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Multipopulation Longevity Risk Modeling: Introducing New Methodologies

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Declaration

This thesis has been submitted to the university of Bergamo in fullfilment of the requirements for the Doctoral degree in Economics, Applied Mathematics and Operational Research.

Dedication

To my parents

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Introduction

The primary goal of this thesis is to contribute on the topic of multipopulation longevity risk with additional literature. We focus specifically on the cointegration approach to forecast mortality indices and life expectancy at birth for respectively Canadian provinces and races in the USA as well as their future trends. Since the last century, human mortality has been declining and life expectancy has been increasing continuously(see Oeppen, 2002). It arises a risk of longevity or mortality faced by pension planners, public finance authorities as well as insurance companies agents which lead to increase the amount of pension liabilities. It appears thus to be vital to quantify these risks in order to face potential losses or avert bankrupcy. The pioneer work on the modeling of single population mortality has been done by Lee Carter model in 1992. The method describes a time series of age specific log-mortality rates as the sum of an age-specific component that is a constant and a product of two terms, in which one component is a time varying parameter reflecting general change in mortality and the second one is an age-specific factor, describing the pattern of deviations from the age profile. Moreover, during the years range 2000, several other extensions of this model have been discussed including Haberman and Renshaw(2006), Currie et al(2004, 2006), Cairns, Blake as well as Dowd(2006b) and its multiple generalizations. However the deficiencies of these models are due to the fact that it takes into account only one temporal factor to explain mortaly rates. New techniques, associated with more common factors, have been involved to

improve these models. They appear to strenghten and increase the accuracy of life table used by actuaries to project future mortality trends and reduce uncertainty of models associated with them.
This thesis aimed at modeling multipopulation longevity risk. With multivariate models, we take advantage of correlations and long run relationships accross various populations. Specifically, we focus on projecting provincial mortality indices and life expectancy within a country. Further, we introduce a new approach to project future life expectancy at birth by races in the United States. The main contributions can be summarized as follows: The Vector Error Correction Model(VECM), derived from the cointegration approach, capture well dynamics of females provincial mortality indices and produce reliable future forecasts from 2010 to 2060. Unlike females group, males mortality indices have been succesful within 15 years horizons of time in the phase of backtesting. In the second chapter, we illustrate successfully the cointegration approach to project life expectancy at birth for a basket of 6 Canadian provinces. Finally, the VECM models have led to improve also the accuracy of provincial life expectancy and also life expectancy at birth by races in the USA.

Chapter 1: **The Historical Review**.

We describe the relevant literature on stochastic mortality models as well as new research area on multipopulation longevity risk. In particular, we describe the most popular models appeared in literature to model mortality rates. We initially present the stochastics models discussed in literature since the pioneer work of Lee Carter(1992) model. In addition, we present multivariate approaches models that take into account more than one population both in mortality and life expectancy. In the second part, we discuss scientific literature on multicountry longevity risk. This is a new area of research in life insurance and pension practice. Also, we present the concept of basis

risk(see Dowd(2011), Jarner Kryger(2011) & Zhou(2013)) which measures the difference between historical and portfolio insurance mortality. Finally we explore the relevant methods reported in literature for the management of mortality. Also we present longevity risks products issued by financial institutions.

Chapter 2: **Multivariate approach to project the long run relationship of mortality indices**

This chapter illustrates a case studies of multipopulation longevity risk. We investigate the common trends accross provincial mortality within Canada. We fit initially mortality rates from each of the 9 most populated provinces in Canada from 1921 to 2009 on the basis of Lee-Carter(1992) model. Time varying mortality indices are retrieved from each fitted mortality by provinces. We apply and compare the two models VAR and VECM for both genders males and females. The various steps of this procedure include: the determination of order of integration through unit root tests such as ADF, PP and KPSS tests. Following, we determine the optimal lag order. We then estimate the VAR model and then derive VECM models. Next, we make use of them to project 50 years ahead future mortality indices for both genders. The results are consistent with D'Amato(2013) for VAR model and Zhou(2013) for VECM. Finally we compute the prices of annuities and life expectancy by cohorts(1960, 1970, 1980,1990,2000) from projected mortality indices.

Chapter 3: **A multivariate approach to project the long run relationship of life expectancy**

In this chapter, we illustrate the cointegration approach to provincial life expectancy at birth in Canada. We retrieve data from Canadian Mortality Database for 6 canadian provinces including Alberta, British Columbia,

Manitoba, New Brunswick, Quebec as well as Ontario from 1921 to 2009. We apply and compare three stochastics models such as ARIMA, VAR and VECM on life expectancy data. In doing this, we first describe various steps to obtain the optimal ARIMA model for each provincial life expectancy. As to the cointegration approach used, we explain the various steps which include the determination of order of integration, determination of the optimal lag order. We estimate then the VAR model and derive the corrispondent VECM models. Finally, we project 50 years future life expectancy ahead over the period 2010-2060.

Chapter 4: **Projecting the long run relationship of life expectancy between races in USA**

We explore life expectancy at birth by races in the United States including black, white as well as other races(a group formed by hispanic, asian) from 1975-2010. Preliminary analysis of the issue involves the principal components analysis to investigate what drives historic life expectancy by races in the USA and existence of potential common trends . We then apply the models ARIMA, VAR, VECM on various populations. We combine males and females group and races in our computations by taking into account the existence of long term relationship between these populations. The method of cointegration which includes the determination of unit root, the optimal lag length and the derivation of both VAR and VECM models is also used here. We then project 50- years ahead future life expectancy for each race group in the United States.This work aim at improving Carter(2010) in predicting future life expectancy by races in the USA.

Chapter 1

Review of prior Literature

1.1 Introduction

Over the centuries, Understanding the dynamic of mortality rates have been a real issue as some studies showed in John Graunt (1662), Edmund Halley(1693), Gompertz(1825) and Heligman-Pollard(1980). Furthermore since the 20th century, mortality data have been improving continuously as many studies such as Tuljapurkar(2000) or Russolillo(2005) reported. The improvements in living standard such as health care, education, technology have impacted human longevity, which has risen with life expectancy at higher ages. It arises that forecasting future mortality pose a real challenge for actuaries in pension funds as well as social security as they are interested in the future payout to be corrisponded to their trustees. It is considered a success for human being to have increased progressively the life expectancy as it is shown in Oeppen(2002). However, this human's evolution causes a real challenge for insurance companies which are interested to mantain the equilibrium of their balance sheet by covering up with premiums received and reserves the future commitments from their insurees. Pensionners pay premiums against protection in the future if the events occur. For insurance companies, it is important to measure the accurate price which reflects the future uncertainty

in life expectancy. The early predictions were based on expert's subjective opinions: This has been found to underpredict future mortality rates and life expectancy. Lee Carter model(1992) is considered as a pioneer model which successfully predict mortality and life expectancy in the USA. Moreover in recent years, several others have explored alternative models such as Lee-Miller(2001), Booth et al(2002), Renshaw and Haberman(2006) all based on a single population. In practice, insurers are exposed on various markets accross regions or countries, the new methods appeared in the literature recently undertake to improve forecasting methods by taking into account these multiple exposures simultaneously allowing for common trends. Also, there is a consensus among academics and practitionners to measure basis risk which is the difference in modeling potential divergence between historical population mortality experience(which traditional models are applied to) and the insured portfolio managed by the insurance company. Equitable Life failure is an example of what may occured to companies which fail from predicting future and manage improvements in mortality and life expectancy. It is incumbent to insurance companies to charge additional premiums or set aside more reserves to face potential losses when they occur. We aim at reviewing the existing literature on how the early mortality models help to quantify mortality risk. Furthermore we present in brief some strategies used by pensions funds or companies to manage longevity. These solutions involve pensions asset buyouts, securitization but also longevity products engineered to help hedging trends improvements issues in mortality faced by insurance industry as well as academics.

1.2 Quantification methods of longevity risk

Since the last century several studies have been appeared in literature to test which model fit best single country mortality. Booth and Thickle(2008) in an attempt to review literature have explored models exposed in Lee-

Miller(2001), Booth et. al(2002), Hyndman(2005) and De Jong(2006) present better fit performance over the Lee Carter model while there is no difference in forecasting the accurate future life expectancy between these various models. Furthermore the model in Renshaw et. al(2006) with a cohort effect best suit for the USA while Cairns(2006b) focus on the UK as we can see in Cairns(2009). We present here an exhaustive list of the stochastic models appeared in literature. The mortality rates are defined as relationship between death rates and the exposure in (1.1):

$$m_{x,t} = \frac{d_{x,t}}{e_{x,t}} \tag{1.1}$$

where $t = 1, 2, ...T$, $d_{x,t}$ is the number of deaths during calendar year t aged x at time last birthday, $e_{x,t}$ is the average population during calendar year t aged x at time last birthday. While the q_x, which varies between 0 and 1, is the probability of dying within age 1 and it is presented here as in the equation (1.2):

$$q_x = \frac{2m_x}{\sum m_x} \tag{1.2}$$

where q_x is the probability of dying within 1 year age old.

Heligman & Pollard(1980) proposed to measure probability of death q_x based on the decomposition of principal component analysis as in the equation (1.3):

$$q_x/p_x = A^{(x+B)^C} + De^{-E(lnx-lnF)^2} + GH^x \tag{1.3}$$

This formula is subdivided in three components such that: $A^{(x+B)^C}$ represents the fall in mortality during the early childhood as the adaptation of the new born to its new environment, $De^{-E(lnx-lnF)^2}$ young adults mortality and reflects the accident mortality(for males) and maternal mortality(females), GH^x is the Gompertz exponential mortality for older ages. It captures the

mortality due to a progressive and natural aging of older people.
A measures the level of mortality at age 1, B refers to infant mortality displacement, C represents the rate of mortality decline in childhood; D, severity of the spread(E) of accident hump(F) or accident, G base level at age 0 mortality & H is the rate of increase in G. This decomposition shows that mortality rates depends on several components rather than 1 as in Lee Carter model(1992).

1.2.1 Single population models

The reference model on mortality modeling is based on Lee Carter(1992). It is a 2 factor model which includes an age(x) specific mortality and a bilinear component which reflect trends over the time(t). The Lee Carter model is described in (1.4) as follows:

$$log\ m(t,x) = a_x + b_x k_t + e_{x,t} \tag{1.4}$$

where $m(t,x)$ is the central death rates at age x at time t; a_x is the age pattern of mortality over time; k_t is the mortality index; b_x deviation from the average pattern as mortality index varies; $e_{x,t}$ is the error of the model. However there is a problem of identifiability in the estimation of parameters. In order to address this problem, two constraints in (1.5) and (1.6) have been imposed to insure the uniqueness of the values of the parameters a and b:

$$\sum b_x = 1 \tag{1.5}$$

$$\sum k_t = 0 \tag{1.6}$$

The two constrains imply that for each age group x the estimate for a_x is the mean over t of the $log\ m(t,x)$. The second constraint has to be taken such that they may not impact the accuracy of the future forecasts of mortality. Since the parameters in this model cannot be estimated through the standard least square model, the singular value decomposition is used to individuate the parameters values inserted in the model. Two steps are important to follow:

1. first, singular value decomposition is applied to the matrix $log(m_{x,t} - a_{x,t})$r to obtain the values of b_x and k_t at each age and at any time t. The mortality indices k_t are modelled also by using their first lag according to ARIMA model $k_t = k_{t-1} + d + e_t$ (see Box and Jenkins(1976) for more details) where d is the drift term, k_t is the mortality index to be forecast on its own lag, e_t is the error term.

2. second, k_t is re-estimated in a second stage to guarantee that observed total number of deaths are equal to the fitted ones. They find new values of k_t such that $D_t = \sum_x exp(a_x + b_x k_t) E_{x,t}$ where D_t is the total number of deaths in year t, $E_{x,t}$ is the exposure population at risk of age x in year t.

The mortality rates are forecasted s period ahead from a based period t through the following equation (1.7):

$$ln\ \hat{m}_{x,t+s} = \hat{a}_x + \hat{k}_{t+s}\hat{b}_x \tag{1.7}$$

where s is the forecasting step for the future, $m_{x,t+s}$ are the forecasted mortality rates

Several other extentions of the Lee Carter model have been conducted to fit mortality rates to other countries(see Cairns, 2009) and Booth et.

al(2002)). We present here the different models proposed recently in literature with their constraints

Lee-Miller Variant. The Lee Miller(2001) variant diverges from Lee Carter model in three ways:

1. the fitting period begins in 1950; this is to avoid problem as in jump-off year and eliminate problems faced by Lee Carter model with underprediction of future mortality;
2. the adjustments on mortality index k_t involves fitting life expectancy at birth in year t;
3. the jum-off-rates are taken to be the actual rates in the jump-off year.

Experiments show bias in fitting period 1900-1989 and 1990-1997, the source of error derives from the fitted rates for the last year(1989) of the fitting period and actual rates in that year. Pattern of change in mortality was not fixed over time diversely to what Lee Carter model assumes.
The authors provide for the US data that the forecasts were biased while using estimation period as 1900-1989 and forecast the period 1990-1997. The source of error was the mismatching between fitted rates in 1989 and actual rate in that same year. The solution was to set a constraint k_t that passes through zero.

Booth Maindonal Smith Variant The Booth Maindonal Smith Variant(2002) investigates a new method that diverges to Lee-Carter model in three ways:

1. the fitting period were based on statistical goodness-of-fit criteria under the assumption of linear mortality index k_t;

2. the adjustments of k_t involves fitting to the age distribution of deaths;

3. the jumps-off rates are taken to be the fitted rates based on this estimation methodology.

Applications on Australian data(1907-1999) which the results are reported in Booth(2006) are the following:

1. pattern of constant mortality decline was not verified;

2. problem in assumption of constant b_x in the underlying Lee Carter;

3. by assuming linearity in mortality index, they seek to maximize the fit and this results also on the values of parameters b_x;

4. k_t is modified on the adjustments and BMS fits age distribution of deaths using Poisson distribution to model death process and the deviance statistic to measure godness of fit.

This approach, based on principal component analysis, has been applied also by Hatzopoulos(2013) which fits mortality improvement by using only two components. An important remark from that study shows that it is not necessary to add multiple components, since the results are not dependent on the number of components, to be successfull. The performance has been found better than usual singular methods including the Lee Carter(1992), Age period cohort as well as Cairns et al(2009) that we will present later on.

B-splines and P-splines models: Currie et. al(2004) proposed B-splines and P-splines to fit the mortality surface. See more details on these models on the appendix of Cairns(2009). A basic illustration is presented in the equation (1.8):

$$log\ m(t,x) = \sum_{i,j} \theta_{i,j} B_{i,j}^{ay}(x,t) \tag{1.8}$$

with $\theta_{i,j}$ smoothing of the $\theta_{i,j}$ in the age and cohort direction, $B^{ay}_{i,j}$ are pre-specified basis function with regularly spaced knots $\theta_{i,j}$ which are parameters to be estimated.

Hyndman-Ullah function: Hyndman and Ullah(2005) proposed new extention of the Lee Carter in (1.9):

$$ln\ m_{x,t} = a(x) + \sum_{j=1} k_{i,j} b_j(x) + e_t(x) + \varepsilon_{x,t} \tag{1.9}$$

where $a(x)$ average pattern of mortality by age accross years represents the smoothed function of age where age is a continuous quantity; $b_j(x)$ basis function; $k_{t,j}$ is a time series coefficient; the pair ($k_{t,j}$, $b_j(x)$) are estimated by using principal component analysis; $\varepsilon_{x,t}$ observational error which appear with age; $e_t(x)$ is a modelling error
Furthermore, Hyndman and Ullah(2005) proposed the following assumptions:

1. mortality are assumed to be like a smooth function; also death rates are estimated by using non parametric smoothing techniques;
2. more components of the pairs (k_t, b_x) are added respect to Lee Carter(1992);
3. more general time series are considered to forecast the coefficient;
4. robust estimation are used to allow for period when there are exceptional events such as wars or epidemics;
5. no adjustments are set up on mortality indices k_t.

De Yong-Tickle model: The approach of De Jong and Tickle(2006) proposed a new model for log death rates by using the following equation in (1.10):

$$y_t = Xa + Xbk_t + \varepsilon_t \tag{1.10}$$

where a, b, k_t, ε_t are similar to the Lee Carter parameters For X=1, Lee Carter model (1992) is a special case of this model, X is a known matrix with more rows than columns. In this general model, there are a and b parameters for each age, k_t time series has an independent impact at each age. In the current analysis, the matrix X is estimated from B-splines (see Hastie ad Tibshirami, 1990). The estimates are based on Maximum likelihood functions derived by using Kalman filtering and smoothing (Harvey, 1989). The fitting period is restricted to 1950 to avoid outliers as for Hyndman-Ullah.

Renshaw and Haberman: An improvement of Lee Carter model for the UK population has been explored through cohort effect by Renshaw and Haberman(2006) as follows (1.11):

$$log\ m(t,x) = a_x + b_x^{(1)} k_t^2 + b_x^{(2)} \gamma_{t-x} \tag{1.11}$$

However several constraints have to be set up as follows in (1.12):

$$\begin{aligned} &\sum k_t^2 = 0 \\ &\sum b_x^{(1)} = 0 \\ &\sum \gamma_{t-x} = 0 \\ &\sum b_x^{(2)} = 1 \end{aligned} \tag{1.12}$$

The first and the third constraint mean that $b_x^{(1)}$ will be equal to the mean over t of the $log\ m(t,x)$, $\sum k_t^2$, $\sum b^{(1)} = 0$ the second and the fourth shift don't have natural choice. Renshaw and Haberman set up (1.13):

$$b_x^{(1)} = \frac{\sum_t log\ m_{x,t}}{N} \tag{1.13}$$

and other parameters in equation are estimated by using an iterative system. Even though Cairns(2009) use another approach to estimate parameters β_x^1 there is still problem with identifiability because parameters value converge slowly to their maximum likelihood(see more in Cairns, 2009) as in the original paper Renshaw and et.al(2006).

Currie model: Currie(2006) introduces a special case of the Renshaw-Haberman(2006) model as follows in (1.14):

$$logit\ m(t,x) = a_{1,x} + k_{2,t} + \gamma_{3,t-x} \tag{1.14}$$

where $a_{1,x}$ is the average shape of mortality, $k_{2,t}$ is the mortality index and finally $\gamma_{3,t-x}$ is the cohort effect. This is a special case of the model Renshaw and Haberman(2006) with $a_x = 1$ & $b_x = 1$. All the three parameters are estimated with P-splines to ensure the smoothness of the parameters. In order to avoid identifiability problems, The following constraints are stipulated as follows (1.15):

$$\begin{aligned} &\textstyle\sum_t k_{2,t} = 0 \\ &\textstyle\sum_{x,t} \gamma_{t-x}^{(3)} = 0 \end{aligned} \tag{1.15}$$

Cairns Blake and Dowd model. Cairns, Blake, *Dowd*(2006*b*)(CBD) fitted models to mortality rates in (1.16):

$$logit\ q(t,x) = a_1 k_1^{(1)} + b_x^2 k_t^{(2)} \tag{1.16}$$

where $\bar{x} = n_\alpha^{-1} \sum_i x_i$ is the mean age in the sample range. A simple parametric form can be assumed as followed in (1.17):

$$logit\ q(t,x) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) \tag{1.17}$$

where $a_{(1)} = 1$; $b_{(1)} = (x - \bar{x})$ and $b_1 = (x - \hat{x})$ $\hat{x}$ is mean of the age of the sample population under study. Unlikely the previous models, the CBD does

not have any problem of identifiability.

Cairns and its three generalization: Cairns(2009) extends three generalization of the CBD model. Cairns Blake and Dowd model 1 is interpreted as follows in (1.18):

$$logit\ q(t,x) = a^{(1)}k_1^{(1)} + b_x^{(2)}k_t^{(2)} + b^{(3)}\gamma_{t-x}^{(3)} \tag{1.18}$$

with the following constraints
$\beta^{(1)} = 1;\ \beta^{(2)} = (x - \bar{x});\ \beta^{(3)} = 1.$
By assuming these three constraints above, we obtain the following final model in (1.19):

$$logit\ q(t,x) = a^{(1)}k_1^{(1)} + b_x^{(2)}k_t^{(2)} + b^{(3)}\gamma_{t-x}^{(3)} \tag{1.19}$$

There is a problem of identifiability and it may be solved by switching $\gamma_{t-x}^{(3)}$ to $\gamma_{t-x}^{(3b)} = \gamma^{(3)} + \phi_1 + \phi_2(t - x - \bar{x})$. We fit with least squares a linear function of $t-x$ to γ_{t-x} and finally the fitted linear function is identically equal to zero.

Cairns Blake Dowd Model and the quadratic term:
The second model derived by Cairns-Blake-Dow(2009) can be written as follows in (1.20):

$$logit\ q(t,x) = k_1^{(1)} + k_t^{(2)}(x - \bar{x}) + k_t^{(3)}((x - \bar{x})^2 - \hat{\sigma}_2^x) + \gamma_{t-x}^{(4)} \tag{1.20}$$

where q_x is the annual probability of death, here $\hat{\sigma}_2^x = meanof\ (x - \hat{x})^2)$, $\gamma_{t-x}^{(4)} to \gamma_{t-x}^{(3b)} = \gamma_{t-x}^{(4)} + \phi_1 + \phi_2(t - x - \bar{x})$, and corresponding adjustments on $k_t^{(1)}$, $k_t^{(2)}$, $k_t^{(3)}$. They fix values of ϕ_1, ϕ_2,ϕ_3 to have estimated $\gamma_{t-x}^{(4)}$ centered around zero.

