ for $t \in (1,2]$ with that bound, thus proving (3.6a).
For $0 \le t \le 1$, we have

$$\begin{aligned}n_{\mathbb{K}}(t;1) &\le N_{\mathbb{K}}(t+1^+) \le N_{\mathbb{K}}(2)\\&\le \Big(\frac{2}{\pi}+d_1\Big)\log\Delta_{\mathbb{K}}+\Big(\frac{2}{\pi}\log\Big(\frac{1}{\pi e}\Big)+d_1\log\Big(\frac{1}{\pi}\Big)+d_2\Big)n_{\mathbb{K}}+d_3\end{aligned}$$

giving (3.6b) when the values for d_j's are introduced. □

lem:C4 **Lemma 3.4.** *Let $c > 0$ and $t \in \mathbb{R}$, with $|t| > c+1$. Let $u \in \mathbb{R}$, $u \ne 0$. Then*

$$\sum_{|\gamma-t|\le c}\frac{1}{|u+i(\gamma-t)|} \le \Big(\frac{\operatorname{asinh}(c/u)}{\pi}+\frac{d_1}{|u|}\Big)W_{\mathbb{K}}(|t|)+\frac{d_2n_{\mathbb{K}}+d_3}{|u|}.$$

Proof. Without loss of generality we can assume u and $t > 0$. We write the sum as an integral in the density of zeros:

$$\sum_{|\gamma-t|\le c}\frac{1}{|u+i(\gamma-t)|} = \int_{t-c^-}^{t+c^+}\frac{\mathrm{d}(N_{\mathbb{K}}(\gamma))}{2|u+i(\gamma-t)|},$$

where the factor $\frac{1}{2}$ appears because only zeros with positive imaginary part matter, because $t-c > 1$. By (3.4) this is

$$= \int_{-c}^{c}\frac{W_{\mathbb{K}}(\gamma+t)}{2\pi|u+i\gamma|}\,\mathrm{d}\gamma+\int_{-c^-}^{c^+}\frac{\mathrm{d}(R_{\mathbb{K}}(\gamma+t))}{2|u+i\gamma|}.$$

$W_{\mathbb{K}}$ is a concave map, thus

$$\int_{-c}^{c}\frac{W_{\mathbb{K}}(\gamma+t)}{2\pi|u+i\gamma|}\,\mathrm{d}\gamma \le \int_{-c}^{c}\frac{\mathrm{d}\gamma}{2\pi|u+i\gamma|}W_{\mathbb{K}}(t) = \frac{\operatorname{asinh}(c/u)}{\pi}W_{\mathbb{K}}(t).$$

Moreover, integrating by parts and using the upper bound $|R_{\mathbb{K}}| \le \tilde{R}_{\mathbb{K}}$ in (3.5) we get

$$\Big|\int_{-c^-}^{c^+}\frac{\mathrm{d}(R_{\mathbb{K}}(\gamma+t))}{2|u+i\gamma|}\Big| \le \frac{\tilde{R}_{\mathbb{K}}(t+c)+\tilde{R}_{\mathbb{K}}(t-c)}{2(u+c^2)^{1/2}}+\frac{1}{2}\int_{-c}^{c}\tilde{R}_{\mathbb{K}}(\gamma+t)\frac{|\gamma|\,\mathrm{d}\gamma}{(u^2+\gamma^2)^{3/2}}$$

and since $\tilde{R}_{\mathbb{K}}$ is also a concave map for positive arguments, we get

$$\leq \frac{\tilde{R}_{\mathbb{K}}(t)}{(u+c^2)^{1/2}}+\frac{\tilde{R}_{\mathbb{K}}(t)}{2}\int_{-c}^{c}\frac{|\gamma|\,\mathrm{d}\gamma}{(u^2+\gamma^2)^{3/2}}=\frac{\tilde{R}_{\mathbb{K}}(t)}{u}.$$

□

lem:C5 **Lemma 3.5.** *For every real t one has:*

$$\Big|\frac{\Gamma'_{\mathbb{K}}}{\Gamma_{\mathbb{K}}}\Big(\frac{1}{4}+it\Big)-\frac{\Gamma'_{\mathbb{K}}}{\Gamma_{\mathbb{K}}}(2+it)\Big|\leq\frac{10n_{\mathbb{K}}}{|1+4it|}.$$

Proof. From the definition of $\Gamma_{\mathbb{K}}$ in (2.4) we have

$$\frac{\Gamma'_{\mathbb{K}}}{\Gamma_{\mathbb{K}}}\Big(\frac{1}{4}+it\Big)-\frac{\Gamma'_{\mathbb{K}}}{\Gamma_{\mathbb{K}}}(2+it)=r_2\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{5}{8}+i\frac{t}{2}\Big)-\frac{\Gamma'}{\Gamma}\Big(\frac{3}{2}+i\frac{t}{2}\Big)\Big)$$
$$+(r_1+r_2)\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{1}{8}+i\frac{t}{2}\Big)-\frac{\Gamma'}{\Gamma}\Big(1+i\frac{t}{2}\Big)\Big).$$