Cairns Blake Dowd Model with constant parameter on cohort effect: The last generalization of the CBD model is given by the equation (1.21):

$$logit\ q(t,x) = \beta^{(1)}k_1^{(1)} + \beta_x^{(2)}k_t^{(2)} + \beta^{(3)}\gamma_{t-x}^{(3)} \tag{1.21}$$

where $\beta_x^{(1)} = 1$, $\beta^{(2)} = (x - \hat{x})$ and $\beta_x^{(3)} = (x_c - x)$ for x_c to be estimated, these results in (1.22):

$$logit\ q(t,x) = k_t^{(1)} + k_t^{(2)}(x - \hat{x}) + \gamma_{t-x}^{(3)}(x_c - x) \tag{1.22}$$

and we fix the following constraint $\sum_{x,t} \gamma_{t-x}^{(3)} = 0$ to avoid any problem of identifiability.

1.2.2 From single population to multiple population mortality models

Most of studies have been attempting to capture the dynamics of single population. In reality, pensions managers or governement authorities are interested to investigate correlations of improvements in mortality for more populations simultaneously. As it is recalled in Cairns(2011) it is necessary to take into account for more populations with the purposes of the pricing of annuities and also for avoiding divergence in the pattern of 2 mortality rates over the time. We present here models that attempts to capture interdepence between linked populations.

Li and Lee(2005) introduce an extention of the Lee Carter model which study a group of populations. They assumed some conditions such that population should have necessary the same b_x and the same drift term for k_t to have long run convergence of mean mortality forecasts. The values of $B(x)$ and $K(t)$ are choosen on the basis of the best fitting model while $a(x)$ is obtained by minimizing the following regression (1.23):

$$min \sum_{t=0}^{T} [log(m(x,t,i)) - a(x,i) - B(x)K(t)]^2 \tag{1.23}$$

Some others contraints can be considered including: $\sum k_t = 0$ then in (1.24)

$$a(x,i) = \frac{\sum_{t=0}^{T} log(m(x,t,i)}{T+1} \tag{1.24}$$

$[a(x,i) + B(x)K(t)]$ is called the common factor of the ith population. The augmented common factor is an added component on the Lee Carter such that (1.25):

$$log\ m((x,t,i)) = a(x,i) + B(x)K(t) + b(x,i)K(t,i) + \varepsilon_{x,t,i} \tag{1.25}$$

where $0 \leq T \leq t$, $\varepsilon_{x,t,i}$ is the modeling error $b(x,i)k(t,i)$ allows for short-medium term difference between the rates of change in country or population i's death reates, $K(t)$ tends toward some constant level over time, $b(x,t)$ describes the difference between the pattern of change by age in mortality for the ith population and for the group as whole.

a)**The stratified Lee Carter model**

The Stratified Lee Carter model is introduced by D'Amato(2012) as follows in (1.26):

$$log(\hat{y}_{xtg}) = log(e_{xtg}) + a_x + a_g + \beta_x k_t \tag{1.26}$$

where $log\ (e_{xtg})$ presents a Poisson error structure age x, period t and an extravariate (g), m_{xtg} given u_{xtg} is the central mortality rates u_{xtg} is the force of mortality for any given subgroup. In this group equation the mortality trend and its deviation β_x with age are the same for the entire population, the main effect which is $\alpha_{xg} + \beta_x b_t$ captures both the effect of age and an additional variate (g). This is a particular example of common factor introduced by Li and Lee(2005). This is a methology that helps to quantify the differences in mortality experience of population subgroups. It considers also the differentiation(not induced by age and period) caused by factors linked to geographical, socio-economic and races differences

b) **The Three Way Lee Cartel**

This new extention of Lee Carter model is proposed by Rusolillo(2011).

It adds a new parameter λ_g to differentiate subpopulation mortality. It is given as follows in (1.27):

$$log_n\ m_{xtg} = \alpha_{xg} + \beta_x \lambda_g k_t \tag{1.27}$$

where α_{xg} measures the age-subpopulation specific pattern of mortality, β_x is respectively the deviation from mortality when k_t, the mortality index varies. In order to ensure the identifiability of the parameters in the model and also better interpretation of mortality differential the initial model has been reparametrized in (1.28) as:

$$log_n\ m_{xtg} = \alpha_x + \alpha_{xg} + \beta_x \lambda_g k_t \tag{1.28}$$

where α_x captures the level of differentiation mortality, λ_g captures the improvements in mortality, $\sum \alpha_{xg} = 0$ for all age groups $\sum \beta_x = 1$, $\sum k_t = 0$, $\sum \lambda_g = 1$ k_t is model and forecasts with ARIMA random walk with drift.
Since models involved in the computations of mortality are applied on the national historical population, however for the insurers it is crucial to calibrate models on their porfolio data. Thus it arises a basis risk faced by insurers which is based on the fact that mortality models cannot be calibrated on the portfolio. Several models have been performed in literature including Cairns(2011) which uses the Bayesian models to explain the dynamics of two related populations. Another modeling technique used is the gravity model(Dowd(2011)) based on two dynamically related populations. The models reviewed previously assume that one population is independent from others. However since there are potential relationships between genders, regions or subgroup and the exposure of insurance companies it is better to allow for more population in order to capture fully mortality dynamics. They assume biological reasonables in the gravity model between the two population under study UK & Wales and the CMI(continuous mortality in-

vestigation) data helps to assure better risk management of longevity basis risk and the correlation between populations. It is based on the case where larger population tend to pull the smaller one. The mortality model of 2 populations here is the Age-Period-cohort from Currie(2006) described above in the single case for each population. This model is based on the relationship of mortality index studied as dynamics of the 2 populations rather than mortality rates given by (1.29) and (1.30):

$$\begin{aligned} k_{1,t} &= k_{1,t-1} + u_1 + B_{11}Z_{1,t} \\ k_{2,t} &= k_{2,t-1} + \phi_{(k)}(K_{1,t-1} - K_{2,t-1}) + u_{(2)} + c_{(21)}Z_{1,t} \end{aligned} \tag{1.29}$$

where $(k_{1,t-1} - k_{2,t-1})$ is the dependence on the spread difference between the mortality index of the two population, ϕ_k is the gravity term and

$$\begin{aligned} &\gamma_{1,c} = (1 + \alpha_{(\gamma 1)})\gamma_{1,c-1} + \alpha_{(\gamma 1)}\gamma_{1,c-2} + u_{\gamma 1}(1 - \alpha_{(\gamma 1)}) \\ &\gamma_{2,c} = (1 + \alpha_{\gamma 2} - \phi_y)\gamma_{2,c-1} - \alpha_{\gamma 2}\gamma_{2,c-2} + \phi_\gamma \gamma_{1,c-1} + u_{\gamma 2}(1 - \alpha_{\gamma 2}) + \\ &d_{\gamma 21}Z_{c,\gamma 1} + d_{\gamma 22}Z_{c,\gamma 2} \end{aligned} \tag{1.30}$$

The first term corresponds to a dependency on the spread between gammas for the two populations. The second term represents a stochastic factor common to $\gamma_{c,1}$ the first of the new term is called a gravity.
The larger population exerts a pull on the small population while the smaller exerts a negligible attraction on the first one. This is the reason why authors explains the parameters of population independently but the second one is explained as dependence of the first. Other assumptions can be made such as ϕ will be varying between [0,1). In contrast, Zhou et. al(2013) introduces a new approach that allow for common trends by using the Vector Error Correction model which assume correlation in improvements of the two populations. They remove the assumption where 1 population is dominant on others and model only the joint dynamics of 2 populations. Process used to model mortality indices are the vector autoregressive model(VAR) and the Vector Error Correction Model(VECM). They provide an extended literature and methodology on the modeling longevity basis risk with the example

of longevity swap. The hypothesis are based on the fact that deaths rates will not diverge in the long term where the coefficient estimated from Lee Carter model of the two populations are equal and the spread between the mortality indices of the two populations are mean reverting. The advantage of two models VAR and VECM over the gravity model are given by:

1. symmetry of the model so that no particular hypothesis is set up on which population is dominant;

2. Ability to capture cross correlation accross the improvement in mortality index.

The VECM presents further benefits including the ability to capture long run relationship between variables and it takes into account additional variable in the equation so that it can improve the model's goodness of fit and backtesting performance as the VECM presents results with least error. Further, several other cases have been explored as Jarner and Kryger(2011) where they study the long run relationship between national mortality of Denmark and a larger population of Europe. Another illustration is given by Biatat and Currie(2010) based on two populations using P-splines models described above. Meanwhile Darkiewicz et. al(2004) tests if the pair mortality age satisfies the structure of Lee Carter. In these cases, the Engle and Granger long run relationship is given by the equation (1.31):

$$ln\ \hat{m}_{x,1t} = a_{1,x} - b\ ln\ \hat{m}_{2,x}(t) + \varepsilon_{x,1t} - b_2 + \varepsilon_{x,2t} \tag{1.31}$$

where $\varepsilon_{x,1t}$, $\varepsilon_{x,2t}$ are integrated of order 0, then it follows that $m_{x,1t}$ and $m_{x,2t}$ are age mortality rates.
They use cointegration between multiple pairs of age mortality group applied to Lee Carter to check the validity of the Lee Carter model for UK & Wales. The conclusion of the lack of cointegration between pair of mortality

group is a sign that predictions are not reliable. In order to obtain better results one should put more weight on recent observations in the model. By checking the cointegration between 2 mortality age group of UK & Wales, in most cases we say that the Lee Carter model is not reliable and we cannot apply the Lee Carter model. In this specific case, one should take into account various events such as wars (1914-1918, 1939-1944) and epidemic diseases occured during the period(1918-1929). Another solution is to use a time breaking-point for greater flexibility of mortality forecasting approach. More emphasis on more recent trends produce better fit as in Renshaw and Haberman(2003). Other solution is to disaggregate data: calculation can be performed separetly on the basis of sex or gender, geographical separation or subdivision in small groups. Also disaggragation is an advantage so that this approach really explains the reasons of the lack of co-integration and the separation. Following, Lazar(2004) found that for pair mortality age group cointegrated greater than 60 years in Romania, Lee Carter is a cointegration relationship that can be used to individuate common factors that drive mortality rates or life expectancy. Njenga and Sherris (2009) attempts to study the long run relationships of common trends between Australia, England, Japan, Norway and USA over the period 1947-2004. Their study provides more information on long run relationship to predict future mortality across these countries. It does not reveals common trends based on standardized mortality rates. In this case, it cannot be applied by insurance companies to exploit potential diversification in portfolio. Rather, they retrieved the 8 parameters in Heligman-Pollard(1980) applied on Australian mortality data and it shows that common trends do exist between the diverse parameters. Lazar(2009) found that there are rather 3 common stochastic trends which contrast with the Lee Carter model assumption of 1 time varying factor, it projects future mortality rate for each of 5 age groups inluding 60-65, 65-70, 70-75, 75-80, 80-85, 85-90, 90-95. Comparing the results from VECM model to Lee Carter on the basis of life expectancy, Lazar(2009) observed values of

period life expectancy at age 60 from Lee Carter are slightly larger than those computed by VECM. Thus, according to her study, the difference between the two approaches lie in the risk management strategies.
The mortality dependence used here through the cointegration approaches measure the appropriate price to improve the accuracy of longevity bonds as showned in Yang-Wang(2013). The multivariate Wang transform is useful to reach to this goal as Cox(2006), Lin et. al(2005), Dowd(2006) and Lino(2007). Applied with several interests such as 1%, 3%, 5% and equal weight are put on risk adjusments parameters. The price of bond is greater when we apply interdependece multicountry dependence age against independence case. In aadition to that, D'Amato et. al(2013) measure the mortality dependence for multiple countries. Another research has been done on modeling mortality trend by using solvency regimes(Borger et. al 2011) and copula factor(see Chen et al(2014) and Lin et.al(2013)). In addition Hatzopoulos and Haberman(2013) have analyzed multi-countries mortality indices by using principal component analysis

1.3 Management of the mortality risk

Over the last years, economic advancements in countries have led to improve longevity of human population. Pensions funds and other financial institutions have underestimated these improvements by failing in their previsions of mortality. This is the case of Equitable life Assurance which had failed in bankrupcy in 2000. Equitable life Assurance society(ELAS) is an example of the consequence of mismagement or measurement of longevity risk. During the period between 1957-1988, ELAS earned profit by selling annuity contracts but due to mainly three reasons, it has failed to evaluate its risks in these following cases:

1. Inability to hedge interest rate risk and longevity risk;

2. Error in the evaluation or quantification of the exposure's entity;

3. Lack of skills or instruments available to hedge its exposure to the two risks.

This situation has given the opportunity for governments or regulators to set up normative for protecting customers in particular and the financial system in general: According to Blake(2006) there are multiple audiences which can be interested in developing a life market to tackle such events:

1. The goverment is interested to promote a market of mortality linked securities to help companies involved hedging their exposure against longevity and at the same time to assist financial institutions via pension plan to save the stability of financial system. As a lender of last resort, government should encourage market of financial derivatives for hedging purposes in order to avoid its commitment, as a lender of last resort, in case of default faced by financial institutions.

2. Regulators are interested in the stability of financial system through the promotion of efficient and fair market. Also, they ensure fair deal for customers.

3. Hedgers are one of the main actor concerned by the risk. In this case financial instituions face improvements both in mortality or in life expectancy. They pay derivatives sellers to fix mortality rates in the future or also introduce reassurance contract to hedge against this type of risk. Another solution is the transfer to the capital market of longevity contracts.

4. General investors such as investments banking or hedge fund may be interested because it is low correlated with traditional risk factors sold on financial markets.

5. Speculators/Arbitraggeurs: these actors are useful for the liquidity of life markets. They make profits from these markets by exploiting the difference in the price of options or future and the information given on these markets.

In order to manage their asset liability, pension funds or companies make use of three different methods such as the following:

1.3.1 Pension buyout

The pension buyout operations involve the transfer of pension liability or asset to a regulated life insurer. This operation allows a company to off-load pension liability. The advantage of this operation is that it would not manage issues such as the related volatility of asset and liabilities. Under prudent assumptions which include taking into account of future improvements of mortality rates, the liability become $\hat{L}$. If $\hat{L} > A$ then the pension is in deficit instead if $\hat{L} < A$ the pension is in surplus. In the first context, a company borrows the amount $\hat{L} - A$ and pays it to an insurer to buy-out its pension assets and liabilities. The preference of insurance companies as counterparty is due to their advanced skills in forecasting and managing longevity risk or building performing hedging strategies. An alternative to the buyout is represented by partial buyout and regards the transfer of liabilities originated from deferred pension or payable over a limited period horizon(see Biffis, 2009). These partial buyout are part of derisking strategy or the changement of the investment strategy toward liability hedging. This

transaction is becoming popular among pension funds or life insurers such as Paternoster, Lucida, Rothesay Life, Legal and General, Prudential and Canada Life.

1.3.2 Securitization of Life insurance

An alternative strategy(see Modu, 2008) for the management of pension's company is represented by the securitization which involves the sale of a portfolio of asset and liabilities to a special purpose vehicle(SPV). This portfolio is then repacked and sold into the capital markets. A common form of securitization involves book of life policies. Pension funds can evaluate asset and liability on the basis of assumptions such as future mortality improvements, inflation rates and market yields. They accumulate also reserves to meet the liabilites in balance sheet. The pool of asset and policies is financed by issuing bonds acquirable in the capital markets and they are secured by promised cash flows or some form of credit insurance such as credit default swap or overcollaterilization. Life insurers are always required to meet liability payments with adequate reserves. There are several examples of such a deal including the senior of life settlements(2004) with Tarrytow second. A secondary market called Life Exchange has been founded in 2005 with the advanced electronic platform in order to conduct all the securitization transactions with efficience.

1.3.3 Longevity products for hedging strategies

Since 2003, many financial products have been considered for the mortality hedging purpose. Actually there have been investigated 4 types of products such as:

Longevity bond

Principal-at-risk: this is a typical Swiss Re bond which the investor may loose the principal entirely. Swiss Re proposes a 3 years life catastrophe bond. The life time was between 1st december 2003-1st january 2007 with the goal of reducing Swiss Re's exposure to a catastrophic mortality such as flu pandemic or natural catastrophes. Investors receive trimestrial coupons of 3 month libor+135 basis points. The principal depended on an index of mortality rates of 5 countries including USA, UK, France, Italy and Switzerland. If mortality index is less than 1.3 time the base level of 2002 level then the principal is repayable full. The principal is reduced by 5% for any increase of 0.01 on the treshold 1.3. The principal is exhausted if the increase in mortality is greater than 1.5 times the level of 2002 level. The bond was issued via a special purpose called the Vita Capital which invested the 400 millions principal in high-quality investments grade bonds. The SPV exchanges the income for a libro-linked cash flow. The advantage of this transfer to the financial market help insurance to remove extreme mortality risk from its balance sheet and at the same time reduce the credit risk. The disadvantage for Swiss Re was due to the finding of sufficient counterparties which can support the off-loaded risk. Beelders and Colarossi(2004) estimated the bond price by assuming Pareto distribution for mortality rates. The appetite from investors for this bond was fully effective due to two reasons mainly: higher coupons offered as well as hedging opportunities from the fact that mortality risk associated with active members of a pension of a pension plan. Then in april 2005, another Swiss Re bond worth 362 millions based on catastrophe using SPV and divided in three tranches class B, class C and Class D were offered. Furthermore, there have been several other bond derivatives including Tartan by Scottish Re, Osiris issued by AXA as reported in Biffis(2009).

Coupon based longevity The second regards coupon based dependent

on mortality such as the EIB/BNP bond. Another bond has been issued by EIB/BNP longevity bond in november 2004 but it didn't generate sufficent demand from investors. The total value was £ 540 millions with maturity of 25 years. The first coupon worth £ 50 millions originated from this bond was linked to a cohort of survivor index S(t) on the realized mortality rates of England & Wales aged 65 years old in 2002. Investors make an initial payment of £ 540 millions and receive each year nearly £ 50 millions *S(t) for 25 years. The S(t) was released on the basis of 2002 mortality's projections calculated by UK government actuaries Department . Each cash flow are priced by discounting at Libor - 35% basis points.The longevity bond involves mainly three components:

1. Floating rate of annuities bond with the payments in Euro by EIB;
2. Cross currency interest rates swap between EIB and BNP with EIB paying floating in Euros and receiving fix sterlings with some foreign exchange rate risk;
3. Mortality swap between the EIB and Partner Re. Concretely there would be floating cash flows which will be received by EIB in exchange of fixed sterling $\hat{S}$ at due payment date each year during 25 years . For BNP this component involves also a credit risk between Partner Re & BNP.

Also the classical longevity bond proposed by Blake(2001) based on coupon payments depend on survivorship rate of the population in the insurance portfolio reference. There is stochastic part on the maturity depending on the last survivor life time.
The third type of longevity regards the zero coupon bond. It works as treasury bill with single coupon at maturity time. There are two caracteristics to be taken into consideration relating to cohort base and to maturity date.

However it results that longevity zero coupon will be illiquid. An example of this type of derivative is the q-forward: It is an agreement between two parties to exchange at a future date(the maturity of the contract) an amount proportional to the realized mortality rate of a given population(or subpopulation) in return for an amount proportional to a fixed mortality rate that has been mutually agreed at inception. Formally a q-forward is a zero coupon swap that exchanges fixed mortality rates for realized mortality at maturity on the reference which is the realized mortality rate. The q-forward are designed in this way in reference to the mortality rate q_x which is the probability of an individual aged x to die within 1 year. The pricing of this instrument is based on a risk premium required by the investor. The spread will be calculated on the difference between effective and the expected mortality. There are two parties involved in this cash flow operation and the operations involves at maturity T that counterparty A pays a notional 100* fixed mortality rate and counterparty B pays realized mortality rate. The q-forward have 2 main functions:

- Hedging mortality risk with q-forwards: mortality risk is the risk that mortality rates to be higher than expected. Thus, If a pension plan enters in q-forward it pays a fixed mortality rate and receive realized mortality rates.
- Hedging longevity risk with q-forward: Longevity risk in contrast is the risk that people survive longer than expected because the mortality rates are lower than expected. Financial institutions which have this type of risk in the liabilities would buy q-forward contract in which they receive fixed mortality rates and pays realized mortality rates. There are still some issues on the development of such a product including mainly basis risk. It reflects the difference in mortality between the population reference(or the exposure) and population which has been associated with the hedging instrument. Basis risk can be managed by actuaries which use efficient tool to minimize it. Based on Life metrics(platform created by JP Morgan), the q forward must observe the 2 criteria of standardization and liquidity.