From the functional equation $s\Gamma(s)=\Gamma(s+1)$ we get

eq:C30 (3.7)
$$\frac{\Gamma'}{\Gamma}\Big(\frac{1}{8}+i\frac{t}{2}\Big)-\frac{\Gamma'}{\Gamma}\Big(1+i\frac{t}{2}\Big)=-\frac{8}{1+4it}+\frac{\Gamma'}{\Gamma}\Big(\frac{9}{8}+i\frac{t}{2}\Big)-\frac{\Gamma'}{\Gamma}\Big(1+i\frac{t}{2}\Big),$$

now we proceed as in Lemma 3.1: from the Euler–Maclaurin summation formula for $\log\Gamma(s)$ the difference of the logarithmic derivatives is at most

$$\Big|\log\Big(\frac{\frac{9}{8}+i\frac{t}{2}}{1+i\frac{t}{2}}\Big)\Big|+\frac{1}{2}\frac{1/8}{|\frac{9}{8}+i\frac{t}{2}|\cdot|1+i\frac{t}{2}|}+\frac{1}{12}\Big|\frac{1}{(\frac{9}{8}+i\frac{t}{2})^2}-\frac{1}{(1+i\frac{t}{2})^2}\Big|+\frac{1}{3}\int_0^{+\infty}\frac{\mathrm{d}u}{|u+1+it/2|^3}.$$

The logarithm is lower than $-\log\big(1-\frac{1/8}{|1+i\frac{t}{2}|}\big)$, and the integral is $4/(4+t^2+2\sqrt{4+t^2})$. Using these bounds on proves that (3.7) is bounded by $10/|1+4it|$. The same argument applied to the other difference of gammas completes the proof. □

In order to prove that $f_{\mathbb{K}}(\sigma+it)\leq\tilde{f}_{\mathbb{K}}(\sigma+it)$ we need an upper for $-\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s)$ in the critical strip, and for this purpose we follow the argument used in [11, Th. 14.4] for the Riemann zeta function. In our setting, however, the argument will be considerably complicated by the need of good explicit constants. Let $\sigma=\mathrm{Re}(s)\in(\frac{1}{2},1)$ and let δ be a parameter in $(0,1)$. Proceeding as for [11, Th. 14.4 p. 340] we get

$$-\sum_{n=1}^{\infty}\frac{\tilde{\Lambda}_{\mathbb{K}}(n)}{n^s}e^{-\delta n}=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(z)\Gamma(z-s)\delta^{s-z}\,\mathrm{d}z.$$

Moving the integration line to $\mathrm{Re}(z)=\frac{1}{4}$ we get the equality

$$-\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s)=\sum_{n=1}^{\infty}\frac{\tilde{\Lambda}_{\mathbb{K}}(n)}{n^s}e^{-\delta n}+\sum_{\rho}\Gamma(\rho-s)\delta^{s-\rho}-\Gamma(1-s)\delta^{s-1}+\frac{1}{2\pi i}\int_{1/4-i\infty}^{1/4+i\infty}\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(z)\Gamma(z-s)\delta^{s-z}\,\mathrm{d}z$$

eq:C31 (3.8) $\quad:=I+II+III+IV;$

here I is the value of the original integral, II comes from the nontrivial zeros, III from the pole of $\zeta_{\mathbb{K}}$ at $z=1$, and $-\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s)$ from the pole of gamma at $z=s$: the Cauchy theorem is applicable here since $-\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s)$ grows polynomially along the vertical lines, while gamma decays

exponentially. The following lemmas provide bounds for I–IV and will be combined into a suitable bound for $\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s)$ in Lemma 3.12.

lem:C6 **Lemma 3.6** (Bound of I)**.** *(RH) Let $\sigma \in (\frac{1}{2}, 1)$ and $\delta > 0$. Then*

$$\sum_{n=1}^{\infty} \frac{\tilde{\Lambda}_{\mathbb{K}}(n)}{n^{\sigma}}\, e^{-\delta n} \le n_{\mathbb{K}} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}}\, e^{-\delta n} \le \Big(\frac{\delta^{\sigma-1}}{1-\sigma} + \frac{0.07}{2\sigma-1} + 4.2\Big) n_{\mathbb{K}}.$$

Proof. The first inequality is an immediate consequence of the inequality $\tilde{\Lambda}_{\mathbb{K}}(n) \le n_{\mathbb{K}}\Lambda(n)$. For the second one, let $\psi^{(1)}(x) := \int_0^x \psi(u)\,\mathrm{d}u = \sum_{n\le x} \Lambda(n)(x-n)$. Then

eq:C32 (3.9)
$$\Big|\psi^{(1)}(x) - \frac{x^2}{2}\Big| \le 0.0462x^{3/2} + 1.838x + 2 \qquad \forall x \ge 1,$$