Other longevity bonds are listed as follows:

The Geared longevity bond and longevity bonds: It facilitates the hedging demands of the investors in a reduced amount . At time t=0, the payment value will be [0,1]*C. At time t, $S(t)$ will be in a range between $[S_l(t), S_u(t)]$ with $u \leq l$ The next step of the geared longevity bond is to set up a SPV(special purpose vehicle) at time zero t=0. The special purpose vehicle (SPV) at time zero holds $S_u(t) - S_l(t)$ units of fixed interest zero coupon bond which matures at time t for each $t = 1, 2, ...T$. Suppose that the SPV is financed by investors A& B then time t SPV will pay: $S(t) - S_l$ to A with a minimum of zero and a maximum of $S_u - S_t$; and $S_u(t) - S(t)$ to B with a minimum of zero and maximum of $S_u - S_l(t)$.

The Deferred longevity bonds or mortality forward contract: The deferred longevity bond is a way of increasing gearing following criticism on EIB longevity bond that the initial coupons are lowly linked to longevity risk(low risk payments linked to initial coupons). A solution of this problem is to add deferments on payments. It has the advantage to save large amount of capital.

Mortality Swaps

This financial derivative introduced by Cox & Lin(2004), Lin & Cox(2005) and Dowd(2006a). It is defined as an agreement to exchange one or more cash flows in the future based on the outcome of the last survivor or mortality index. They have similarities with reinsurance treaties. Many of these products are traded over the Counter(OTC) and the underlying is linked to mortality projections. There are two types of mortality swaps which are:

- One payment mortality swaps: It works similarly to an interest rate swap and involves exchanges between 1 fix part and another floating lag tied to a LIBOR market such as LIBOR

- Vanilla Mortality swap: This mortality product is similar to the vanilla interest rate swap called as IRS where two counterparties are involved: one with fixed interest rate and another with floating leg tied to a market interest rate such as LIBOR. The fixed part of the contract requires preset declining payments related to a survivor index anticipated. The floating part of this depend on realized value of the survivor index at time t.

- Other Mortality swaps: This other type of mortality swap which consists of the exchange of one floating rate payment for another. There are also other types of swap such as: swaps on mortality spreads, cross-currency mortality swap, mortality swaps with embedded options.

Mortality swaps are used in three situations:

- Exchange between 2 institutions one wishing to manage risk on its annuity book and another looking for exposure on longevity risk

- mortality swaps are used potentially by firm to manage the hedge implicit in the exposure on annuity market

- use for speculation object on longevity risk.

Mortality Futures

The basic form of a future asset involves:

- Underlying process $X(t)$ which define also the payoff on the future contract
- The maturity T of the contract set up on the future or a range of delivery dates.

The future marketpricing works as follows:

- Future contract has value equal 0 at time t=0;
- A time t the future price is $F(t,T)$ for delivery of X(T) at time T;
- At time t+1, the future price will be F(t+1,T) and the change in price will be divided in two parts $F(t+1,T)-F(t,T)$ to a long position on short will be the opposite;
- At time T, the final margin at time T will be: $F(T,T)-F(T-1,T)=X(T)-F(T-1,T)$ is equivalent to the expected value of the underlying X(T) in return for the price F(T-1,T). There are also longevity futures or also Annuity Futures.

Mortality Options

There are three types of mortality options:

- Survivor cap and floors they refer to survivor index $S(t)$ as the underlying;

- Annuity future options: It includes options on the annuity futures market;

- OTC options and embedded options.

It could be an agreement between hedgers and counterparties. An example according to Blake(2006b) will be to arrange a survivor cap linked to its portfolio mortality experience. It gives the holder the right to enter into the swap one or other side. It might be American, European or Bermudan. If the underlyinging can be swap such as vanilla mortality swap, the swaption might be payer swaption, giving the holder the right to enter as the fixed rate receiver. The underlying can be of three types: vanilla swap, interest rate swaption as well as mortality swaption(use for risk management purposes).

1.4 Life expectancy modeling

Since the last century, experts have argued that life expectancy is approaching a ceiling(see Dublin(1928, 1990, 2001)). In 1928, Dublin calculated life expectancy could have been reached 64.75 for both males and females. The calculations were based on unsofisticated tool available at that time and no hypothesis on future innovation such as improvements in medicine. This statistic has been contradicted by New Zealand non Maori population with life expectancy of 65.93 years. Dublin, Olshanky et.al(1990, 2001) the assessed life expectancy would be of 85 years without any controlling tool of population aging. However Japan brook this statistic in 1996. Oeppen(2002) show that the two studies cited have been violated with an increasing life expectancy by 5 years. For the USA it has more than doubled to 65 and 70 years respectively for males and females sexes. Moreover lagging countries are catching up the level of advanced countries. In the past 160 years, life expectancy has been increased by 25 years which is great achievement in human longevity. In Lee-Carter(1992), life expectancy have been derived from

mortality indices which the procedure, are described here in (1.32):

$$m_{x,X+s} = m_{x,year} exp(\hat{\beta}_x K_{X+s} - K_{year}) \tag{1.32}$$

X is year taken is arbitrary, s the forecasting window, $\hat{\beta}$ forecast deviation then mortality rates are converted into probability of death in (1.33) such that :

$$q_x = \frac{w_x m_x}{(1 + f'_x w_x m_x} \tag{1.33}$$

where $x = 0, 1, ...$ represents the age, $w_{x,t} = x_{i+1} - x_i$, $k = 22$, $f^{'} = 1 - f_x$,$f_x = 1/2$ for all ages

Let continuing by calculating the probability of surviving from age x into age x+1 as it is described in Rusollilo(2005) in the equation (1.34)

$$p_x = 1 - q_x \tag{1.34}$$

Then from q and arbitrary l_0 we construct the life table as in (1.35) such that:

$$\begin{aligned} l_{x+wx} &= l_x(1 - q_x) \\ d_x &= l_x - l_{x+wx} = l_x q_x \end{aligned} \tag{1.35}$$

where $x = x_0, x_1, ...x_{k-2}$, l_x indicates the number of survivor and d_x the distribution of death

The person-years lived by the life-table population are written in (1.36):

$$T_{xi} = \sum_{x=x_i}^{x_{k-1}} L_x \tag{1.36}$$

and from the latter equation, the life expectancy is given by (1.37):

$$e_{xi} = \frac{T_{xi}}{l_{xi}} \tag{1.37}$$

Russolillo(2005) introduces a new approach by forecasting life expectancy at birth directly using the standard box and Jenkins(1976) by finding the appropriate ARIMA (p,d,q) from time series in both genders males and females the results shows for Italy 1950-2000 in sample life expectancy. Her study shows that the forecast from 2001-2025 life expectancy at birth with ARIMA present better performance than forecasts from the Lee Carter model. Further improvements have been led by Torri(2009), Carter(2010), Ntamjokouen(2014b) describe the cointegration approach to improve not only the accuracy of future life expectancy but also their future trend.

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Chapter 2

Multivariate approach to project the long run relationship of mortality indices

2.1 Introduction and Context

Over the last century, life expectancy of human population has been increasing. People are living longer and safer than expected according to Cairns et al(2006). Furthermore, mortality rates have been declining. These reductions are illustrated by Russolillo and Haberman (2005), Chou-Wen Wang et al. (2011), Lee and Carter(1992), Tulgapurkar et al.(2007) which refer respectively to Italy, Taiwan, USA as well as G7 countries in general. Accordingly, longevity risk is faced by insurers and pension funds as well as by governments in terms of the increasing amounts of annuity paid back to the policyholders as shown by Cocco and Gomes(2012). Cairns and Blake(2008b) have found that mortality rates have been declining at all ages and significant volatility persists over from one year to another. Thus, these improvements in mortality attracted the curiosity of researchers in actuarial science over the world. Many articles have been investigating new modeling methods and

forecasting mortality rates. To illustrate this, Booth and Tickle(2008) review 3 different and broad groups of approaches including: expectation which is based on expert's opinions but which is subjective; explanation based on the use of structural epidemiology models and finally extrapolation, which the core idea is to exploit the pattern of the past to predict the future path of mortality rates as well as age of mortality.

We focus on the latter approach from which the Lee-Carter model(1992) is based. Various extensions of this model have been carried out: The extension by sex(Carter and Lee 1992), disaggregation by cause(Wilmoth 1995), the matching of latest death rates(Lee Carter 1992), by cohort (Renshaw and Haberman 2006), the addition of higher order effect by Gomez de Leon(1990). In addition, Wilmoth(1996) has carried out research on comparison between lagging and leader countries while Villegas et al(2014) analyze the disaggregation by socio-economic factors by using a range of models while the disaggregation by region or provinces in Canada have been examined by Lee and Nault(1993).

The Lee Carter model has the structure of an age period model and it also highlights trends in the evolution of mortality by time. Moreover, Lee-Carter discuss their model for forecasting mortality differential on subpopulation. However, it presents drawbacks such as the uncertainty deriving from errors in the quantification of pattern of coefficients parameters as well as the number of factors to be included. A growing literature, based on the cointegration approach, which tackles these deficiencies, is being explored. Several authors such as Lazar and Denuit(2009), Darkiewicz and Hoedemakers(2004) as well as Li et al(2011) have developed important analysis. Other analyses have been made by Njenga and Sherris(2011) which have focused on the long term relationship of the mortality trends between developed countries by applying the Heligman-Pollard(see Heligman and Pollard(1980)) on mortality rates. They used either the mortality rates for different ages groups and for different countries or mortality indices. Furthermore S.S. Yang et

al(2013) analyse a multi-country longevity risk by using cointegration analysis applied on mortality indices to price longevity derivatives while D'Amato et. al(2013) measure the mortality dependency for multiple countries bu using Sieve bootstrapping. In addition Hatzopoulos and Haberman(2013) and Yang et al(2008) have analyzed multi-countries mortality indices by using principal component analysis. Another research has been done on modeling mortality trend by using solvency regimes(Borger et. al 2011) and copula factors(see Chen et al(2014) and Lin et.al(2013)).

Another growing literature is the use of co-integration techniques for tackling the basis risk issue which arises when an insured portfolio is a subset of a national population. Salhi and Loisel (2010) and Zhou et al(2014) model mortality indices from England & Wales and CMI portfolio by using the cointegration approach. The latter shows that it presents a better performance than the gravity model from Dowd et al(2011). Also Jarner and Kryger(2011) have investigated the basis risk between Denmark and a pool of several countries

Issues come out regarding long term relationship in regions within a country. An application of the long run relationship in regional population data can be applied to Canada where there are 12 provinces. Our goal is to explore evidence of common trend in mortality indices between the 9 most populous provinces in the long term. It is the first study accomplished on the modelling of joint mortality between provinces within a country. Given that studies have been done only on two populations, we want to extend the cointegration approach to more than two. In this way, we take into account of the correlation of mortality improvements accross provinces. Applications on the modelling multipopulation are useful not only for the pricing of annuities, as it is showed in this framework, but also in respect the financial engineering of longevity derivatives(see Yang, 2013).

In order to address these questions, we will apply the cointegration framework for determining whether there is a long term relationship between 2 or

more variables. This econometric technique has been successfully employed in economics and financial times series fields(see Committee, Nobel Prize 2003). Two software like Eviews and R are useful to analyse our data.
Mortality data are provided by the Canadian Human Mortality Database(CHMD) which the website is www.bdlc.umontreal.ca/chmd. It is managed by the Department of Demography of the Université de Montreal in collaboration with the Max Plank Institute for Demographic Research and the Department of demography of the University of California at Berkeley. We have used data from provinces of Canada which include: Prince Edward Island(PEI), Nova Scotia(NS), News Brunswick(NB), Quebec(Q), Ontario(Q), Manitoba(M), Saskatchewan(S), Alberta(A), British Columbia(BC). This database provides information regarding births, deaths, population size, exposure-to-risk, death rates as well as life expectancy at birth. Data are 1 year age and span through 1921 and 2009. In the first section, we retrieve from the Lee Carter model the mortality indices from each individual provincial mortality data. The mortality index(figures 2.19 and 2.20) shows a decreasing trend for all the provinces from 1921 to 2009. Observing the figures of mortality indices, decreasing trends are amplificated from 1970 onwards. We can see naturally common trends between the provinces since they share the same tendency to decline over the years. However the Prince Edward Island mortality index show a different pattern from others even if it shows a decreasing trend. Overall, it can be observed that mortality is decreasing in all of the provinces of Canada during the period 1921 to 2009. The methodology, which applies the cointegration, includes several steps as follows:
- the determination of order of integration for each of the 9 mortality indices using the Augmented Dickey Fuller, Philips-Perron as well as KPSS Test;
- the computation of the optimal value of lag of the Vector Autoregressive model;
- the Johansen cointegration test which investigates the cointegration rank and specify which variable will enter in the cointegrated equations and in the

Vector Error Correction model;
- the estimation of VECM and the VAR models and the forecasting of derived model.
- furthermore, third part will be reserved to the comparison between the VAR model forecasting and the VECM relatively to their goodness of fit and robustness. Finally these models will be applied to calculate actuarial annuity value by group of cohorts, particularly from 1960, 1970, 1980, 1990 as well as 2000 although it can be extended to more cohorts.
The detailed steps of cointegration analysis are described by Juselius(2007), Harris and Sollis(2002) as well as Lutkepohl(2005). The empirical analysis will be done by using R statistical software package developed by Pfaff(2008).

2.2 Cointegration analysis: methodology and theory

In this case study we consider mortality indices by provinces in Canada taken from Canadian Human Mortality Database which provides data on Death and Exposure for both male and female. We compute them by using the quotient between these two variables. Data are available for 9 provinces from 1921-2009. North Territories and Yukon data start from 1950 to 2009. NewFoundland data start from 1949 extend until 2009. We exclude these three provinces because of the mismatching of the sample with the 9 others. We compute the ADF by using the Eviews 7.0. In our study, we define $m_{t,x}$ to be a mortality rate at age x at time t in 2.1 by:

$$m_{x,t} = \frac{d_{x,t}}{e_{x,t}} \tag{2.1}$$

where t=1,2,....T $d_{x,t}$ is the number of deaths during calendar year t aged x at time last birthday $e_{x,t}$ is the average population during calendar year t

aged x at time last birthday

2.2.1 Unit root Test

Preliminary visualization of plots from mortality indices show improvements for each province of Canada as they present decreasing trend. Bigger size provinces such as Alberta, British Columbia, Nova Scotia, Ontario as well as Quebec show up improvements in mortality 1970 onward.
In order to test the non stationarity of provincial mortality, we perform the ADF test, the PP test as well as the KPSS test for the mortality indices k_t derived from the Lee Carter model. In the case study, mortality indices for all provinces as well as national level, under ADF test, are all integrated of order 1 as p-value of these tests at the levels is greater than 0.05. Further, the first difference computed present values which are less than 0.05, these empirical results lead us to conclude that all provinces are integrated of order 1. We compute also in tables 2.13 and 2.16 the Durbin Watson statistics and their lag length and most of them are nearly 2. The PP test confirms the same results as ADF for provinces and Canada. The results show that the mortality index are integrated of order 1. The results are summarized in tables 2.14 and 2.17. Finally, we performed the KPSS test for both sexes males and females. The critical statistics are indicated in tables 2.15 and 2.18. All Provinces including Alberta, British Columbia, Manitoba, New Brunswick, Nova Scotia, Ontario, Prince Edward Island, Quebec and Sakastchewan are integrated of order I(1) according to the criteria of constant.
Before starting analysis, we want to explore briefly the decomposition of the variances from mortality rates dataset. We undertake to model the Lee Carter Model for 9 most populated Canadian provinces which include: Alberta, British Columbia, Manitoba, New Brunswick, Nova Scotia, Ontario, Prince Edward Island, Quebec, Sakastchewan. We describe the mortality indices from the provinces. Let's consider n=1...9 canadian regions, then from

the Lee carter model we obtain the 9 mortality indices $k = k_1, ...k_9$ from this decomposition. The following equations(2.2 to 2.6) described the Lee carter model's parameters from each province mortality experience:

$$ln\ m_1(t,1) = a_{1,x} + b_1 k_{1,t} + e_{1,t} \tag{2.2}$$

$$ln\ m_2(t,1) = a_{1,x} + b_2 k_{2,t} + e_{2,t} \tag{2.3}$$

$$ln\ m_3(t,1) = a_{1,x} + b_3 k_{3,t} + e_{3,t} \tag{2.4}$$

$$..... \tag{2.5}$$

$$ln\ m_9(t,x) = a_{9,x} + b_{9,x} k_{9,t} + e_{9,t} \tag{2.6}$$

Since the equations cannot be fitted by ordinary least square(OLS) estimators because the components are not observable, the Singular Value Decomposition(SVD) is used to find the various parameters of the model. In order to obtain a particular solution and overcome the identifiability problem, 2 constraints have to be set up in (2.7) as follows :

$$\sum b_x = 1, \sum k_t = 0 \tag{2.7}$$

2.2.2 VAR and VECM Theory for Modelling Mortality Indices

The Vector Autoregression(hence VAR) is a valid representation of a multivariate model as Lutkepohl(2005) describes in (2.8) in general as:

$$k_t = \nu + \eta_1 k_{t-1} + \eta_1 k_{t-2} + ...\eta_p k_{t-p} + e_t \tag{2.8}$$

where $k_t = (k_{1,t}, k_{2,t}, ...k_{N,t})'$, is a N-dimentional time series, η_i are matrix with coefficient $(S * S)$, $\nu = (\nu_1, \nu_2, ...\nu_p)'$ is the intercept term, e_t is the residuals part with white noise and $t = 0, 1, ...T$ and p the maximum lag

order.

For estimating the VAR with the objective of making prediction, we need to select the optimal lag p which corresponds to the best predictive model with least error. This particular lag length of variables in the VAR model (and the VECM see later) is derived by choosing the order p which minimizes the value of the information criteria model such as Akaike(AIC), the HQ(Hannan-Quinn), the Schwarz(SC) as well as Final Prediction criteria(FPE) as stated here below in the equations (2.9), (2.10), (2.11) and (2.12):

$$AIC(p) = log\ det(\sum(p)) + \frac{2}{T}pK^2 \tag{2.9}$$

$$HQ(p) = log\ det(\sum(p)) + \frac{2logT}{T}pK^2 \tag{2.10}$$

$$SC(p) = log\ det \sum(p) + \frac{logT}{T}pK^2 \tag{2.11}$$

$$FPE(p) = \frac{T+p^*}{T-p^*})(\frac{T+p^*}{T-p^*})^K det\ \sum(p) \tag{2.12}$$

where $\sum p$ is estimated by $T^{-1}\sum_{i=1}^{n} u_t u_t^{'}$, p^* is the total number of parameters in each equation of the model when p is the lag order of the endegeneous variables.

The different information criteria may identify different choices for the lag order p. However, Lutkepohl(2005) suggests that, in the case where the criteria choose different lags, SC criteria is the more parsimonious in choosing the good forecasting models than others because it produces the lowest forecast error variance. p_* is the local number of the parameters in each equation and p assigns the lag order VAR(p) is said to be stationary if and only if : $det(I_k - A_1 z - .... - A_p z^p) \neq 0$ with $-1 \leq z \leq 1$. If $z = 1$ then at least one of these variables inserted into VAR(p) are integrated of order 1 and therefore cointegration does exist between variables. This conclusion introduces the Vector Error Correction Model(VECM). According to Pfaff(2008) and Engle(1987), the VAR (p) in the equation (2.7) can be converted into a

VECM.There are two versions of this model: the short run version and the long run version. The latter form of VECM, which will be used in our case studies is defined, in (2.13) as follows:

$$\Delta k_t = \Gamma_1 \Delta k_{t-1} + \Gamma_2 \Delta k_{t-2} + ... + \Gamma_{p-1} \Delta k_{t-p+1} + \Pi k_{t-p} + \nu + \varepsilon_t \quad (2.13)$$

where $\Gamma_i = -(I - \eta_1 + ... - \eta_i)$, for $i = 1, ...p-1$ and $\Pi = -(I - \eta_1 - ..... - \eta_p)$.

The forecasts of mortality indices are yielded through the Johansen Maximum Likelihood methodology. The decisions on whether there are cointegrated equations between variables are taken according to the following hypotheses.
If $r = K$, the number of cointegrated variables, r, which is stationary equals the rank(K) of Π and then the model will be estimated by using the standard statistical model.
If $r = 0$, this means that there are no cointegrated relationships between the variables. The variables are stationary if we take the differences of variables above.
If $0 < r < K$ there exists 2 matrices, α and β, such that $\Gamma = \alpha\beta'$, there will be r cointegrating relationship or $n - r$ common trends. The test of cointegration is reduced to the two following hypotheses:
The rank test, which is specified by the hypotheses in ((2.14) as follows:

$$H_0 : rank(\Pi) = r, H_1 : rank(\Pi) > r \quad (2.14)$$

and the likelihood ratio value of such tests are measured in ((2.15)):

$$\begin{aligned} LR(r) &= -(T-p)\sum(1-\lambda_i) \\ LR(r) &= -(T-p)\sum(1-\lambda_i) \end{aligned} \quad (2.15)$$

where T is the length size of the sample and r equals the number of cointegrated relationships, $\lambda_1 > \lambda_2 > .... > \lambda_i$ are the eigenvalue that are associated with the linear relationship

The decision for the cointegration rank is based on the trace test and the maximum eigenvalue test of Johansen(1988 and 1991). In addition, the test on maximum eigenvalue is specified here in ((2.16)) as follows:

$$H_0 : rank(\Pi) = r, H_1 : rank(\Pi) = m + 1 \tag{2.16}$$

The statistic value of this test is specified in ((2.17)):

$$LR(r) = -(T - p) \sum ln(1 - \hat{\lambda}_{m+1}) \tag{2.17}$$

2.2.3 Diagnostic of residuals

Vector Error Correction and the VAR models have to be subjected to the diagnostic tests on residuals. Three main tests have been developed over the years: test for autocorrelation, conditional heteroskedasticity as well as autocorrelation in Lutkepohl(2005).