(see [4, Th. 1.1]; there the claim is stated for $x \ge 3$, but it actually holds true for $x \ge 1$). Thus, by partial summation we get

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}}\, e^{-\delta n} = \int_{2^-}^{+\infty} \psi^{(1)}(x) \Big[\frac{e^{-\delta x}}{x^{\sigma}}\Big]''\, \mathrm{d}x$$

which we write as

$$\begin{aligned}
&= \int_{2}^{+\infty} \frac{x^2}{2} \Big[\frac{e^{-\delta x}}{x^{\sigma}}\Big]''\, \mathrm{d}x + \int_{2^-}^{+\infty} \Big[\psi^{(1)}(x) - \frac{x^2}{2}\Big] \Big[\frac{e^{-\delta x}}{x^{\sigma}}\Big]''\, \mathrm{d}x \\
&= (2+\sigma+2\delta)\frac{e^{-2\delta}}{2^{\sigma}} + \delta^{\sigma-1} \int_{2\delta}^{+\infty} \frac{e^{-u}}{u^{\sigma}}\, \mathrm{d}u + \int_{2^-}^{+\infty} \Big[\psi^{(1)}(x) - \frac{x^2}{2}\Big] \Big[\frac{e^{-\delta x}}{x^{\sigma}}\Big]''\, \mathrm{d}x \\
&\le (2+\sigma+2\delta)\frac{e^{-2\delta}}{2^{\sigma}} + \delta^{\sigma-1}\Gamma(1-\sigma) + \int_{2^-}^{+\infty} \Big[\psi^{(1)}(x) - \frac{x^2}{2}\Big] \Big[\frac{e^{-\delta x}}{x^{\sigma}}\Big]''\, \mathrm{d}x.
\end{aligned}$$

The function $e^{-\delta x}x^{-\sigma}$ is completely monotone, thus

$$\begin{aligned}
\int_{2}^{+\infty} x^{3/2} \Big|\Big[\frac{e^{-\delta x}}{x^{\sigma}}\Big]''\Big|\, \mathrm{d}x &= \int_{2}^{+\infty} x^{3/2} \Big[\frac{e^{-\delta x}}{x^{\sigma}}\Big]''\, \mathrm{d}x \\
&= \sqrt{2}\Big(\frac{3}{2} + \sigma + 2\delta\Big)\frac{e^{-2\delta}}{2^{\sigma}} + \frac{3}{4}\int_{2}^{+\infty} e^{-\delta x} x^{-1/2-\sigma}\, \mathrm{d}x.
\end{aligned}$$

The last integral is at most $\min(e^{-2\delta}\delta^{-1}, e^{-2\delta}(\sigma-1/2)^{-1})$, but later we will choose σ and δ such that the minimum comes from the term in σ, thus we write

$$\le \sqrt{2}\Big(\frac{3}{2} + \sigma + 2\delta\Big)\frac{e^{-2\delta}}{2^{\sigma}} + \frac{3}{2}\frac{e^{-2\delta}}{2\sigma-1}.$$

Moreover,

$$\begin{aligned}
\int_{2}^{+\infty} x \Big|\Big[\frac{e^{-\delta x}}{x^{\sigma}}\Big]''\Big|\, \mathrm{d}x &= \int_{2}^{+\infty} x \Big[\frac{e^{-\delta x}}{x^{\sigma}}\Big]''\, \mathrm{d}x = (1+\sigma+2\delta)\frac{e^{-2\delta}}{2^{\sigma}}, \\
\int_{2}^{+\infty} \Big|\Big[\frac{e^{-\delta x}}{x^{\sigma}}\Big]''\Big|\, \mathrm{d}x &= \int_{2}^{+\infty} \Big[\frac{e^{-\delta x}}{x^{\sigma}}\Big]''\, \mathrm{d}x = (\sigma+2\delta)\frac{e^{-2\delta}}{2^{\sigma+1}},
\end{aligned}$$

hence, recalling (3.9) and using the inequality $\Gamma(1-\sigma) \le (1-\sigma)^{-1}$ for $\sigma \in (0,1)$, we get

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} e^{-\delta n} \le (2+\sigma+2\delta)\frac{e^{-2\delta}}{2^\sigma}+\frac{\delta^{\sigma-1}}{1-\sigma}+0.0462\Big(\sqrt{2}\Big(\frac{3}{2}+\sigma+2\delta\Big)\frac{e^{-2\delta}}{2^\sigma}+\frac{3}{2}\frac{e^{-2\delta}}{2\sigma-1}\Big)$$
$$+1.838(1+\sigma+2\delta)\frac{e^{-2\delta}}{2^\sigma}+(\sigma+2\delta)\frac{e^{-2\delta}}{2^\sigma} \le \frac{\delta^{\sigma-1}}{1-\sigma}+\frac{0.07}{2\sigma-1}+4.2.$$