This Portmanteau test is run for deciding whether residuals are aucorrelated with 2 alternatives such that:

$H_0 : E(u_t u_{t-i}) = 0$

$H1$: *At least one autocovariance between error terms is non zero*

The decision will be made on the basis of p-values. If p-value is greater than 0.05 then we cannot reject the null hypothesis and the residuals are not autocorrelated(see tables 2.2, 2.3, 2.7 and 2.8). We perform the results for optimal VAR lag choosen above.

LM Test The Breusch Pagan(1979) test for the autocorrelation of residuals of order h is carried out. The related model which considers coefficients equals zero as in (2.18) is based on:

$$u_t = B_1 u_{t-1} + ...B_h u_{t-h} + \varepsilon_t \tag{2.18}$$

with ε_t is considered as white noise error term. The Breusch Pagan test is specified as follows:

$H_0 : B_1 = B_2 = ...B_h = 0,$

$H_1 : B_n \neq 0 for\ at\ least\ one\ n = 0, 1, ...h$

This test is based on the idea of Jarque and Berra(1987) for univariate time series which has been extended to multivariate time series. They are useful to check the skewness and the Kurtosis of standardized residuals errors. The test of normality is based on these two latter indicators. In our framework we will perform the normality test based on Cholesky decomposition of residuals(see Lutkepohl,2005). The hypothesis formulated are presented here in (2.19) below:

$$H_0 : \varepsilon_t\ are\ normal, H_1 : \varepsilon_t\ are\ non\ normal \tag{2.19}$$

2.3 Estimation procedure of the two models

In order to start the estimation of the models based on the Canadian provincial mortality indices, we performed the Augmented Dickey Fuller(see Dickey and Fuller(1979)), the Philips Perron (see Philipps and Perron(1988)) as well as the KPSS(Kwiatkowski et.al(1992)) tests for the mortality indices k_t derived from the Lee Carter model. These tests allow us to determinate whether the variables are stationary or no also their level of stationarity. Further, the first difference tests under the three tests are also perfomed. The first difference from the mortality index shows that the mortality index are stationarity. The PP and KPSS tests confirm the same results as the ADF. The critical statistics indicate that the mortality indices are integrated of order 1 for the three tests performed above. Finally on the basis of the integration analysis we deduce from the empirical results that the mortality index from provinces for both genders females and males, including Alberta(A), British Columbia(BC), Manitoba(M), New Brunswick(NB), Nova Scotia(NS), On-

tario(O), Prince Edward Island(PEI), Quebec(Q) and Sakastchewan(S), are integrated of order 1 or I(1).

2.4 Cointegration analysis of Mortality rates for the Canadian provinces

Following Pfaff(2008), we investigated the VAR optimal lag for the two sex groups by selecting up to 6 lags. Given these assumptions, we compute the lag length for both sexes. We observe that the AIC indicates the optimal lag to be 6 for both females and males whereas other the information criterion indicate only 1 optimal lag. Information criteria can identify the same lag or different levels. However, Lutkepohl(2005) suggests that in case the criteria identify different lags, SC criteria is the more parsimonious in choosing the good forecasting models than others because it produces the lowest forecast error variance. Accordingly if our intent is to estimate the variables included in a VAR model, we select the best order p to get the best predictive model with the least error. Following Lutkepohl(2005), the optimal lag length is 1 as suggested by SC criteria, as in table 2.1, for both males and females because SC presents the smallest information criterion values.

Sex	AIC	HQ	SC	FPE
Males	6	1	1	1
Females	6	1	1	1

Table 2.1: The diagnostics tests of residuals under VAR model

As we can observe, the mortality index of Alberta depend mainly on the first lag of mortality index of Alberta(coefficient is 0.43) and also on the Quebec provinces(coefficient is 0.10) and Manitoba(0.09). There exists also

negative relationship with Sakastchewan and Prince Edward Island with coefficient respectively -0.22 and -0.11. In the same way, we can also explain the mortality index from other provinces.

The model involves also a trend component denoted as λ:

$$\begin{bmatrix} k_{A,t} \\ k_{BC,t} \\ k_{M,t} \\ k_{NB,t} \\ k_{NS,t} \\ k_{O,t} \\ k_{PEI,t} \\ k_{Q,t} \\ k_{S,t} \end{bmatrix} = \begin{bmatrix} 11.74 \\ 8.59 \\ 10.93 \\ 9.46 \\ -8.55 \\ .096 \\ -12.76 \\ -0.37 \\ 7.91 \end{bmatrix} + \begin{bmatrix} .43 & -.03 & .09 & .03 & .06 & .30 & -.11 & -.10 & -.22 \\ .0008 & .57 & .12 & .19 & .08 & .50 & .0035 & -.25 & -.40 \\ 041 & .15 & -.11 & .30 & .10 & .22 & .01 & -.07 & .02 \\ .048 & .10 & .03 & .28 & .28 & -.09 & .10 & .17 & -.13 \\ .34 & -.04 & .26 & .05 & -.54 & .26 & 0.23 & .10 & .10 \\ .14 & .20 & -.12 & -.06 & .01 & .86 & .08 & -.001 & -.17 \\ .038 & -.07 & .07 & .23 & -.08 & .31 & .15 & .32 & -.32 \\ .20 & .09 & -.05 & .04 & .10 & .09 & .14 & .701 & -.03 \\ .14 & .06 & .17 & .32 & 0.25 & 0.12 & -.06 & -.12 & -.04 \end{bmatrix} \begin{bmatrix} k_{A,t-1} \\ k_{BC,t-1} \\ k_{M,t-1} \\ k_{NB,t-1} \\ k_{NS,t-1} \\ k_{O,t-1} \\ k_{PEI,t-1} \\ k_{Q,t-1} \\ k_{S,t-1} \end{bmatrix}$$

$$+ \lambda \begin{bmatrix} -.26 \\ -.227 \\ -.26 \\ -.23 \\ .13 \\ -.05 \\ 0.34 \\ -0.05 \\ -.18 \end{bmatrix}$$

The same computations are done for females and the results are the following:

$$
\begin{bmatrix} k_{A,t} \\ k_{BC,t} \\ k_{M,t} \\ k_{NB,t} \\ k_{NS,t} \\ k_{O,t} \\ k_{PEI,t} \\ k_{Q,t} \\ k_{S,t} \end{bmatrix} = \begin{bmatrix} 40.28 \\ 27.64 \\ 15.28 \\ -8.71 \\ 25.71 \\ 49.29 \\ 12.79 \\ 33.59 \\ 38.54 \end{bmatrix} + \begin{bmatrix} .46 & -.13 & .10 & .11 & .049 & .11 & -.005 & -.15 & -.05 \\ -.14 & .25 & .24 & .10 & .08 & .07 & .02 & .21 & -.17 \\ .14 & .006 & -.48 & .38 & .058 & .18 & .08 & -.35 & -.14 \\ .31 & .14 & .16 & .16 & .10 & -.09 & .18 & .20 & -.04 \\ .25 & -.025 & .13 & .061 & .19 & -.08 & 0.11 & .30 & -.32 \\ .06 & .08 & .19 & .002 & .047 & .31 & .02 & -.0007 & -.26 \\ -.18 & -.02 & .67 & .10 & .15 & -.008 & .04 & -.08 & -.11 \\ .005 & .24 & -.015 & .13 & .04 & .04 & .01 & .46 & -.21 \\ .06 & -.04 & .47 & .12 & -.02 & -0.15 & -.04 & -.15 & -.25 \end{bmatrix} \begin{bmatrix} k_{A,t-1} \\ k_{BC,t-1} \\ k_{M,t-1} \\ k_{NB,t-1} \\ k_{NS,t-1} \\ k_{O,t-1} \\ k_{PEI,t-1} \\ k_{Q,t-1} \\ k_{S,t-1} \end{bmatrix}
$$

$$
+ \lambda \begin{bmatrix} -.91 \\ -.63 \\ -.39 \\ -.14 \\ .58 \\ -1.12 \\ 0.21 \\ -0.79 \\ .88 \end{bmatrix}
$$

In the following figures, we show the diagram of residuals for Alberta province for both males and females respectively. VAR model in both cases fits very well. Also, the ACF and PACF (from figures 2.1 and 2.2) are appropriated to capture the autocorrelation structure of the residuals. This is confirmed by the test statistics presented in tables 2.2 and 2.3, where the Portmanteau(autocorrelations of residuals) test and Jarque & Berra(normality test for residuals) test are all significant. Additionally, the same results can be observed for males. Computations of the residuals for diagnostics purposes has been done from others provinces but in order to save space we will not show the graphs derived from the other provinces.

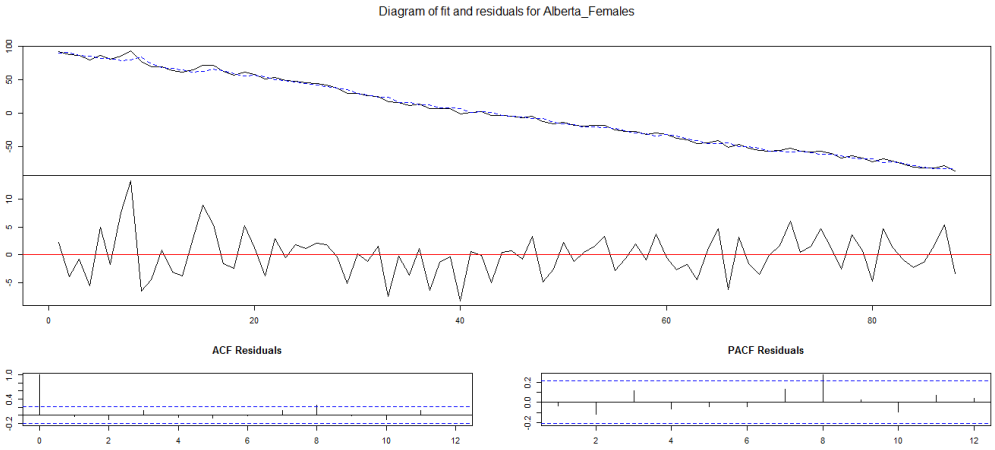

Figure 2.1: Diagnostics of residuals with reference to females for Alberta

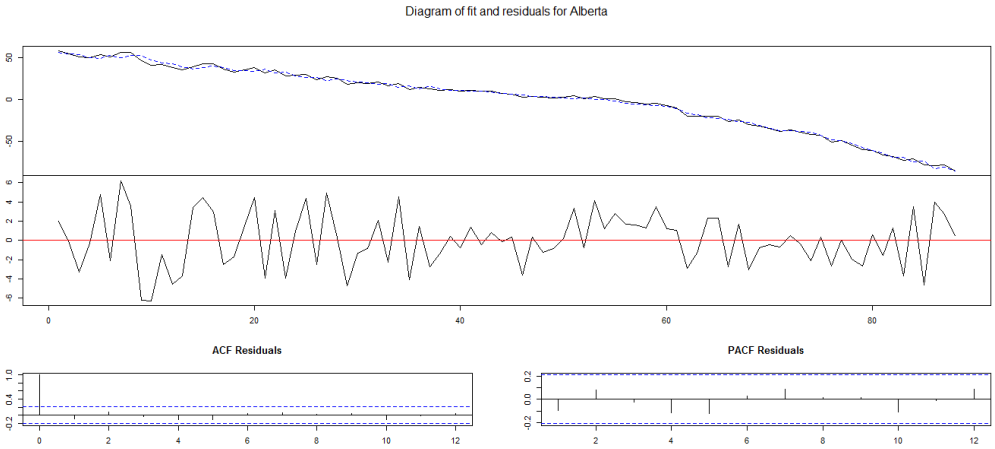

Figure 2.2: Diagnostics of residuals with reference to males in Alberta

Females	Type of test	Pvalue	Chi-squared
Autocorrelation test	Portmanteau test	0.68	1190.60
Normality	JB Multivariate	0.31	20.36
	Skewness	0.17	12.66
	Kurtosis	0.56	7.70

Table 2.2: Diagnostic test indicators for females: Portmanteau test for Autocorrelations, Normality test with VAR(1)

Males	Type of test	pvalue	Chi-squared
Autocorrelation test	Portmanteau test	0.81	1171
Normality	JB Multivariate	0.18	23.07
	Skewness	0.88	4.34
	Kurtosis	0.02	18.72

Table 2.3: Diagnostic test indicators: Portmanteau test for Autocorrelations, Normality test

2.4.1 Estimation of the Vector of Error Correction Model and diagnostic test

The Vector Error correction model is fitted for the 9 most important provinces over the period from 1921 to 2009. We performed the trace test and the Eigenvalue maximum for the cointegrated equations. We test the Johansen Cointegration on the mortality levels for Canadian provinces which are all integrated of order 1. We find out that there are 5 cointegrated equations at the 5% significant level driven by 4 common stochastic trends(see table 2.4 below). They are estimated by using the maximum likelihood estimators and the results are presented here below for females

The p-values show that the number of cointegrated equations are 5 for females derived from the VECM. For any r, the test value is less than the

r	test value	5% critical values	10% critical values	1% critical values
$r <= 8$	3.34	9.24	7.52	12.97
$r <= 7$	11.38	19.96	17.85	24.6
$r <= 6$	25.50	34.91	32	41.07
$r <= 5$	46.40	53.12	49.65	60.16
$r <= 4$	84.23	76.07	71.86	84.45
$r <= 3$	127.73	102.14	97.18	111.01
$r <= 2$	175.99	131.7	126.58	143.09
$r <= 1$	229.25	165.58	159.48	117.2
r = 0	300.68	202.92	196.37	215.74

Table 2.4: Evidence of the cointegrated equations for Canadian female provincial mortality level

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critical values then the corresponding r represents the number of cointegrated equations. For example, from r=0 to r=4 there are no cointegrated equation. The r=5 test value equals 46.40 is less than critical value(53.12), the number of cointegrated equations is 5 at 5% significance level. These equations measure the long run relationship between the 9 variables along the period 1921 to 2009.

The equation in matrix form below for females is the Vector Error Correction form which explains the variation of the improvements mortality indices by pattern observed from other provinces at the first lag level and at the variation of the first lag level is given below. The variation in Alberta is explained by others provinces in their first lag level(coefficients are -0.38, -0.06, 0.06, 0.13 and so on) and also by the first difference of each province mortality as can be observed from the coefficient of the matrices(coefficients are -0.12, 0.07, 0.03 -0.13 and so on). The same interpretation can be applied

for the other provinces.

$$
\begin{bmatrix} \Delta k_{A,t} \\ \Delta k_{BC,t} \\ \Delta k_{M,t} \\ \Delta k_{NB,t} \\ \Delta k_{NS,t} \\ \Delta k_{O,t} \\ \Delta k_{PEI,t} \\ \Delta k_{Q,t} \\ \Delta k_{S,t} \end{bmatrix} = \begin{bmatrix} -.12 & .07 & .03 & -.13 & -.13 & .08 & .08 & -.03 & -.08 \\ -.01 & -.09 & .01 & -.005 & .01 & .48 & .48 & -.26 & -.05 \\ -.06 & -.017 & .16 & -.18 & .003 & .34 & -.06 & -.23 & -.16 \\ -.13 & -.24 & .20 & -.30 & .03 & -.01 & .01 & .17 & -.07 \\ .17 & .07 & -.04 & -.56 & -.23 & .17 & -.05 & .24 & .16 \\ -.001 & .04 & .13 & -.10 & -.06 & -.24 & .02 & -.02 & .04 \\ .40 & -.21 & .20 & .13 & -.20 & -.02 & -.24 & -.25 & .14 \\ -.05 & 0.06 & .12 & -.14 & -.10 & .05 & -.05 & -.23 & -.08 \\ .12 & .07 & .14 & -.08 & -.03 & -.17 & -.01 & .15 & -.32 \end{bmatrix} \begin{bmatrix} \Delta k_{A,t-1} \\ \Delta k_{BC,t-1} \\ \Delta k_{M,t-1} \\ \Delta k_{NB,t-1} \\ \Delta k_{NS,t-1} \\ \Delta k_{O,t-1} \\ \Delta k_{PEI,t-1} \\ \Delta k_{Q,t-1} \\ \Delta k_{S,t-1} \end{bmatrix} +
$$

$$
\begin{bmatrix} -.38 & .06 & .06 & .13 & .01 & .34 & -.24 & -.08 & .12 \\ .04 & -.25 & 0.08 & .20 & .03 & .30 & -.12 & -.09 & -.29 \\ .26 & .27 & -1.38 & .44 & -.02 & .12 & .01 & -.06 & .23 \\ .46 & .18 & -.23 & -.41 & .14 & -.13 & -.13 & -.01 & -.12 \\ .25 & -.19 & -.32 & .44 & -.35 & .52 & .24 & -.19 & -.33 \\ .19 & .16 & -.24 & .02 & -.01 & .01 & .03 & -.04 & -.16 \\ -.19 & .06 & -.03 & .09 & .24 & -.58 & .36 & -.48 & .002 \\ .29 & .04 & -.18 & .13 & .07 & .20 & .15 & -.32 & .31 \\ .23 & .04 & -.9 & .40 & -.05 & .17 & -.07 & .17 & -.72 \end{bmatrix} \begin{bmatrix} A(-1) \\ BC(-1) \\ M(-1) \\ NB(-1) \\ NS(-1) \\ O(-1) \\ PEI(-1) \\ Q(-1) \\ S(-1) \end{bmatrix} + \begin{bmatrix} -.77 \\ -1.11 \\ -1.82 \\ -2.42 \\ -2.83 \\ -2.50 \\ 1.96 \\ -3.56 \\ -.94 \end{bmatrix}
$$

The trace test reveals 5 cointegrated equations between the various females mortality indices which have described in the following 5 equations:

$$Z_{A,t-1} = k_{A,t-1} - .20k_{BC,t-1} - .27k_{M,t-1} + .04k_{NB,t-1} - 1.06k_{NS,t-1} - .08k_{O,t-1} + .85k_{PEI,t-1} + .69k_{Q,t-1} - .83k_{S,t-1} - 8.38$$

$$Z_{BC,t-1} = -.39k_{A,t-1} + k_{BC,t-1} + 1.10k_{M,t-1} - 1.24k_{NB,t-1} + 1.09k_{NS,t-1} - .11k_{O,t-1} + -.93k_{PEI,t-1} - .75k_{Q,t-1} + .20k_{S,t-1} - 3.18$$

$$Z_{M,t-1} = 3.86k_{A,t-1} + 2.19k_{BC,t-1} + k_{M,t-1} - 6.66k_{NB,t-1} - .59k_{NS,t-1} - 5.69k_{O,t-1} + 3.04k_{PEI,t-1} + 4.11k_{Q,t-1} - .35k_{S,t-1} - 2.93$$

$$Z_{A,t-1} = .79k_{A,t-1} + 2.53k_{BC,t-1} - 1.64k_{M,t-1} + k_{NB,t-1} - 1.53k_{NS,t-1} - .18k_{O,t-1} + -.27k_{PEI,t-1} - 1.14k_{Q,t-1} + .43k_{S,t-1} - .49$$

$$Z_{A,t-1} = .58k_{A,t-1} + .74k_{BC,t-1} - 2.04k_{M,t-1} + .11k_{NB,t-1} + k_{NS,t-1} - .012k_{O,t-1} + .53k_{PEI,t-1} - 1.19k_{Q,t-1} + .62k_{S,t-1} - 6.69$$

Here $Z_{i,t}$ represents the stationary variable which quantifies the deviation from the equilibrium of various mortality indices analyzed. Changes in each provincial mortality are reflected in $Z_{i,t}$ which involves as we said above the changing trends in mortality indices.
The normalization is used in each equation for all the 5 first variables(Alberta, British Columbia, Manitoba, New Brunswick and Nova Scotia) such that the coefficient for the first 5 variables in each equation is equal to 1. The model measures the variations in trends of mortality indices. The VECM shows a clear dependence between province's mortality indices over the past 90 years. It takes into consideration also the cross correlation between the 9 Canadian provinces. This relationship can be translated into the mortality rates and it shows accordingly also a dependence between the provinces.

r	test value	5% critical values	10% critical values	1% critical values
$r <= 8$	3.55	9.24	7.52	12.97
$r <= 7$	8.85	19.96	17.85	24.6
$r <= 6$	19.96	34.91	32	41.07
$r <= 5$	34.46	53.12	49.65	60.16
$r <= 4$	53.94.23	76.07	71.86	84.45
$r <= 3$	96.19	102.14	97.18	111.01
$r <= 2$	140.52	131.7	126.58	143.09
$r <= 1$	195.07	165.58	159.48	117.2
r = 0	264.02	202.92	196.37	215.74

Table 2.5: Evidence of the cointegrated equations for males Canadian provincial mortality level: Male experience

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The cointegration equations shows the dependence of each province with others. In the first equation, for both males and females genders, it's obvious to see that there is an interdependence between mortality indices across Canadian provinces. Since this study is carried on the mortality indices of different provinces, it implies also that there exist a dependence between mortality rates(from which the mortality indices are derived) of Canadian provinces. The mortality improvement in one province is not only influenced by the lagged mortality index of the same province but also by improvement from another province.

In table 2.5 we show results from the Johansen cointegration test. Careful analysis of table highlights 3 cointegrated equations for the 9 various groups. We computed also the number of common factors for each group and we find that all these groups present more than 1 common factor. The cointegration approach involves more common trends in modelling mortality indices.

$$\begin{bmatrix} \Delta k_{A,t} \\ \Delta k_{BC,t} \\ \Delta k_{M,t} \\ \Delta k_{NB,t} \\ \Delta k_{NS,t} \\ \Delta k_{O,t} \\ \Delta k_{PEI,t} \\ \Delta k_{Q,t} \\ \Delta k_{S,t} \end{bmatrix} = \begin{bmatrix} -.02 & .28 & -.19 & -.02 & -.17 & .19 & .12 & -.01 & -.19 \\ .23 & -.05 & .09 & .03 & -.15 & .03 & -.05 & -.14 & -.04 \\ .19 & .21 & -.009 & -.10 & -.09 & .21 & -.03 & .05 & -.26 \\ .23 & -.11 & -.11 & .005 & -.17 & .33 & -.07 & -.08 & -.13 \\ .05 & .16 & -.05 & -.16 & -.06 & -.18 & -.14 & -.23 & .06 \\ -.01 & .13 & .11 & .04 & .03 & -.41 & .01 & -.06 & -.07 \\ .48 & -.33 & -.08 & -.10 & -.11 & -.26 & -.09 & -.49 & -.21 \\ .12 & .09 & .07 & .06 & -.09 & -.06 & .01 & -.31 & -.16 \\ -.06 & .41 & -.10 & -.04 & -.21 & .14 & -.02 & .04 & -.33 \end{bmatrix} \begin{bmatrix} \Delta k_{A,t-1} \\ \Delta k_{BC,t-1} \\ \Delta k_{M,t-1} \\ \Delta k_{NB,t-1} \\ \Delta k_{NS,t-1} \\ \Delta k_{O,t-1} \\ \Delta k_{PEI,t-1} \\ \Delta k_{Q,t-1} \\ \Delta k_{S,t-1} \end{bmatrix} +$$

$$\begin{bmatrix} -.53 & -.35 & .20 & .01 & .26 & .28 & -.17 & .04 & .20 \\ -.21 & -.77 & .14 & .01 & .26 & .27 & .005 & .26 & -.04 \\ .02 & -.13 & -.54 & .38 & .21 & .21 & .06 & -.25 & .03 \\ .23 & .16 & .22 & -.92 & .28 & -.32 & .22 & .22 & .01 \\ .31 & -.20 & .02 & .06 & -.81 & .19 & .24 & .46 & -.32 \\ .06 & -.02 & .07 & -.08 & .04 & -.08 & .001 & .11 & -.11 \\ -.44 & .17 & .60 & .13 & .35 & .32 & -.92 & -.48 & .002 \\ -.07 & .08 & -.03 & .01 & .14 & .36 & .04 & -.36 & .02 \\ -.01 & -.42 & .44 & .007 & .23 & .08 & .03 & .08 & -.48 \end{bmatrix} \begin{bmatrix} A(-1) \\ BC(-1) \\ M(-1) \\ NB(-1) \\ NS(-1) \\ O(-1) \\ PEI(-1) \\ Q(-1) \\ S(-1) \end{bmatrix} + \begin{bmatrix} -.82 \\ -1.79 \\ -2.42 \\ -2.67 \\ -2.98 \\ -2.66 \\ 2.17 \\ -3.42 \\ -2.09 \end{bmatrix}$$

$Z_{A,t-1} = k_{A,t-1} + 0.38k_{BC,t-1} - 2.35k_{M,t-1} + 0.75k_{NB,t-1} - 0.08k_{NS,t-1} + 0.47k_{O,t-1} + 0.56k_{PEI,t-1} - 0.67k_{Q,t-1} + 0.12k_{S,t-1} - 8.81$

$Z_{BC,t-1} = 1.70k_{A,t-1} + k_{BC,t-1} - 12.70k_{M,t-1} + 5.44k_{NB,t-1} + 1.61k_{NS,t-1} + 0.03k_{O,t-1} + +1.32k_{PEI,t-1} - 0.48k_{Q,t-1} + 5.73k_{S,t-1} - 12.54$

$Z_{M,t-1} = -67.13k_{A,t-1} - 23.61k_{BC,t-1} - 12.70k_{M,t-1} + 5.44k_{NB,t-1} - 1.61k_{NS,t-1} - 0.03k_{O,t-1} + 1.32k_{PEI,t-1} - 0.48k_{Q,t-1} + 5.73k_{S,t-1} + 12.54$

Here $Z_{i,t}$ represents the stationary variable which quantifies the deviation from the equilibrium of various mortality indices analyzed. The normalization is used in each equation for the first 3 variables(Alberta, British

Columbia and Manitoba) such that the coefficient for the variables is equal to 1.

Sex group	cointegrated equation	common factors
Indices	Trace — Eigen	Trace — Eigen
Females	5 — 5	4— 4
Males	3 — 4	6—5

Table 2.6: Summary of the cointegration Johansen Test

We experiment with the Johansen test for trace test and Eigenvalue test on the 9 mortality indices which are integrated of order 1. The results are summarized in table 2.6. Females mortality indices identify 5 cointegrated equations based on the trace and eigenvalue tests. Similar analysis for mortality indices for males shows 3 and 4 for the trace and eigenvalue respectively. It can be deduced that there are 4 common stochastic trends in the case of females. The analysis for males analysis indicates rather 6 and 5 common factors using trace and eigenvalues test. To sum up, it's clear that there exists in both tests trace and eigen common trends and dependency of mortality indices(mortality rates) between provinces within Canada.
In the next section of this paper, we focus only on the results based from the trace test to interpret the diagnostics of residuals, compute forecasts and applications to annuities. Obviously, the same procedure can be accomplished with the eigenvalue test.
In conclusion, The Portmanteau test, as well as Jarque Berra number(JB) obtained from the normality test suggests that Vector Error Correction model satisfy conditions for forecasting mortality indices for both sexes and for different age groups studied. Assuming that the relationship holds in the future, the dynamic VECM can be use appropriately to forecast future mortality indices as diagnostic test show up robust results with diagnostic testing. The decision in taken by looking at p-value. Results for the three tables have

been summarized into the following tables 2.7 and 2.8.

Females	Type of test	Pvalue	Chi-squared
Autocorrelation test	Portmanteau test	0.75	1372.56
Normality	JB Multivariate	0.16	25.95
	Skewness	0.062	17.6
	Kurtosis	0.59	8.34

Table 2.7: Diagnostic test indicators: Portmanteau test for Autocorrelations, Normality test for females

Males	Type of test	Pvalue	Chi-squared
Autocorrelation test	Portmanteau test	0.97	1308.60
Normality	JB Multivariate	0.04	32.27
	Skewness	0.17	14.01
	Kurtosis	0.0507	18.26

Table 2.8: Diagnostic test indicators: Portmanteau test for Autocorrelations, Normality test for males

For the males p-values are greater than 0.05, so we cannot reject the null hypothesis for each test. Accordingly, residuals are non autocorrelated and normal. In the same way, we analyze the males group. The Jarque and Berra p-value normality test is equal to $JB = 0.04$ but the skewness and the kurtosis are significant. The residuals are non autocorrelated. The two tests are valid and well behaved for forecasting mortality indices for Males even if there exists a non normality of residuals for males.

2.4.2 Backtesting of the Models VAR and VECM

We have backtested the models VAR and VECM by using the following out-of-sample model 2005-2009, 2003-2009, 2000-2009, 1990-2009, 1995-2009, 1985-2009 for both sexes males and females. As to females, it is clear that VECM is more precise than VAR models for all the samples cited above. Regarding the males groups it seems uncertain which model is better for the sample 2005-2009, 2003-2009 and 2000-2009. For other intervals, VECM model appears more robust than VAR model. We summarize the general table 2.9 here.

Sex group	Females	Males
Out-of-samples	VAR — VECM	VAR — VECM
h=2005-2009	5.63% — 5.13%	6.85%— 5.73%
h=2002-2009	6.66% — 6.52%	9.47%—10.96%
h=2000-2009	12.89%—7.43%	8.42%—22.91%
h=1995-2009	16.38%—9.79%	10.66%—2.45%
h=1990-2009	19.36%—15.14%	29.67%—24.51%
h=1984-2009	21.77%—16.80%	39.80%—30.01%

Table 2.9: The average MAPE for models VAR and VECM for the 9 provinces

We calculate the average MAPE indicator for the 9 provinces(the detailed MAPE and backetesting results for each singular province are available upon request). We observe that VECM is more precise than the VAR model particularly for females. However, it's uncertain for 3 periods as to males which model is the best following intervals: 2005-2009, 2002-2009, 2000-2009 there is uncertainty in robustness models. But beyond the length of 15 years, it is clear that for periods 1995-2009, 1990-2009, 1984-2009, the model VECM is more robust than the VAR model. Overall the VECM presents a better

predictive ability. A large part of the variation in the mortality index by both models remained unexplained. The errors are too large, almost 40% of the variation in the historical data remains unexplained for males in the 25 year length sample. This is due to the fact that historically mortality rates volatility in long term have improved more quickly than the forecasts produced by the two models when applied out-of-sample are beyond 15 years. Moreover the forecasts from the models VAR and VECM do not match the volatility of future mortality indices for males(see table 2.10) especially for males where the variance is far from the variation in the historical data. The same observation can be observed for VAR since there 25.17, 41.34 ans 57 and therefore we conclude that volality of historical mortality are greater than we get from the forecasts obtained with the two models.

Out-of-samples	Sex	Historic	VAR Forecast	VECM Forecast
h=1995-2009	Males	166.31	37.23	48.10
	Females	98.16	91.19	78.51
h=1990-2009	Males	172.9	52.17	59.75
	Females	107.77	114.88	107.72
h=1984-2009	Males	213.93	67.46	69.44
	Females	124.45	139.94	136.18

Table 2.10: Comparison of volatility of historical mortality with mean of out-of-sample forecasts produced by models VAR and VECM with in sample

In the following part we illustrate graphically the models in sample and out-of-sample for Alberta. We can also do the same for the other 8 provinces to evaluate their goodness of fitting of the two multivariate models on historical data. Here below, we can observe the models in sample in the following graphs:

We fit VAR and VECM from 1921 to 2004 then we project models on from 2005 to 2009(see graphs 2.3 and 2.4). As we can see, the VECM is