□

lem:C7 **Lemma 3.7** (Bound of *II*). *(GRH) Let $\sigma \in (\frac{1}{2},1)$, $|t| \ge 10$ and $\delta \in (0,1)$. Then*

$$\Big|\sum_\rho \Gamma(s-\rho)\delta^{s-\rho}\Big| \le \delta^{\sigma-\frac{1}{2}}\Big[\Big(\frac{\log(\frac{1}{2\sigma-1})}{\pi}+\frac{0.64}{2\sigma-1}+0.82\Big)W_{\mathbb{K}}(t)+\Big(\frac{13.9}{2\sigma-1}+1.6\Big)n_{\mathbb{K}}+\frac{6.9}{2\sigma-1}+0.8\Big].$$

Proof. We are assuming GRH, thus $\big|\sum_\rho \Gamma(s-\rho)\delta^{s-\rho}\big| \le \delta^{\sigma-\frac{1}{2}}\sum_\rho |\Gamma(s-\rho)|$. We separate the contribution of zeros close to t, since in this case the weight $\Gamma(s-\rho)$ is large because of the pole of Γ at 0. We chose 2 as threshold value, which appears to be near the optimal value ≈ 2.3. Since $|(u+iv)\Gamma(u+iv)| \le 1$ for $u \in (0,1/2)$ and every $v \in \mathbb{R}$, we have for every $t > 3$ (and setting $u := \sigma-\frac{1}{2} \in (0,\frac{1}{2})$)

$$\sum_{|\gamma-t|\le 2} |\Gamma(u+i(\gamma-t))| \le \sum_{|\gamma-t|\le 2} \frac{1}{|u+i(\gamma-t)|}$$

so that by Lemma 3.4 we get

eq:C33 (3.10) $$\sum_{|\gamma-t|\le 2} |\Gamma(u+i(\gamma-t))| \le \Big(\frac{\operatorname{asinh}(\frac{4}{2\sigma-1})}{\pi}+\frac{2d_1}{2\sigma-1}\Big)W_{\mathbb{K}}(t)+\frac{2d_2 n_{\mathbb{K}}+2d_3}{2\sigma-1}.$$

To estimate the contribution of zeros with $|\gamma-t| \ge 2$ we use the bound $|\Gamma(u+iv)| \le \sqrt{2\pi}e^{-\frac{\pi}{2}|v|}$ for $u \in (0,1/2)$, $|v| \ge 1$. Thus we get (assuming $t > 3$) that

$$\sum_{|\gamma-t|\ge 2} |\Gamma(s-\rho)| \le \sqrt{2\pi}e^{-\pi}\sum_{j=0}^{\infty} e^{-j\pi}\big(n_{\mathbb{K}}(t+2j+3;1)+n_{\mathbb{K}}(t-2j-3;1)\big).$$

Without loss of generality we can assume $t \in \mathbb{R}\backslash\mathbb{Z}$; then the claim for $t \in \mathbb{Z}$ will follow by continuity. Under this hypothesis the quantity $|t-2j-3|$ is smaller than 1 only for $j = \bar{j} := \lfloor \frac{t-2}{2} \rfloor$. Thus from (3.6a) and (3.6b) we deduce that

$$\begin{aligned}\sum_{|\gamma-t|\ge 2} |\Gamma(s-\rho)| \le &\sqrt{2\pi}e^{-\pi}\sum_{j=0}^{\infty} e^{-j\pi}(0.64\log\Delta_{\mathbb{K}}+(0.64\log(t+2j+3)+5.75)n_{\mathbb{K}}+3.49)\\ &+\sqrt{2\pi}e^{-\pi}\sum_{\substack{j=0\\ j\ne\bar{j}}}^{\infty} e^{-j\pi}(0.64\log\Delta_{\mathbb{K}}+(0.64\log(|t-2j-3|)+5.75)n_{\mathbb{K}}+3.49)\\ &+\sqrt{2\pi}e^{-\pi}e^{-\bar{j}\pi}(0.96\log\Delta_{\mathbb{K}}+5.19n_{\mathbb{K}}+3.49)\\ \le &\frac{\sqrt{2\pi}e^{-\pi}}{1-e^{-\pi}}(1.28\log\Delta_{\mathbb{K}}+11.5n_{\mathbb{K}}+6.98)+0.32\sqrt{2\pi}e^{-\frac{\pi}{2}(t-2)}\log\Delta_{\mathbb{K}}\end{aligned}$$

eq:C34 (3.11)
$$+0.64\sqrt{2\pi}e^{-\pi}\Big(\sum_{j=0}^{\infty}e^{-j\pi}\log(t+2j+3)+\sum_{\substack{j=0\\ j\neq\bar{j}}}^{\infty}e^{-j\pi}\log(|t-2j-3|)\Big)n_{\mathbb{K}}.$$