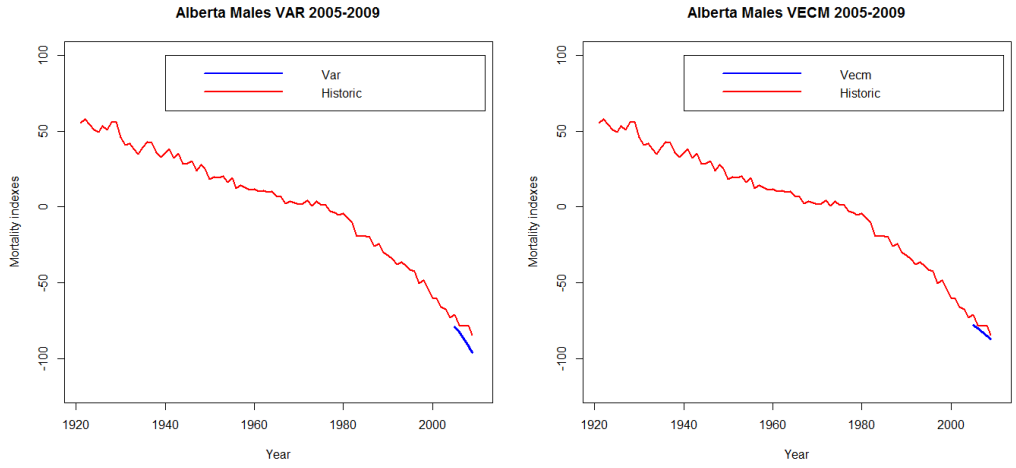

Figure 2.3: Results out-of-sample of the two models VAR and VECM with males

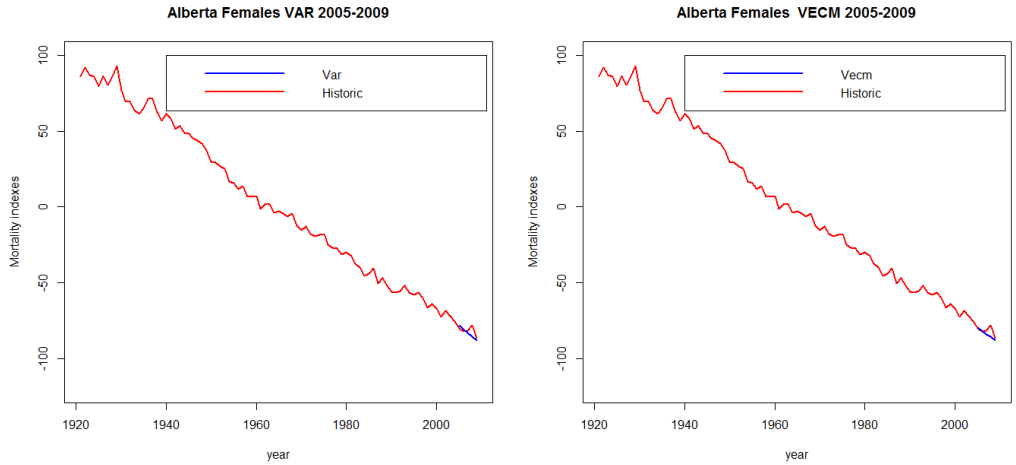

Figure 2.4: Results out-of-sample of the two models VAR and VECM on females

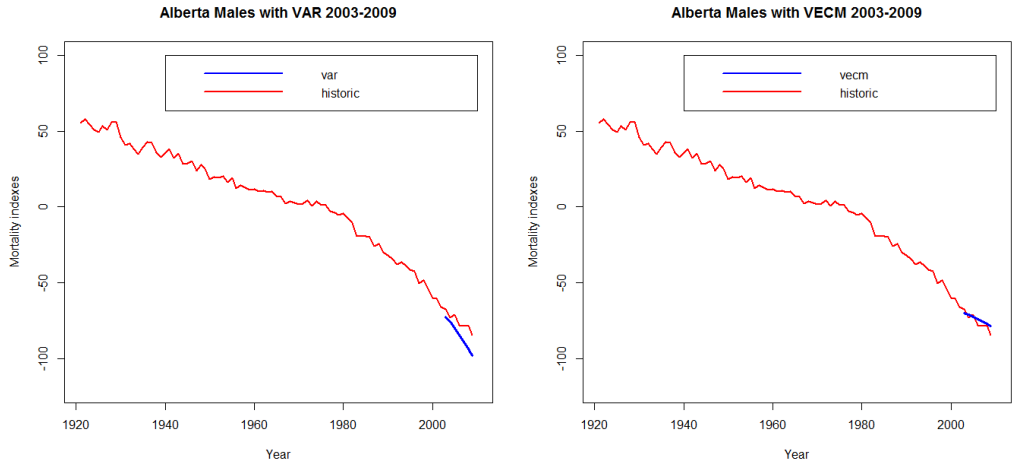

Figure 2.5: VAR and VECM projections on the period 2002-2009 for males

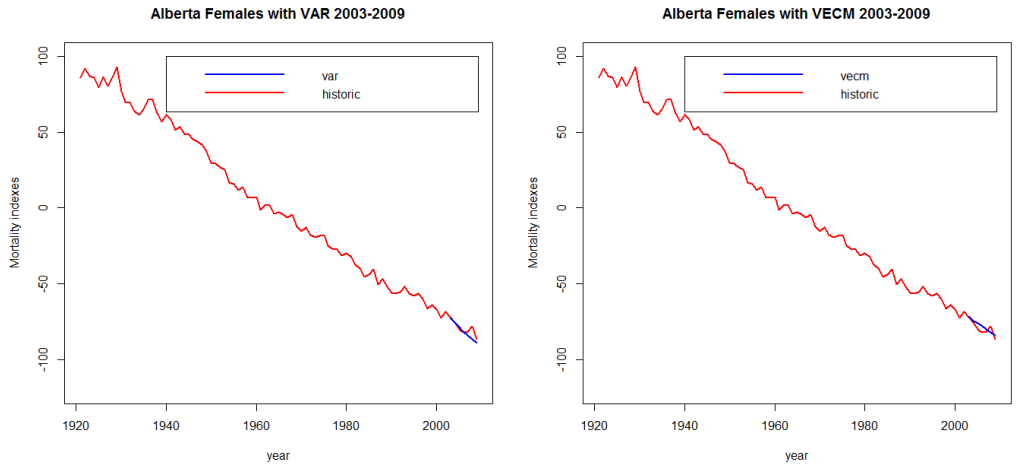

Figure 2.6: VAR and VECM projections on the period 2002-2009 for females

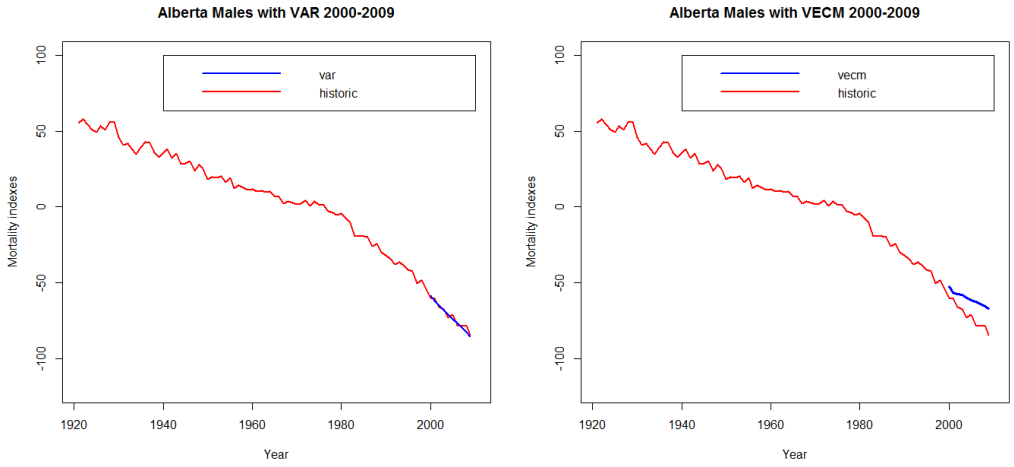

Figure 2.7: VAR and VECM projections on the period 2000-2009 for males

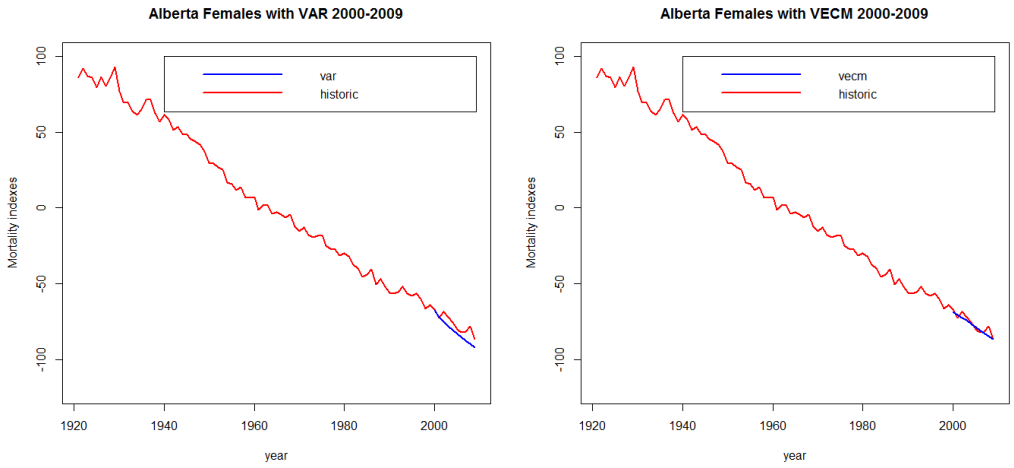

Figure 2.8: VAR and VECM projections on the period 2000-2009 for females

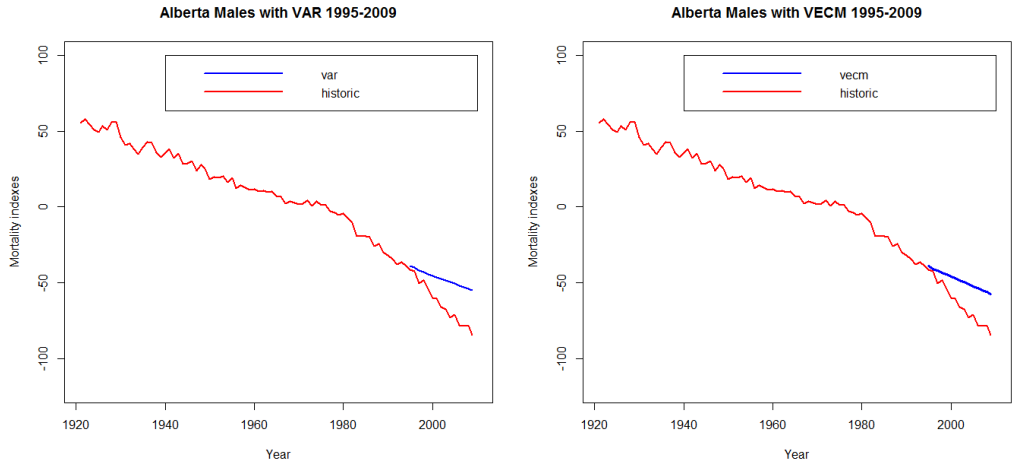

Figure 2.9: VAR and VECM projections on the period 1995-2009 for males

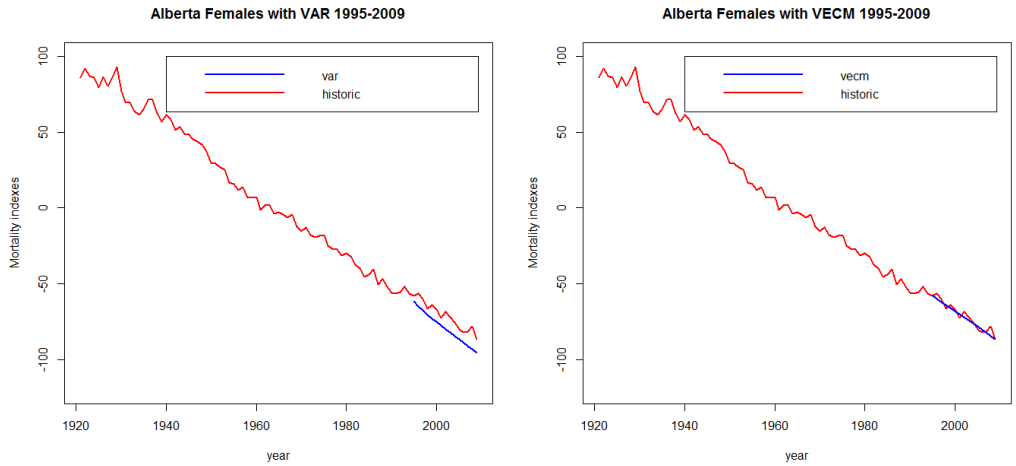

Figure 2.10: VAR and VECM projections on the period 1995-2009 for females

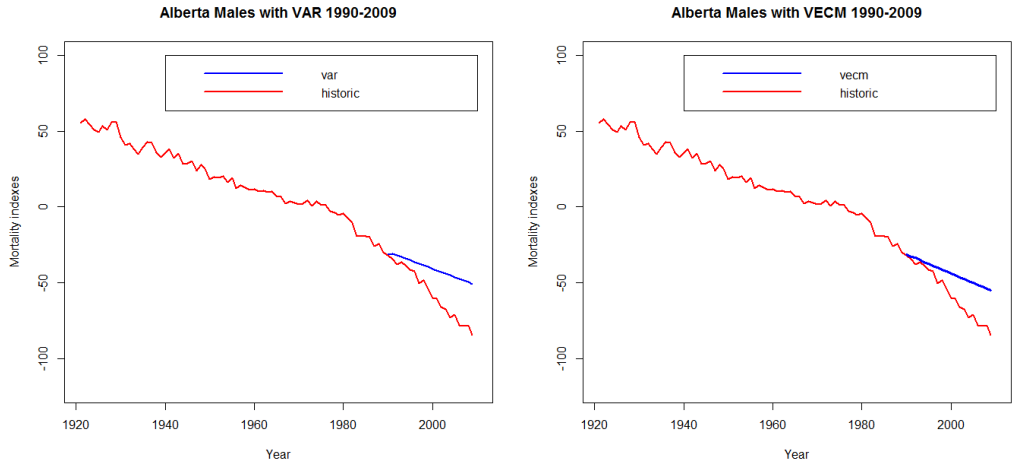

Figure 2.11: VAR and VECM projections on the period 1990-2009 for males

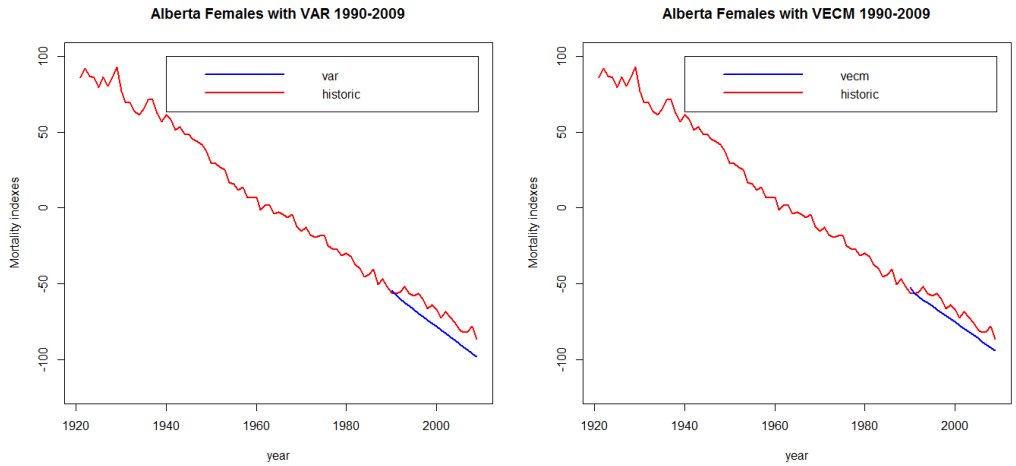

Figure 2.12: VAR and VECM projections on the period 1990-2009 for females

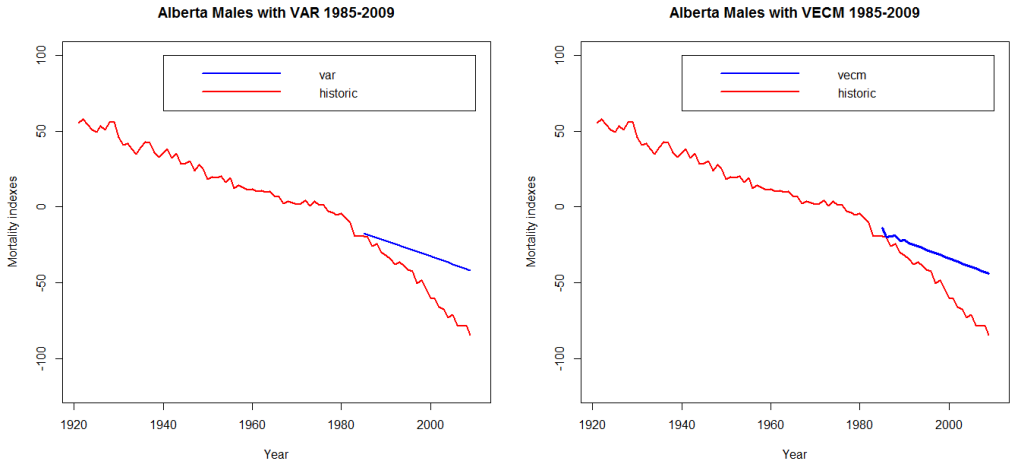

Figure 2.13: VAR and VECM projections on the period 1985-2009 for males

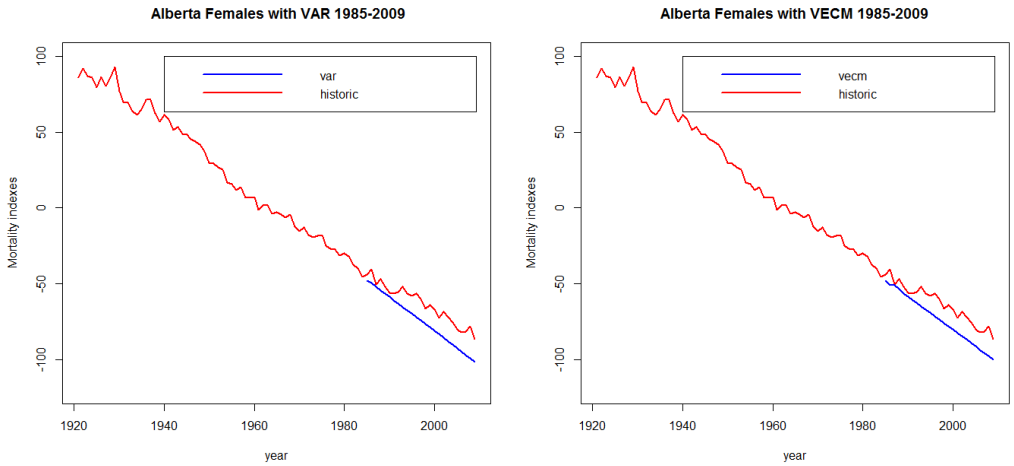

Figure 2.14: VAR and VECM projections on the period 1985-2009 for females

more precise on the 5 years period. It fits very well with the time series of mortality indices. We quantify the errors and the VECM models are 3.09% and 5.96% respectively for females and males. Compare to VECM models, the VAR models present errors of 3.73% and 11.67% respectively for females and males. It's clear that VECM models is better than VAR. In red line, we represent historical data of mortality indices from Alberta for both males and females. The blue curve represents the point forecasts time series as projected by the two models VAR and VECM on the period 2005-2009.

These 4 graphs show analysis in sample and out-of-sample((see graphs 2.5 and 2.6). In red line, we represent historical data of mortality indices from Alberta for both males and females. The blue curve represents the point forecasts time series as projected by the two models VAR and VECM on the period 2002-2009 out-of-sample.
We fit VAR and VECM from 1921 to 2002 then we project models on from 2003 to 2009. As we can see, the VECM is more precise on the 7 years periods sample. It fits very well with the time series of mortality indices. We quantify the errors and the VECM models are 3% and 3.59% respectively for females and males. Compare to VECM models, the VAR models present errors of 3.09% and 11.59% respectively for females and males. It's not clear that in this case VAR model is better than VECM.

These 4 graphs show analysis in sample and out-of-sample(see graphs 2.7 and 2.8). In red line, we represent historical data of mortality indices from Alberta for both males and females. The blue curve represents the point forecasts time series as projected by the two models VAR and VECM on the period 2000-2009.
We fit VAR and VECM from 1921 to 1999 then we project models on from 2000 to 2009. As we can see, the VECM is more precise on the 10 years period out-of-sample for females. However, the VECM is less precise with 15.14% compared to 2.35% of VAR models. It's clear that in this case VAR

model is better than VECM.

These 4 graphs show analysis in sample and out-of-sample(see graphs 2.9 and 2.10). In red line, we represent historical data of mortality indices from Alberta for both males and females. We fit two models VAR and VECM on the in sample period from 1921-1994 and project them out-of-sample from 1995-2009. The blue curve represents the point forecasts time series as projected by the two models VAR and VECM on the period 1995-2009. As we can see, the VECM is more precise on the 15 years period windows. We quantify the errors and the VECM models are 3.07% and 22.17% respectively for females and males. Compare to VECM models, the VAR models present errors of 11.63% and 23.63% respectively for females and males. It's clear that in this case VECM model is better than VAR.

In red line, we represent historical data of mortality indices from Alberta for both males and females(see graphs 2.11 and 2.12). We fit two models VAR and VECM on the in sample period from 1921-1989 and project them out-of-sample from 1990-2009. The blue curve represents the point forecasts time series as projected by the two models VAR and VECM on the period 1990-2009 As we can see, the VECM is more precise on the 20 years period windows. We quantify the errors and the VECM models are 10.90% and 20.41% respectively for females and males. Compare to VECM models, the VAR models present errors of 14.65% and 25.65% respectively for females and males. It's clear that also in this case VECM model is more robust than Var model.

In red line, we represent historical data of mortality indexes from Alberta for both males and females(see graphs 2.13 and 2.14). The blue curve represents the point forecasts time series as projected by the two models VAR and VECM on the period 1985-2009. We estimate the two models VAR and VECM from 1921-1984 and project them on the remaining sample from 1985-2009. As we can see, the VECM is more precise on the 25 years period out–of-sample. It fits very well with the time series of mortality indices. We

quantify the errors and the VECM models are 16.58

2.4.3 Projecting mortality indices for Canadian provinces

We project mortality indices from the 9 provinces for models VAR and VECM models with confidence interval of 95%. We project them in 50 years ahead. We plot for each province until 2059 the mortality provinces projections. We can observe at a first glance that the forecasts confidence interval of mortality indices for all the provinces in Canada are narrow(see figure 2.15 and 2.16) using the VAR model. It is due to the fact that the confidence interval does not allow for more quantification of future mortality risks and improvements in mortality in the long term beyond 15 years horizons. However, the prediction intervals are wider for both sexes with the VECM models(see figures 2.17 and 2.18). The VECM model improves VAR model and also the risk quantification is better off in terms of explaining future trends of mortality indices. The additional variables included in the former improve definitely the model as well as the future life improvements trends as the VAR model only considers variables in terms of their levels. These models can be used to price policies accurately within 15 years. Insurers could consider reinsurance contract with frequency of 15 years.

We observe that the mortality in the future will be taking into account more factor risk for males than females as we can see from figure 2.14 and 2.15 where confidence intervals are wider than in those projected by VAR in both cases. As we can observe the mortality indices for Alberta, Manitoba, Prince Edward Island and Sakastchewan. Moreover, the improvements in mortality will be more intense for Nova Scotia, British Columbia, Quebec. While the confidence interval for VAR models is narrow, the related results for the Vector Error Correction model is larger than what we have obtained precedently.

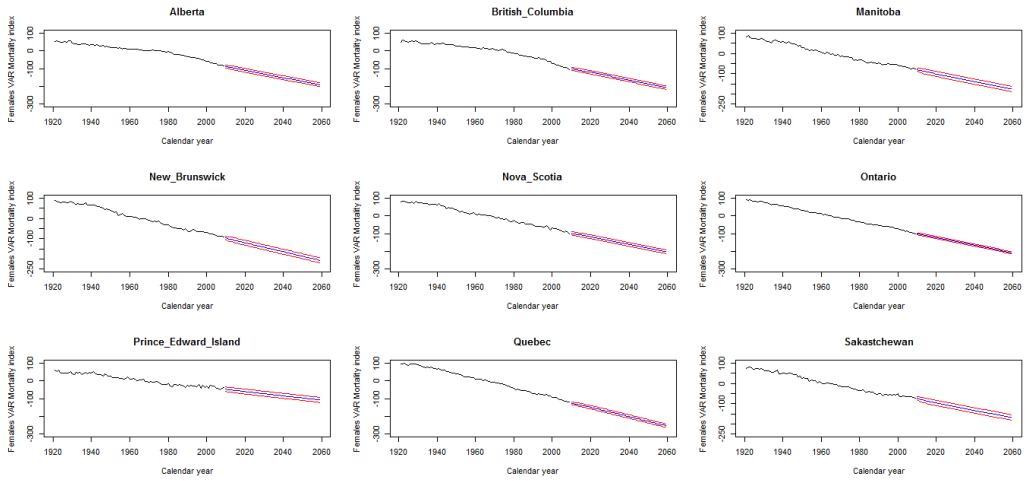

Figure 2.15: Projecting Females mortality indices for all other provinces with VAR models

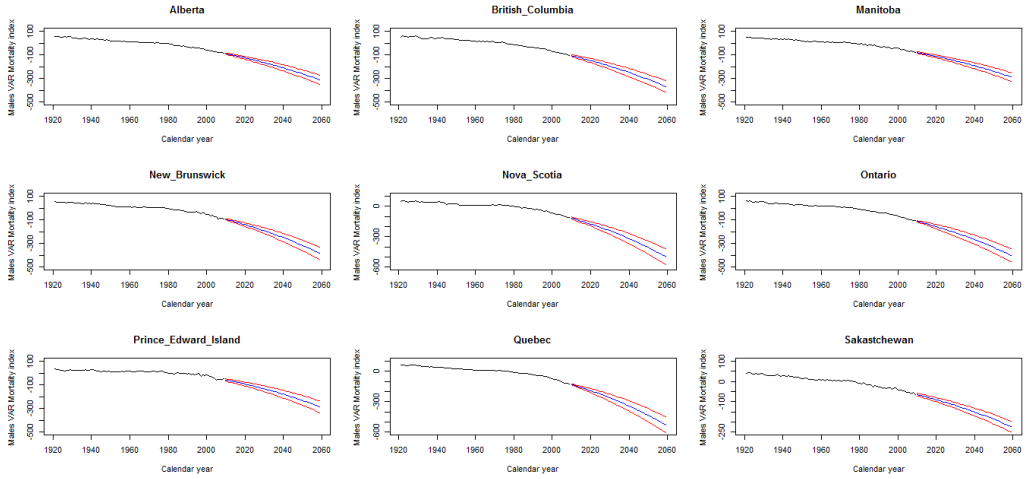

Figure 2.16: Projecting Males mortality indices for all other provinces with VAR models

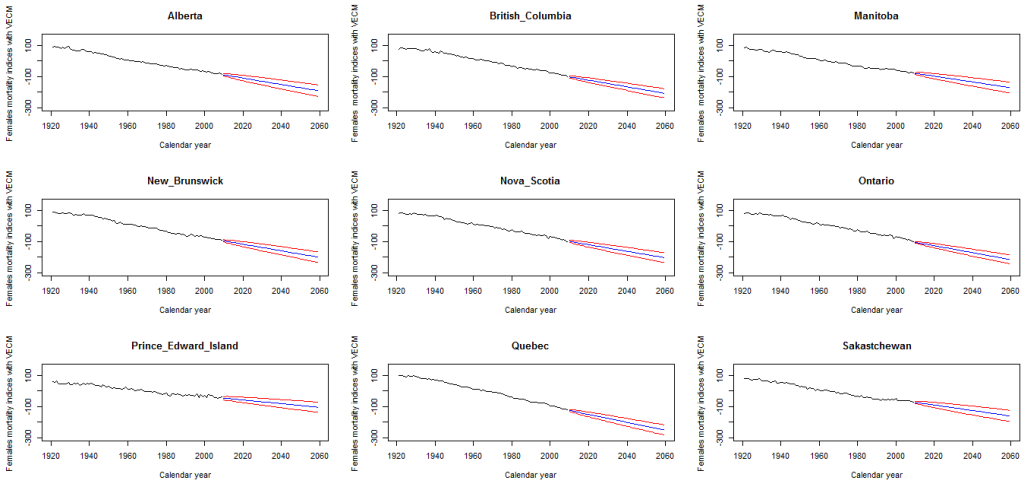

Figure 2.17: Forecasting Canadian females Mortality indices from the Vector of Error Correction model

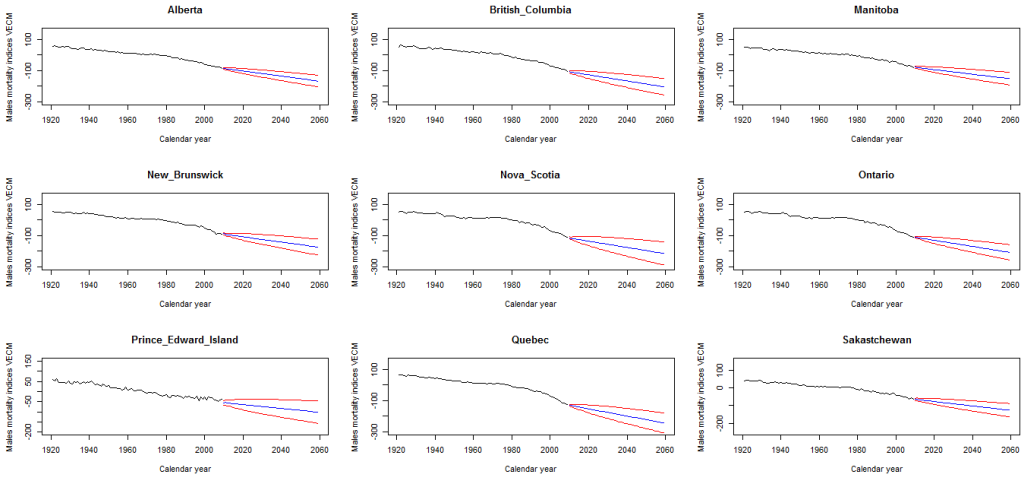

Figure 2.18: Forecasting Canadian males Mortality indices from the Vector of Error Correction model

2.5 Applications of models on Annuities

In the last section of this work, we attempt to apply three models - ARIMA(see Hyndman, 2013 and (Box and Jenkins, 1976) which considers independently mortality indices, VAR and VECM models for both genders in order to compute the actuarial pricing value for annuities by cohorts as well as the corresponding life expectancy. The existing literature shows the calculation of the annuity only with ARIMA(see Lee Carter(1992)) based on a single mortality index separately from other provinces. In this paper, we emphasize the illustrations of the calculations for multiple population through the VAR and VECM and evaluate their incidence on the pricing of risk compared to the ARIMA. We first project mortality indices on 100 years ahead for the 3 models. Relying on the life contingencies package from Spedicato(2013), after projecting the mortality indices for Alberta, we calculate the cost of a pension annuity for the cohorts of 1960, 1970, 1980, 1990 and 2000. The following assumptions are made:

- The retirement age will be set to 65 years old regardless of the cohort.
- the number of fractional payments are 12 based on a monthly frequency.
- The actuarial present value of an annual annuity of yearly annuity is 1 monetary unit.
- The interest rate for the evaluation is 4% and the inflation rate is 2%.

In table 2.11, the expected future life time indicated by the VECM are 17.81, 19,5, 20.96, 22.29, 23.21 respectively for the 5 cohorts examined. However, the results from ARIMA(16.65, 18.25, 19.56, 20.68, 21.54) and VAR(16.73, 18.42, 19.67, 20.86, 21.7) are found to be less than for the first model. We obtain similar results when calculating annuity values with these 3 models. For the VECM there are 8.38, 8.79, 9.14, 9.45, 9.71 for the VECM but less

than for the VAR which are 7.91, 8.23, 8.52, 8.79, 9.03 and 7.85, 8.16, 8.45, 8.72, 8.97 for ARIMA. To sum up, the female results indicate that expected life time provided by the VECM model is greater than those computed by the VAR or the mortality index taken independently. This has a direct consequence on the annuity value which is greater for the VECM than the value obtained for the other models. Regarding males in table 12, VECM show small differences over VAR model but it still computes greater values of expected life expectancy and annuity. The current example is illustrated for Alberta in tables 2.11 and 2.12. Moreover results from the other provinces, which are available on request, present the same conclusions as in the case of Alberta province. As a result, it is better to use a multipopulation model especially the VECM model than the ARIMA(each province taken indipendently) and the VAR because the former captures better longevity risk in terms of higher life expectancy and annuities values.

Females	ARIMA	VAR	VECM
Cohorts	Life time — APV	Life time — APV	Life time — APV
1960	16.65 — 7.85	16.73— 7.91	17.81— 8.38
1970	18.25 — 8.16	18.42— 8.23	19.5— 8.79
1980	19.56 — 8.45	19.67— 8.52	20.96— 9.14
1990	20.68 — 8.72	20.86— 8.79	22.29— 9.45
2000	21.54 — 8.97	21.7— 9.03	23.21— 9.71

Table 2.11: Price of annuity and life time after 65 years old from Alberta provinces cohorts 1960, 1970,1980,1990 and 2000

2.6 Conclusions

This paper investigates the dependence of mortality indices between Canadian provinces through the cointegration approach. Mortality multivariate

Males	ARIMA	VAR	VECM
Cohorts	Life time — APV	Life time — APV	Life time — APV
1960	11.39 — 6.65	12.34— 7.29	12.58— 7.43
1970	13.63 —7.08	15.26 — 8.02	15.54 — 8.15
1980	15.62 — 7.49	17.89 — 8.7	18.15 — 8.81
1990	17.91 — 7.87	20.9 — 9.33	21.11— 9.4
2000	19.53 — 8.22	23.08— 9.88	23.22 — 9.91

Table 2.12: Pricing annuities and life time after 65 years old from Alberta province of male cohorts 1960, 1970,1980,1990 and 2000

time series have been retrieved from the Lee Carter Model for both males and females. Specifically, the Johansen test has been used to show dependence between provinces. We compute the corresponding Vector Error Correction model that clearly shows a dependence between mortality indices. This econometric analysis allows us to capture common trends and also the correlation structure from mortality accross provinces for each sex group. We compute the Vector Autoregression model and compare it to the VECM. The two models applied to mortality indices work better for females than for males. The latter sex group present some deficiencies as the forecasts from models do not capture fully historical mortality indices. The two models were used then to project future mortality indices for Canadian provinces. The VECM allows for more quantification of mortality risks than VAR. The results from these analysis have been applied to calculate annuity values per cohorts and the corresponding life expectancy. For both genders, the price of annuities for females from the model VECM is greater than from other models. This current work will help insurers or pension funds to price fairly longevity risk.

This three figures(figures 2.19, 2.20, 2.21) represents the parameters of

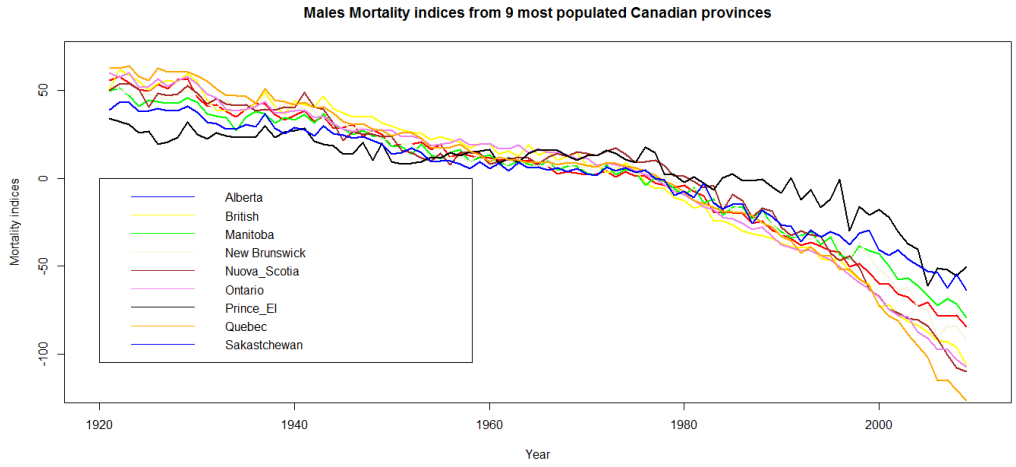

Figure 2.19: Parameters k_t mortality index for Males with Lee Carter model

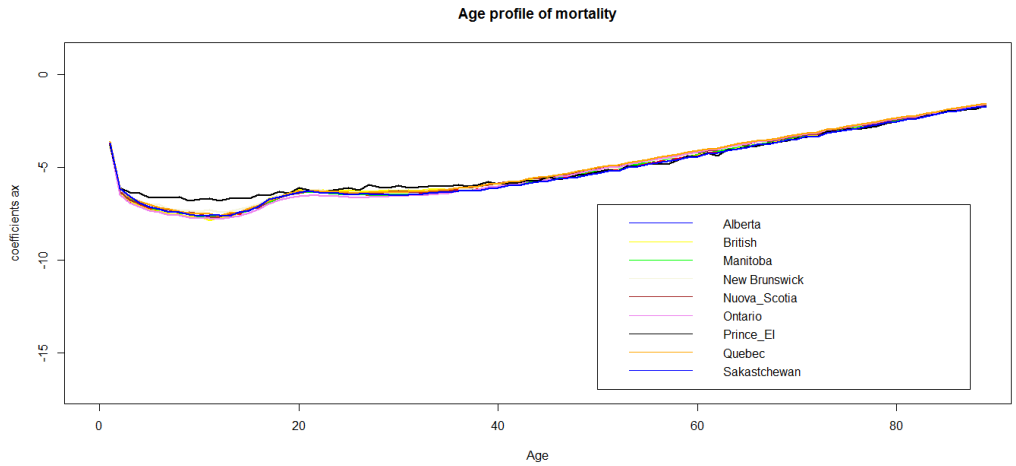

Figure 2.20: Parameters of a_x age shape of mortality profile for Males

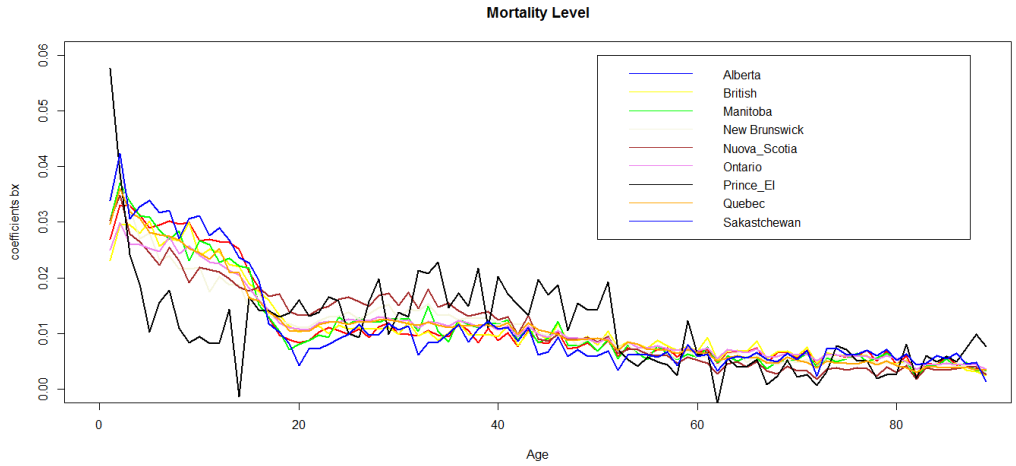

Figure 2.21: Results of b_x deviation of Males mortality rates

Lee Carter model for females Canadian provinces. The first figure represents the mortality indices, the second in the middle represents the ax coefficients and finally the figure at the bottom represent the b_x

This three figures(figures 2.22, 2.23, 2.24) represents the parameters of Lee Carter model for females of each Canadian provinces. The first figure represents the mortality indices, the second in the middle represents the ax coefficients and finally the figure at the bottom represent the bx.

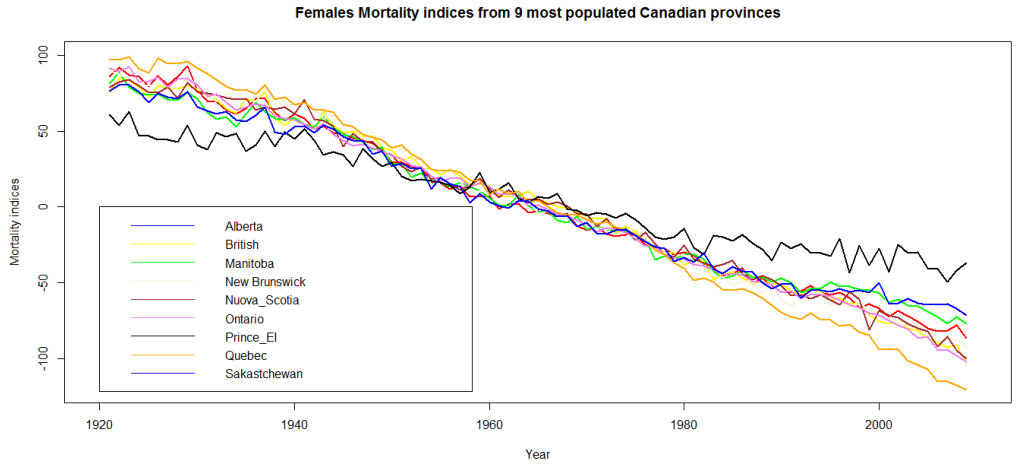

Figure 2.22: Parameters k_t mortality index females

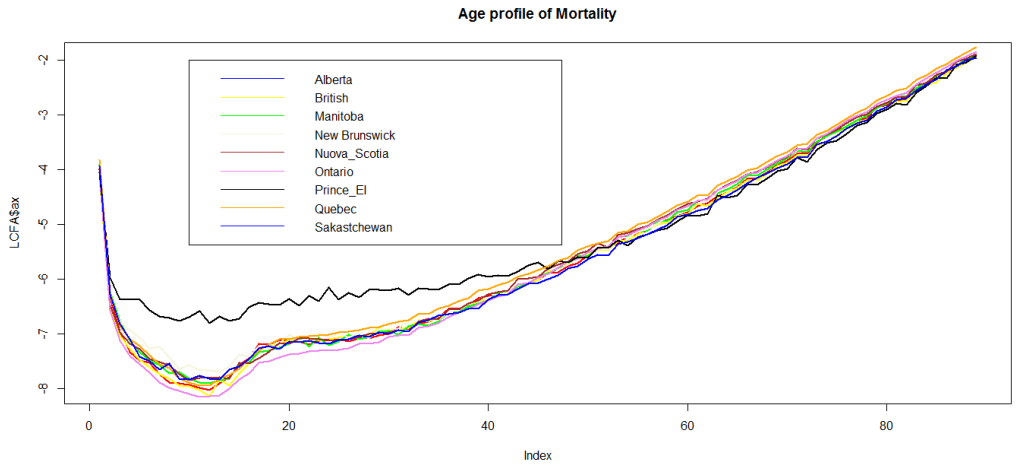

Figure 2.23: Parameters of a_x age shape of mortality profile for females

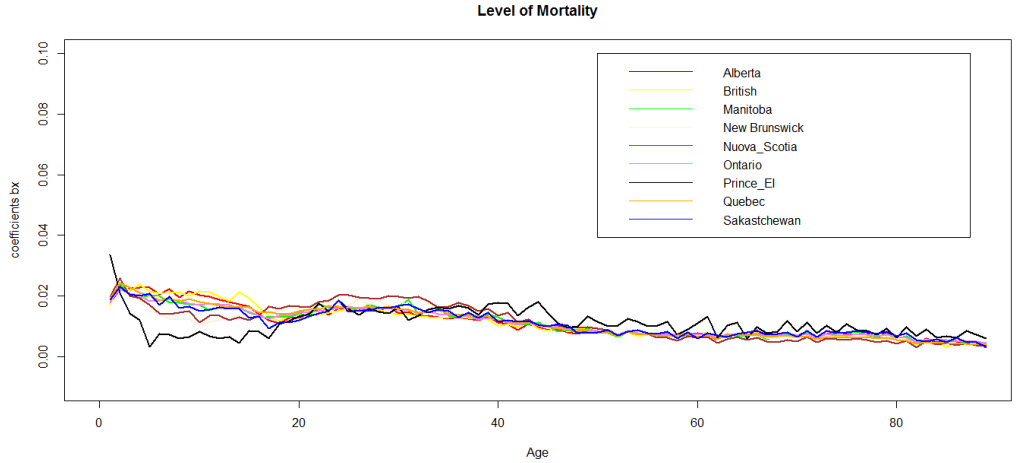

Figure 2.24: Results of b_x deviation of females mortality rates

Mortality Indices	Constant	lags	Dw stat	trend	Lags	Dw stat
Alberta	0.89	2	1.99	0.005	0	2.08
$\Delta Alberta$	0	1	1.99	0	1	2
British Columbia	0.99	2	2.05	0	0	2.12
$\Delta BritishColumbia$	0	1	2.02	0	1	2.05
Manitoba	0.8	1	1.98	0.056	0	2.19
$\Delta Manitoba$	0	0	1.98	0	0	1.98
New Brunswick	0.97	2	2.08	0.14	1	2.08
$\Delta New_Brunswick$	0	1	2.08	0	1	2.08
$Nova_Scotia$	0.99	2	2.1	0	0	2.3
$\Delta Nova_Scotia$	0	1	2.08	0	1	2.1
Ontario	0.99	1	2.14	0	0	2.17
$\Delta Ontario$	0	0	2.13	0	0	2.14
Prince Edward Island	0.79	1	2.13	0.04	1	1.97
$\Delta PrinceEI$	0	0	2.14	0	0	2.14
Quebec	0.99	1	2	0.08	1	1.97
$\Delta Quebec$	0	0	1.97	0	0	2
Sakastchewan	0.71	3	1.87	0.87	4	1.95
$\Delta Sakastchewan$	0	2	1.87	0	2	1.87
Canada	0.99	1	2.07	0.01	0	2.2
$\Delta Canada$	0	0	2.06	0	0	2.07

Table 2.13: Performance of the Augmented Dickey Fuller of mortality indices for Canadian provinces for Female

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Mortality Indices	Constant	lags	Dw stat	trend	Lags	Dw stat
Alberta	0.96	7	2.45	0.005	4	2.08
$\Delta Alberta$	0	5	1.92	0	7	2.14
British Columbia	0.99	17	2.49	0	0	2.12
$\Delta BritishColumbia$	0	5	1.927	0	13	2.19
Manitoba	0.91	20	2.48	0.08	1	2.19
$\Delta Manitoba$	0	4	1.86	0	11	1.98
New Brunswick	0.98	29	2.67	0.0151	0	2.33
$\Delta New_Brunswick$	0	4	1.97	0	16	2.16
$Nova_Scotia$	0.99	14	2.91	0	4	2.3
$\Delta Nova_Scotia$	0	5	1.97	0	10	2.3
Ontario	0.99	3	2.73	0	4	2.17
$\Delta Ontario$	0	6	2.04	0	0	2.14
Prince Edward Island	0.69	87	3.06	0	4	2.19
$\Delta PrinceEI$	0	6	2.06	0	29	2.14
Quebec	0.99	6	2.75	0.01	0	2.5
$\Delta Quebec$	0	6	2.09	0	1	2
Sakastchewan	0.9	37	2.75	0.039	1	2.36
$\Delta Sakastchewan$	0	4	2	0	20	2.23
Canada	0.99	3	2.45	0.01	3	2.2
$\Delta Canada$	0	6	2.18	0	1	2.07

Table 2.14: Performance of the Phillips Perron test of mortality indices for Canadian provinces for Females.

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Mortality Indices	Constant	lags	Dw stat	trend	Lags	Dw stat
Alberta	1.21	7	0.007	0.219	6	0.67
$\Delta Alberta$	0.09	7	2.45	0.09	7	2.45
British Columbia	1.21	7	0.008	0.16	6	0.73
$\Delta BritishColumbia$	0.39	13	2.46	0.11	18	2.5
Manitoba	1.216	7	0.008	0.17	6	0.48
$\Delta Manitoba$	0.12	21	2.48	0.12	21	2.48
New Brunswick	1.21	7	0.008	0.11	6	0.56
$\Delta New_Brunswick$	0.19	31	2.67	0.16	34	2.68
$Nova_Scotia$	1.21	7	0.01	0.09	6	0.96
$\Delta Nova_Scotia$	0.26	12	2.89	0.08	14	2.92
Ontario	1.22	7	0.004	0.06	6	0.89
$\Delta Ontario$	0.13	2	2.72	0.05	3	2.73
Prince Edward Island	1.21	7	0.06	0.13	6	1.32
$\Delta PrinceEI$	0.22	26	3.1	0.15	26	3.1
Quebec	1.21	7	0.003	0.18	6	0.41
$\Delta Quebec$	0.28	4	2.71	0.07	7	2.76
Sakastchewan	1.21	7	0.01	0.19	6	0.6
$\Delta Sakastchewan$	0.36	16	2.76	0.16	35	2.76
Canada	1.21	7	0.003	0.09	6	0.48
$\Delta Canada$	0.18	2	2.42	0.07	3	2.45

Table 2.15: Performance of the KPSS test of mortality indices for Canadian provinces for Females

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Mortality Indices	Constant	lags	Dw stat	trend	Lags	Dw stat
Alberta	0.99	1	2.07	0.99	1	2.05
$\Delta Alberta$	0	2	2.01	0	0	2.06
British Columbia	1.	1	2.06	0.99	1	2.04
$\Delta BritishColumbia$	0	0	1.97	0	1	2.04
Manitoba	1	2	2.04	0.99	2	2.04
$\Delta Manitoba$	0	0	2.13	0	1	2.02
New Brunswick	1	1	2.11	0.99	1	2.1
$\Delta New_B runswick$	0	0	1.98	0	0	2.08
Nova Scotia	1	1	2.09	0.99	1	2.08
$\Delta Nova_S cotia$	0	0	2.02	0	0	2.06
Ontario	1	0	2.41	1	0	2.41
$\Delta Ontario$	0	0	1.94	0	0	1.97
Prince Edward Island	0.99	1	2.14	0.96	1	2.11
$\Delta PrinceEI$	0	0	2.11	0	0	2.14
Quebec	1	1	1.96	1	1	1.98
$\Delta Quebec$	0	1	2	0	0	1.94
Sakastchewan	0.99	2	2.06	0.98	2	2.04
$\Delta Sakastchewan$	0	1	2.01	0	1	2.05
Canada	1	0	2.29	1	0	2.31
$\Delta Canada$	0	1	1.94	0	0	1.97

Table 2.16: Performance of the ADF test of mortality indices for Canadian provinces for Males

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Mortality Indices	Constant	lags	Dw stat	trend	Lags	Dw stat
Alberta	1	9	2.62	0.99	4	2.53
$\Delta Alberta$	0	5	1.92	0	5	2.06
British Columbia	1	9	2.3	0.99	5	2.24
$\Delta BritishColumbia$	0	5	1.89	0	4	2.03
Manitoba	1	11	2.8	0.8	0	2.62
$\Delta Manitoba$	0	5	1.94	0	7	2.19
New Brunswick	1	19	2.74	0.99	9	2.67
$\Delta New_Brunswick$	0	5	1.92	0	2	2.08
Nova Scotia	1	3	2.47	0.99	1	2.48
$\Delta Nova_Scotia$	0	5	2	0	0	2.06
Ontario	1	1	2.41	1	1	2.41
$\Delta Ontario$	0	6	2.18	0	3	1.97
Prince Edward Island	0.96	1	2.87	0.57	4	2.67
$\Delta PrinceEI$	0	3	2.05	0	2	2.14
Quebec	1	4	2.44	1	7	2.48
$\Delta Quebec$	0	5	2	0	0	1.94
Sakastchewan	1	18	2.85	0.55	2	2.56
$\Delta Sakastchewan$	0	4	2	0	7	2.26
Canada	1	2	2.29	1	3	2.31
$\Delta Canada$	0	5	2	0	3	1.97

Table 2.17: Performance of the Phillips Perron test of mortality indices for Canadian provinces for Males

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Mortality Indices	Constant	lags	Dw stat	trend	Lags	Dw stat
Alberta	1.16	7	0.009	0.25	7	0.14
$\Delta Alberta$	0.56	3	2.52	0.13	8	2.61
British Columbia	1.15	7	0.01	0.28	7	0.1
$\Delta BritishColumbia$	0.7	0	2.1	0.07	8	2.28
Manitoba	1.17	7	0.01	0.26	7	0.26
$\Delta Manitoba$	0.58	5	2.72	0.13	11	2.79
New Brunswick	1.14	7	0.01	0.22	7	0.13
$\Delta New_Brunswick$	0.71	3	2.59	0.13	13	2.71
Nova Scotia	1.07	7	0.014	0.23	7	0.08
$\Delta Nova_Scotia$	0.81	3	2.24	0.1	6	2.42
Ontario	1.13	7	0.005	0.27	7	0.04
$\Delta Ontario$	0.91	5	2.01	0.19	3	2.32
Prince Edward Island	1.02	7	0.09	0.24	6	0.36
$\Delta PrinceEI$	0.17	1	2.91	0.06	3	2.93
Quebec	1.12	7	0.006	0.23	7	5
$\Delta Quebec$	0	1	2	0	0	1.94
Sakastchewan	1.17	7	0.02	0.26	6	0.41
$\Delta Sakastchewan$	0.42	11	2.82	0.12	20	2.86
Canada	1.14	7	0.005	0.26	6	0.04
$\Delta Canada$	0.91	5	1.81	0.22	2	2.17

Table 2.18: Performance of the KPSS test of mortality indices for Canadian provinces for males

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Chapter 3

A multivariate approach to project the long run relationship of life expectancy

3.1 Introduction

The improvements in life expectancy and mortality rates have been investigated in many studies across the 20th century, especially in developed countries (see Lee and Carter, 1992), (Russolillo and Haberman, 2005), (Tuljapurkar, 2007) and (Oeppen and Vaupel, 2002). These progressions in human life have been induced by higher life quality, typically associated with structural improvements of medical healthcare systems(Shaw et al, 2005), social advancements and economic development(Chen and Ching, 2000). Due to increasing levels of life expectancy, longevity risk is borne by insurance companies, pension funds, and social security. Furthermore, young active taxpayers are facing issues about their future pension payouts after retirement. Researchers are working to improve the accuracy of life expectancy computations with the goal of reducing the incidence of policy payouts as pension liability amounts are increasing. Two main methods of quantifying

future life expectancy have been identified in the literature: the biological techniques and extrapolative methods(Whiteford, 2006).