To bound the sums we use the inequalities $\log(t+2j+3)\le\log t+\frac{2j+3}{t}$ for the first one and and $\log(|t-2j-3|)\le\log t$ when $j\le J$ and $\log(|t-2j-3|)\le\log t+\frac{2(j-J)}{t}$ when $j>J$, where $J:=\lfloor t-\frac{3}{2}\rfloor$ for the second. Thus

$$\begin{aligned}&\sum_{j=0}^{\infty}e^{-j\pi}\log(t+2j+3)+\sum_{\substack{j=0\\ j\neq\bar{j}}}^{\infty}e^{-j\pi}\log(|t-2j-3|)\\ &\le\sum_{j=0}^{\infty}e^{-j\pi}\Big(\log t+\frac{2j+3}{t}\Big)+\sum_{j=0}^{\infty}e^{-j\pi}\log t+\frac{2}{t}\sum_{j=J+1}^{\infty}e^{-j\pi}(j-J)\\ &=\frac{2\log t+3}{1-e^{-\pi}}+2\frac{e^{-\pi}(1+e^{-J\pi})}{t(1-e^{-\pi})^2}.\end{aligned}$$

Moving this bound into (3.11) we get for $t\ge 10$ that

$$\sum_{|\gamma-t|\ge 2}|\Gamma(s-\rho)|\le 0.145\log\Delta_{\mathbb{K}}+(0.145\log t+1.33)n_{\mathbb{K}}+0.8\le 0.15W_{\mathbb{K}}(t)+1.6n_{\mathbb{K}}+0.8,$$

that with (3.10) gives

$$\sum_{\rho}|\Gamma(s-\rho)|\le\Big(\frac{\operatorname{asinh}(\frac{4}{2\sigma-1})}{\pi}+\frac{2d_1}{2\sigma-1}+0.15\Big)W_{\mathbb{K}}(t)+\frac{2d_2n_{\mathbb{K}}+2d_3}{2\sigma-1}+1.6n_{\mathbb{K}}+0.8.$$

We get the claim using the bound $\operatorname{asinh}(4/u)\le\log(1/u)+\operatorname{asinh}4$ which holds for $u\in(0,1]$, and the known values for d_j's. □

lem:C8 **Lemma 3.8** (Bound of III)**.** *Let $\sigma\in(\frac{1}{2},1)$, $|t|\ge 10$ and $\delta\in(0,1)$. Then*

$$|\Gamma(1-s)\delta^{s-1}|\le\sqrt{2\pi}e^{-\frac{\pi}{2}|t|}\delta^{\sigma-1}.$$

Proof. It is sufficient to prove that $|\Gamma(s)e^{\pi s/2}|\le\sqrt{2\pi}$ for any $s\in D:=\{s\colon \operatorname{Re}(s)\in[0,\frac{1}{2}],\ \operatorname{Im}(s)\ge 10\}$. By the maximum modulus principle it is sufficient to prove it for $s\in\partial D$. The claim for $\operatorname{Re}(s)=\frac{1}{2}$ follows immediately from the complementation formula $\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s}$. The claim for the other two lines may be proved using the Euler–Maclaurin formula for gamma. □

To bound IV efficiently we split it in two

$$\begin{aligned}IV&=\frac{1}{2\pi i}\int_{1/4-i\infty}^{1/4+i\infty}\sum_{|\gamma-y|\le 1}\frac{\Gamma(z-s)}{z-\rho}\delta^{s-z}\,\mathrm{d}z+\frac{1}{2\pi i}\int_{1/4-i\infty}^{1/4+i\infty}\Big[\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(z)-\sum_{|\gamma-y|\le 1}\frac{1}{z-\rho}\Big]\Gamma(z-s)\delta^{s-z}\,\mathrm{d}z\\ &=:IVa+IVb,\end{aligned}$$

where $y:=\operatorname{Im}(z)$, which we estimate separately.

lem:C9 **Lemma 3.9** (Bound of IVa)**.** *(GRH) Let $\sigma\in(\frac{1}{2},1)$, $|t|\ge 10$ and $\delta\in(0,1)$. Then*

$$|IVa|\le\frac{\delta^{\sigma-1/4}}{2\pi}\big(9.16\log\Delta_{\mathbb{K}}+(9.16\log(|t|+1)+114.03)n_{\mathbb{K}}+65.88\big).$$

Proof. It is sufficient to prove that

$$\int_{\mathbb{R}} \sum_{|\gamma-y|\le 1} \frac{|\Gamma(u+i(y-t))|}{|\frac14+i(\gamma-y)|}\,\mathrm{d}y \le 9.16\log\Delta_{\mathbb{K}}+(9.16\log(|t|+1)+89.73)n_{\mathbb{K}}+65.88$$

when $u\in[-\frac34,-\frac14]$ and $t\ge 10$.
We split the integral into the regions $|y|\ge 2$ where the sum on the zeros is estimated by Lemma (3.4) with $c=1$, and the remaining part $|y|<2$ where the sum is estimated simply by $4n_{\mathbb{K}}(y;1)$, getting:

$$\int_{\mathbb{R}} \sum_{|\gamma-y|\le 1} \frac{|\Gamma(u+i(y-t))|}{|\frac14+i(\gamma-y)|}\,\mathrm{d}y \le \int_{|y|<2} 4n_{\mathbb{K}}(y;1)|\Gamma(u+i(y-t))|\,\mathrm{d}y$$
$$+\int_{|y|\ge 2}\Big(\Big(\frac{\operatorname{asinh}4}{\pi}+4d_1\Big)W_{\mathbb{K}}(|y|)+4(d_2n_{\mathbb{K}}+d_3)\Big)|\Gamma(u+i(y-t))|\,\mathrm{d}y.$$

Now we restore the part of the integral with $|y|<2$, getting

eq:C35

$$\begin{aligned}(3.12)\qquad &= \int_{|y|<2}\Big(4n_{\mathbb{K}}(y;1)-\Big(\frac{\operatorname{asinh}4}{\pi}+4d_1\Big)W_{\mathbb{K}}(|y|+1)-4(d_2n_{\mathbb{K}}+d_3)\Big)|\Gamma(u+i(y-t))|\,\mathrm{d}y\\ &+\int_{\mathbb{R}}\Big(\Big(\frac{\operatorname{asinh}4}{\pi}+4d_1\Big)W_{\mathbb{K}}(|y|+1)+4(d_2n_{\mathbb{K}}+d_3)\Big)|\Gamma(u+i(y-t))|\,\mathrm{d}y.\end{aligned}$$

The exponential decay of the gamma function and the assumption $t\ge 10$ allow to bound the first integral trivially, without affecting the strength of the result. From (3.6a) and (3.6b) we get

$$4n_{\mathbb{K}}(y;1)-\Big(\frac{\operatorname{asinh}4}{\pi}+4d_1\Big)W_{\mathbb{K}}(|y|+1)-4(d_2n_{\mathbb{K}}+d_3)\le 2\log\Delta_{\mathbb{K}}$$

in $y<2$. Moreover, $|\Gamma(u+iv)|\,\mathrm{d}y\le 2\cdot 10^{-6}$ when $u\in[-\frac34,-\frac14]$ and $|v|\ge 8$. With bounds (3.1a) and (3.1b) these facts prove that (3.12) is bounded by

$$2\cdot 10^{-5}\log\Delta_{\mathbb{K}}+4.73\Big(\Big(\frac{\operatorname{asinh}4}{\pi}+4d_1\Big)W_{\mathbb{K}}(|t|+1)+4(d_2n_{\mathbb{K}}+d_3)\Big)$$

which is the claim, once the values for d_j's are considered. □

The following lemma bounds the integrand in IVb.

lem:C10

Lemma 3.10. *(GRH) For $s=\frac14+it$ with $t\notin\mathbb{Z}$ we have*

$$\Big|\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s)-\sum_{\substack{\rho\\|\gamma-t|\le 1}}\frac{1}{s-\rho}\Big|\le\Big(2.18+\frac{3.2}{1+t^2}\Big)\log\Delta_{\mathbb{K}}+11.7+\frac{7}{1+2t^2}$$
$$+\Big(2.18\log(|t|+1)+21.6+\frac{10}{|1+4it|}\Big)n_{\mathbb{K}}.$$

Proof. We can assume $t\ge 0$. We subtract (2.6) at $s=\frac14+it$ and $2+it$, obtaining

$$\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s)-\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(2+it)=\sum_{\rho}\Big(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\Big)-\frac{\Gamma_{\mathbb{K}}'}{\Gamma_{\mathbb{K}}}(s)+\frac{\Gamma_{\mathbb{K}}'}{\Gamma_{\mathbb{K}}}(2+it)-\Big(\frac1s+\frac{1}{s-1}-\frac{1}{2+it}-\frac{1}{1+it}\Big).$$

We use (2.1) and Lemma 3.5 to estimate $\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(2+it)$ and the gamma factors respectively, and the bound $|\frac{1}{s}-\frac{1}{2+it}+\frac{1}{s-1}-\frac{1}{1+it}|\le\frac{7}{1+2t^2}$. In this way we get

eq:C36
$$\Big|\frac{\zeta'_{\mathbb{K}}}{\zeta_{\mathbb{K}}}(s)-\sum_{|\gamma-t|\le 1}\frac{1}{s-\rho}\Big|\le n_{\mathbb{K}}\Big|\frac{\zeta'}{\zeta}(2)\Big|+\sum_{|\gamma-t|>1}\Big|\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\Big| + \sum_{|\gamma-t|\le 1}\Big|\frac{1}{2+it-\rho}\Big|+\frac{10n_{\mathbb{K}}}{|1+4it|}+\frac{7}{1+2t^2}. \tag{3.13}$$

Moreover, for the first sum on the right-hand side of (3.13) we have

$$\frac{4}{7}\sum_{|\gamma-t|>1}\Big|\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\Big|=\sum_{|\gamma-t|>1}\frac{1}{|s-\rho||2+it-\rho|}$$

eq:C37
$$\le\sum_{j=1}^{\infty}\frac{n_{\mathbb{K}}(t+2j;1)+n_{\mathbb{K}}(t-2j;1)}{|\frac14+i(2j-1)||\frac32+i(2j-1)|}. \tag{3.14}$$

By hypothesis t is not an integer, then $|t-2j|$ is in $(0,1)$ only for $j=\bar{\jmath}:=\lfloor(t+1)/2\rfloor$. Thus using the bound in (3.6a) for $j\ne\bar{\jmath}$ and (3.6b) when $\bar{\jmath}$ we deduce that (3.14) is

$$\le\sum_{\substack{j=1\\ j\ne\bar{\jmath}}}^{\infty}\frac{0.64W_{\mathbb{K}}(t+2j)+0.64W_{\mathbb{K}}(|t-2j|)+2\cdot 6.92n_{\mathbb{K}}+2\cdot 3.49}{|\frac14+i(2j-1)||\frac32+i(2j-1)|} +\frac{0.64W_{\mathbb{K}}(t+2j)+6.92n_{\mathbb{K}}+3.49+(0.96\log\Delta_{\mathbb{K}}+5.19n_{\mathbb{K}}+3.49)}{|\frac14+i(2j-1)||\frac32+i(2j-1)|}\Big|_{j=\bar{\jmath}\text{ if }t>1}.$$

Using $\log(t+2j)+\log(|t-2j|)\le 2\log(2(t+1)j)$ (for $t\ge 0$ and $j\ge 1$):

$$\le 2\sum_{\substack{j=1\\ j\ne\bar{\jmath}}}^{\infty}\frac{0.64(\log\Delta_{\mathbb{K}}+\log(2(t+1)j/2\pi)n_{\mathbb{K}})+6.92n_{\mathbb{K}}+3.49}{|\frac14+i(2j-1)||\frac32+i(2j-1)|} +\frac{0.64(\log\Delta_{\mathbb{K}}+\log((t+2j)/2\pi)n_{\mathbb{K}})+6.92n_{\mathbb{K}}+3.49+(0.96\log\Delta_{\mathbb{K}}+5.19n_{\mathbb{K}}+3.49)}{|\frac14+i(2j-1)||\frac32+i(2j-1)|}\Big|_{j=\bar{\jmath}\text{ if }t>1}.$$

Suppose $t>1$. Then restoring the missing term in the first sum we get

$$\le 2\sum_{j=1}^{\infty}\frac{0.64(\log\Delta_{\mathbb{K}}+\log(2(t+1)j/2\pi)n_{\mathbb{K}})+6.92n_{\mathbb{K}}+3.49}{|\frac14+i(2j-1)||\frac32+i(2j-1)|} +\frac{0.64(\log(2\pi(t+2j))-2\log(2(t+1)j))n_{\mathbb{K}}+(0.96-0.64)\log\Delta_{\mathbb{K}}+(5.19-6.92)n_{\mathbb{K}}}{|\frac14+i(2j-1)||\frac32+i(2j-1)|}\Big|_{j=\bar{\jmath}}.$$

Since

$$\sum_{j=1}^{\infty}\frac{1}{|\frac14+i(2j-1)||\frac32+i(2j-1)|}\le 0.76 \qquad \sum_{j=1}^{\infty}\frac{\log(2j)}{|\frac14+i(2j-1)||\frac32+i(2j-1)|}\le 0.82$$

and since for $t\ge 1$ one has

$$\log(2\pi(t+2j))-2\log(2(t+1)j)+5.19-6.92<0 \qquad \text{for}\quad j=\bar{\jmath},$$

$$\frac{1}{|\frac{1}{4}+i(2j-1)||\frac{3}{2}+i(2j-1)|} \leq \frac{5.4}{1+t^2} \qquad \text{for} \quad j=\bar{\jmath},$$

we get:

$$\leq 2\Big(0.64\cdot 0.76\log\Delta_{\mathbb{K}}+0.64\cdot 0.76 n_{\mathbb{K}}\log\Big(\frac{t+1}{2\pi}\Big)+0.64\cdot 0.82 n_{\mathbb{K}}+6.92\cdot 0.76 n_{\mathbb{K}}+3.49\cdot 0.76\Big)$$
$$+\frac{5.4}{1+t^2}\,(0.96-0.64)\log\Delta_{\mathbb{K}}$$

which is

eq:C38 (3.15)
$$\leq \Big(1+\frac{1.8}{1+t^2}\Big)\log\Delta_{\mathbb{K}}+(\log(t+1)+9.8)n_{\mathbb{K}}+5.31.$$

Suppose $0<t<1$. Then the term for $j=\bar{\jmath}$ disappears and (3.14) is

eq:C39 (3.16)
$$\leq \log\Delta_{\mathbb{K}}+(\log(t+1)+9.8)n_{\mathbb{K}}+5.31.$$

Lastly we note that

$$\sum_{|\gamma-t|\leq 1}\Big|\frac{1}{2+it-\rho}\Big| \leq \frac{2}{3}n_{\mathbb{K}}(t;1)$$

which can be bounded with Lemma 3.3. The claim follows putting all together in (3.13) and using (3.15) and 3.6a for $t>1$, and (3.16) and (3.6b) for $0<t<1$. The proof concludes by noticing that the first bound is worst than the second in $0<t<1$ and that therefore its range can be extended to $t>0$. □

lem:C11 **Lemma 3.11** (Bound of IV)**.** *(GRH) let $\sigma\in(\frac{1}{2},1)$, $|t|\geq 10$ and $\delta\in(0,1)$. Then*

$$|IV| \leq \big(3.11\log\Delta_{\mathbb{K}}+(3.11\log(|t|)+35)n_{\mathbb{K}}+20\big)\delta^{\sigma-1/4}.$$

Proof. By Lemmas 3.2 and 3.10 we get

$$|IVb| \leq \frac{\delta^{\sigma-1/4}}{2\pi}\Big((4.73\cdot 2.18+0.013\cdot 3.2)\log\Delta_{\mathbb{K}}+4.73\cdot 11.7+0.007\cdot 7$$
$$+(4.73\cdot 2.18\log(|t|+1)+4.73\cdot 21.6+0.171\cdot 10)n_{\mathbb{K}}\Big).$$

We get the claim adding $|IVa|$ as estimated in Lemma 3.9. □

We are finally able to prove the bound of $\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s)$ in the critical strip.

lem:C12 **Lemma 3.12.** *(GRH) Let $\sigma\in(\frac{1}{2},1)$ and $|t|\geq 10$. Then*

$$\Big|\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s)\Big| \leq \Big(\frac{n_{\mathbb{K}}}{1-\sigma}+\frac{\log(\frac{1}{2\sigma-1})}{\pi}+\frac{0.64}{2\sigma-1}+1.37\Big)Q^{2-2\sigma}+\Big(\frac{0.07}{2\sigma-1}+4.2\Big)n_{\mathbb{K}}$$

with $Q:=\log\Delta_{\mathbb{K}}+(\log|t|+20)n_{\mathbb{K}}+11$.

Proof. From (3.8) and Lemmas 3.6, 3.7, 3.8 and 3.11 we get

$$\Big|\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s)\Big| \leq \delta^{\sigma-\frac{1}{2}}\Big(\Big(\frac{\log(\frac{1}{2\sigma-1})}{\pi}+\frac{0.64}{2\sigma-1}+0.82\Big)W_{\mathbb{K}}(t)+\Big(\frac{13.9}{2\sigma-1}+1.6\Big)n_{\mathbb{K}}+\frac{6.9}{2\sigma-1}+0.8\Big)$$
$$+\sqrt{2\pi}e^{-\frac{\pi}{2}|t|}\delta^{\sigma-1}+\big(3.11\log\Delta_{\mathbb{K}}+(3.11\log|t|+35)n_{\mathbb{K}}+20\big)\delta^{\sigma-1/4}$$
$$+\Big(\frac{\delta^{\sigma-1}}{1-\sigma}+\frac{0.07}{2\sigma-1}+4.2\Big)n_{\mathbb{K}}$$

which we simplify to

$$\Big|\frac{\zeta_{\mathbb{K}}'}{\zeta_{\mathbb{K}}}(s)\Big| \le \Big(\frac{\log(\frac{1}{2\sigma-1})}{\pi}+\frac{0.64}{2\sigma-1}+0.82\Big)Q\delta^{\sigma-\frac{1}{2}}+\sqrt{2\pi}e^{-\frac{\pi}{2}|t|}\delta^{\sigma-1}+3.11Q\delta^{\sigma-1/4}$$
$$+\Big(\frac{\delta^{\sigma-1}}{1-\sigma}+\frac{0.07}{2\sigma-1}+4.2\Big)n_{\mathbb{K}}$$

where $Q := \log\Delta_{\mathbb{K}}+(\log|t|+20)n_{\mathbb{K}}+11$ (thus $W_{\mathbb{K}} \le Q-21.8n_{\mathbb{K}}-11$).
We get the claim by setting $\delta := Q^{-2}$ and with some minor simplifications which come from the assumption $|t| \ge 10$ and the lower bound $Q \ge 33$. □

Finally, the inequality $f_{\mathbb{K}}(s) \le \tilde{f}_{\mathbb{K}}(s)$ with the $\tilde{f}_{\mathbb{K}}(s)$ given in the theorem follows plugging the estimates of Lemmas 3.1 and 3.12 in (2.7), and simplifying the resulting inequality using the bound $n_{\mathbb{K}}\log\big(\frac{\sqrt{\sigma^2+t^2}}{2\pi}\big)+\frac{2\sigma}{\sigma^2+t^2}+\frac{2\sigma-2}{(\sigma-1)^2+t^2} \le n_{\mathbb{K}}\log t$ which holds true when $\sigma \in (\frac{1}{2},1)$ and $t \ge 1$.

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