As for the first group, calculations are based on medical scenarios(Oeppen and Vaupel, 2002) to project life expectancy. However, this results in underestimation of future life expectancy. (Olshansky et al, 2005) explained that diseases such as obesity slow down human longevity especially in developed countries, and that this is one as one of the reasons for the underestimation of future life expectancy.

The main example of the extrapolative method is the Lee Carter model(1992), which model life expectancy and mortality rates. It is based on the extrapolation of past mortality trends. This method has been adopted by the US Social Security. Another extrapolative method (Whitehouse, 2007) is based on three steps to generate a 50 year forecast. (Rusolillo and Haberman, 2005) improved the life expectancy forecasts by using ARIMA(see also (Torri, 2011)) which presents better results over the Lee Carter model. Several other papers, including (De Beer and Alders, 1999), (Keilman et al, 2001), (Maarten, 2007), Booth(2006), (Alho and Spencer, 2005) and (Denton, 2005) and (Torri, 2012) have explored this approach. In addition (Adekola, 2002) used a generalized model to explain life expectancy while (Raftery et al, 2013) projected life expectancy at birth for all the countries in the world using the Bayesian probabilistic model. Other forecasting approaches, including(Andreev and Vaupel, 2006) and (Lee, 2006) have explored methods based on the hypothesis of a non-stochastic component of the life expectancy variable.

In an international context, (Torri, 2011) used the cointegration approach methodology to examine future life expectancy in several countries. The results showed that the VAR model increases the accuracy of life expectancy predictions over ARIMA and VECM models for four countries, France, Italy, Norway and Sweden . Other research led by (Babel, 2007) has investigated life improvements in Australia, Europe, Japan, and North America. In this

chapter, we implement the cointegration analysis, which takes into account the long run historical relationships across groups. This method, examines the potential dependency between regions within a country with the aim of extrapolating their future life expectancies. We examine male life expectancy, but similar analysis can be conducted on females groups. We apply the cointegration method, which has proved to be successfull in modeling time series(see commitee Nobel prize, 2003), to life expectancy data from six heavily populated Canadian provinces by taking into account their correlation structure. In the literature some models, including VAR(Torri, 2011) and ARIMA have shown improvements in explaining the dynamics of life expectancy. The purpose of this paper is to explore how VECM performs on data. In addition, the use of the VAR model is related to previous literature, which has shown strong performance in explaining multiple time series. We will not a priori eliminate one model, but compare different models in order to show which is suitable to explain life expectancy time series among the Canadian provinces.

Life expectancy data were provided by the Canadian Human Mortality Database(CHMD) through the website www.bdlc.umontreal.ca/chmd, which is managed by the Department of Demography of the Université de Montréal in collaboration with the Max Planck Institute for Demographic Research and the Department of Demography at the University of California in Berkeley(CHMD). This database was created to provide information on human longevity in Canada to researchers, students, journalists and policy makers. It supplies the data used here, which has a frequency of 1 year spanning 1921-2009, and covers the six Canadian provinces of Nova Scotia, New Brunswick, Quebec, Ontario, Alberta, and British Columbia. It also provides detailed information regarding births, population size, exposure-to-risk, death rates, and life expectancy at birth.

Life expectancy (Figure 3.1) shows an increasing trend for all of the

provinces analyzed from 1921 to 2009. The historical pattern of life expectancy can be subdivided into two main periods. From 1921 to 1960 we observe a divergence in provincial life expectancy. However, after 1960, we can clearly observe a convergence as the six canadian provinces show common trends, as seen in the Figure 1. British Columbia shows the highest life expectancy level for the sample period 1960-2009, followed by Ontario. Quebec recovers from its low level during the period 1921-1960 to become one of the provinces where residents live longest. New Brunswick, in contrast, is the province where people have lower life expectancy. We observe the common trends across the provinces to increase over the years. Accordingly, it can be deduced that life expectancy is converging to the same level in all parts of Canada. In the rest of this of this paper, we present the features of the three models (ARIMA, VAR, and VECM) used in the analysis. The methodology of cointegration applied here includes several steps:

- the computation of the optimal value of lag of the vector autoregressive model
- the Johansen cointegration test, which estimates the dynamic relationship among the regional life expectancies
- the estimation of the VAR and VECM models and the forecasting of the derived model
- we also present the backtesting output from the different models, and finally we generate the values of future life expectancy 50 years ahead, using VECM.

The procedure for VECM also involves determining the order of integration for each of the six life expectancy data sets by using the Augmented Dickey Fuller, Phillips-Perron, and KPSS tests The steps of cointegration analysis are described by (Hamilton, 1994) (Juselius, 2007) and (Harris and Sollis, 2002). The empirical analysis will be done using R statistical software package developed by (Pfaff, 2008)

3.2 Explanation of various models

3.2.1 ARIMA Model: description and fit

In the ARIMA model, life expectancy is modeled as a stochastic process. The methodology consists of three phases: identification, estimation, and diagnostics. These three steps are all described in (Box and Jenkins, 1976) and (Hyndman and Athanasopoulos, 2013) who explain that the process involves choosing an appropriate ARIMA(p,d,q) when modeling a variable. The goal is to identify the correct model that will best fit the time series under study. Two options outlined by the literature may help to select the most appropriate model: selection of the model by the user, or the automatic ARIMA that will be used from here. In general, ARIMA is described as in (3.1):

$$L_t = a_0 + a_1 L_{t-1} + \varepsilon_t \tag{3.1}$$

where a_0 is the drift term, a_1 is a coefficient, L_{t-1} is the time series, and ε_t is the error term distributed with $\varepsilon \sim (0, \sigma^2)$. The principal steps in selecting the best model as follows:

Identification of the model: consists of plotting data and identifying the pattern of the time series. As we can observe in Figure 3.1, life expectancy presents an increasing trend, with drift, for all the six provinces. The basic analysis also, consists of differencing the data until they appear to be stationary. The unit root tests, including Augmented Dickey Fuller(ADF), Phillips-Perron(PP) and Kwiatkowski-Phillips-Schmidt-Shin(KPSS) are useful in determining the level of stationarity. Results obtained from the three unit root tests are used to determine the order of integration which corresponds to the value of the parameter d. The best model corresponds to the lowest Akaike Information criterion(AIC).

The first visualization of life expectancy from the six provinces(Figure 3.1) indicates that the variables are non-stationary. In order to confirm this, we compute the unit root test for life expectancy through the Augmented Dickey Fuller(see Dickey and Fuller, 1979)), the Phillips Perron (see Phillips and Perron, 1988) and KPSS tests(Kwiatkowski et.al, 1992). The values for the KPSS are greater than the critical value of the test and KPSS also confirms the hypothesis of non stationarity. The order of integration analyzed through the ADF test(Table 3.10), PP test(Table 3.11), and KPSS test(see Table 3.12) shows that life expectancy time series are non stationary at 5%. Under the criterion of constant, all the p-values are greater than the critical values. However, the analysis under the criterion of trend and constant, show significance only for New Brunswick, Nova Scotia, Ontario, and Quebec, where the p-values are greater than the critical values(see the ADF results from the Table 3.10). Under the PP test(see Table 3.11) only Quebec, Ontario and Nova Scotia are significant. Last, the analysis of Alberta, New Brunswick, Nova Scotia, Ontario and Quebec with the KPSS shows that life expectancy are non stationary(see results in table 3.12 and the relative critical values in Table 3.13) . Furthermore, the p-values from the three models measured on first difference data from each life expectancy are less than the critical values. Overall, the three tests accept the null hypothesis that life expectancy for each province is integrated of order 1 under the constant criterion and the criterion of trend.

Estimation of the order of the model: after derivation of the order of stationarity, it is necessary to experiment with various combinations of p, d, and q where p is the number of autoregressive parameters d is the drift, and q is the moving average parameter. A Box-Cox transformation may also be necessary to stabilize the variance.
It is recommended at this stage to examine the autocorrelation(ACF), the partial autocorrelation(PACF), and the diagnostics of residuals graph to

choose the appropriate model. (Hyndman and Athanasopoulos, 2013) developed an automated algorithm which consists of the inclusion of a constant. (Box and Jenkins, 1976) advised relying on the AIC(Akaike) and SIC(Schwarz criterion) to choose the best model.

Model validation checks the diagnostics of residuals from the chosen models by plotting and conducting a Portmanteau test of the residuals. The residuals diagnostics are investigated to see whether there is white noise. The procedure is completed by computing the forecasts through the choice of the best fitting model. The best numerical results of the ARIMA are described in Table 3.1. The Portmanteau test(see Table 3.2) indicates non autocorrelation of residuals with 4, 10, 15, or 20 lags for each of the provinces life expectancy. These results suggest that ARIMA appears to behave well with white noise disturbances.

models	Alberta	Columbia	Brunswick	Scotia	Ontario	Quebec
ARIMA(p,d,q)	(1,1,1)	(0,1,2)	(0,1,1)	(1,1,2)	(0,1,0)	(0,1,1)
ar1	0.44	-0.35	-	-0.83	-	-
(se)	(0.14)	(0.11)	-	(0.10)	-	-
ma1	-0.79	-0.46	-0.42	0.58	-	-0.34
(se)	(0.09)	(0.14)	(0.10)	(0.16)	(0.10)	
ma2	-	-	-	-	-0.38	-
(se)	-	-	-	-	(0.12)	-
drift	0.22	0.23	0.25	0.22	0.24	0.31
(se)	(0.02)	(0.01)	(0.04)	(0.04)	(0.045)	(0.04)

Table 3.1: The best ARIMA models from the analysis of life expectancy

lags	Alberta	British Columbia	New Brunswick	Nova Scotia	Ontario	Quebec
4	0.83	0.57	0.63	0.23	0.19	0.91
10	0.55	0.54	0.39	0.092	0.55	0.91
15	0.67	0.52	0.57	0.11	0.67	0.26
20	0.83	0.67	0.76	0.83	0.83	0.35

Table 3.2: The p-values of the Portmanteau test from ARIMA models over the period 1921-2009

3.2.2 The Vector Autoregressive model theory

To forecast and explain the historical pattern and forecast of each variable as a function of others in the system, the Vector Autoregressive Model of order p is used. The optimal lag length of the variables in the VAR model is derived by choosing the lag of order p that minimizes the value of information criteria models such as Akaike(AIC), HQ(Hannan-Quinn), Schwarz(SC), and the Final Prediction criteria(FPE). When these information criteria choose different values of p, (Lutkepohl, 2005) recommends considering only the lag chosen by the SC criterion. The VAR(p) models behave well with white noise in forecasting whether the residuals are normally distributed and non-autocorrelated. We start by determining whether they are non stationary(see results from the previous sections). We then derive the optimal lag order of these variables.

Optimal lag length

We analyze the optimal lag length of the VAR model.The Information criteria show contradictory results: AIC and FPE indicate three optimal lags while HQ indicates a lag order of two and finally SC indicates a lag order of only one. Since they differ, following (Lutkepohl, 2005), preference will be given to SC. Consequently, the lag length is 1.

Estimation of the VAR model

The VAR model is derived in (3.2):

$$L_t = b_0 + b_1 L_{t-1} + b_2 L_{t-2} + ...b_p L_{t-p} + \varepsilon_t \tag{3.2}$$

where $L_t = (L_{1t}, L_{2t}, ..., L_{kt})$ for $k = 1, ..., K$ time series, $(b_0.....b_i)$ are the coefficients and ε_t is the error term distributed with $\varepsilon \sim (0, \sigma^2)$.

The following equations describe the VAR(p) of each of the variables included in the model(A = ALBERTA; BC = BRITISH COLUMBIA; NB = NEW BRUNSWCIK; NS= NOVA SCOTIA; O=ONTARIO; Q= QUEBEC):

$$\begin{bmatrix} L_{A,t} \\ L_{BC,t} \\ L_{NB,t} \\ L_{S,t} \\ L_{O,t} \\ L_{Q,t} \end{bmatrix} = \begin{bmatrix} 21.98 \\ 9.80 \\ 10.16 \\ -0.73 \\ 4.43 \\ -4 \end{bmatrix} + \begin{bmatrix} .54 & -.16 & .07 & -.06 & .13 & -.12 \\ .53 & .13 & .22 & -.01 & -.29 & -.36 \\ .58 & -.13 & .36 & .36 & -.40 & .02 \\ .16 & -.05 & .22 & .45 & .29 & -.07 \\ .27 & .05 & -.03 & -.08 & .80 & -.09 \\ .49 & -.19 & .13 & -.14 & 0.15 & 0.59 \end{bmatrix} \begin{bmatrix} L_{A,t-1} \\ L_{BC,t-1} \\ L_{NB,t-1} \\ L_{S,t-1} \\ k_{O,t-1} \\ k_{Q,t-1} \end{bmatrix}$$

$$+ \lambda \begin{bmatrix} .06 \\ .07 \\ .06 \\ -.005 \\ .03 \\ .02 \end{bmatrix}$$

Diagnostic tests of residuals are computed for both Portmanteau and Normality. The results in Table 3.1 show remaining autocorrelation(p =

0.0009), but normality on the residuals, as the p value is equal to 0.23. These results can be expected since we use only a few parameters. However, for the purpose of forecasting, it is better to use as few lags as possible.

Type of test	Specific name	p-values
Autocorrelation	Portmanteau(4 lags)	0.0009
Normality	Both	0.23
	Kurtosis	0.195
	Skewness	0.36

Table 3.3: The diagnostics tests of residuals under the VAR model

The autocorrelation(ACF) and partial autocorrelation functions (PACF) are performed on residuals as shown in (Figure 3.2) which shows that the residuals for life expectancy in Alberta are an appropriate fit and do not present autocorrelation.

3.2.3 The Vector Error Correction model

Once the test of unit root and the optimal lag are determined, the VECM is determined by conversion of the VAR(p). It can be described in two versions of this model: the short run and the long run version, where each variable in the vector system is explained by its own past values, lagged changes in other variables, and residuals. Each lagged difference of the L_{t-1} variable included must be stationary. The long version of VECM, which will be used here, is defined in (3.3) as follows:

$$\Delta L_t = \Gamma_1 \Delta L_{t-1} + \Gamma_2 \Delta L_{t-2} + ... + \Gamma_{p-1} \Delta L_{t-p+1} + A_0 + e_t \qquad (3.3)$$

where $\Gamma_i = -(I - A_1 - ..... - Ai)$, $i = 1, ..., (p-1)$ $\Pi = -(I - A_i, -... - A_p)$ is N-dimensional time series, A_0 is the intercept term, and e_t is white noise.

The Vector Error Correction model is used for forecasting and estimations, performed with the Johansen Maximum Likelihood methodology. It is

used to determine the number of common trends(or cointegrated equations) derived from multiple data. The presence of cointegrated equations between variables is determined according to the three following hypotheses.

If $r = K$, the number of cointegrated variables, r, which is stationary, equals the rank(K) of Π, then the model will be estimated by using the standard statistical model.
If $r = 0$, this means that there are no cointegrated relationships between the variables. The variables are stationary if we take the differences of variables above.
If $0 < r < K$ there are two matrices, α and β, such that $\Gamma = \alpha\beta'$, and there will be r cointegrating relationships or $n - r$ common trends. The test of cointegration is reduced to the two following hypotheses:
The rank test is specified in the following form as in (3.4):

$$\begin{aligned} H_0 &: rank(\Pi) = r \\ H_1 &: rank(\Pi) > r \end{aligned} \tag{3.4}$$

and the Likelihood Ratio statistic is described in (3.5) as:

$$LR(r) = -(T - p)\Sigma ln(1 - \lambda_i) \tag{3.5}$$

where r represents the number of cointegrated relationships and λ is the eigenvalue associated with the linear relationship.
The cointegration rank is determined in the trace test and the maximum eigenvalue test of (Johansen, 1988 and 1991). In addition, the test on the maximum eigenvalue test is specified as follows in (3.6):

$$\begin{aligned} H_0 &: rank(\Pi) = r \\ H_1 &: rank(\Pi) = m + 1, r = 0, 1, ...n - 1 \end{aligned} \tag{3.6}$$

The statistic value is written here in (3.7):

$$LR(r) = -(T-p)\Sigma ln(1-\lambda_{m+1}) \tag{3.7}$$

The eigenvalue statistic value tests the null hypothesis of m cointegrated relations against the alternative m+1. For example, the null hypothesis of five cointegrated relations is accepted against the alternative of six cointegrated relations.

Model fitting

Cointegrating relationship	critical values	5%	1%
5	3.09	9.24	12.97
4	7.20	15.67	20.20
3	22.16	22.00	26.81
2	38.87	28.14	33.24
1	47.18	34.40	39.79
0	74.58	40.30	46.82

Table 3.4: The cointegration relations under eigen test

Cointegrating relationship	critical values	5pct	1pct
5	3.09	9.24	12.97
4	10.29	19.96	24.60
3	32.45	34.91	41.07
2	71.31	53.12	60.16
1	118.49	76.07	84.45
0	193.08	102.14	111.01

Table 3.5: The cointegration relations under trace test

The eigenvalue and trace test results from Johansen's procedure are reported in tables 3.4 and 3.5. In the remaining sections of this paper, our computations will be given based on the trace test. Obviously, the same procedure can be accomplished with the eigenvalue test. But in order to save space we will illustrate the results obtained under the trace test. For any r, if the test value is less than the critical values then the corresponding r represents the number of cointegrated equations. For example, from r=0 to r=2 there are no cointegrated equations. The r=3 test value equals 32.45 which is less than the critical value(34.91), therefore the number of cointegrated equations is three at a 5% significance level. We can say that the null hypothesis of three cointegrating relations is accepted against the alternative of two, while the null hypothesis of zero cointegrated relations is rejected. Consequently, according to these two tests, there are three cointegrated relations under the trace test and four for the eigen test among the six groups of regional life expectancy data used in this study. The results of the fitted VECM are presented below as:

$$\begin{bmatrix} \Delta A \\ \Delta BC \\ \Delta NB \\ \Delta NS \\ \Delta ON \\ \Delta Q \end{bmatrix} = \begin{bmatrix} 0.36 & 0.09 & 0.04 & -0.40 & 0.26 & -0.36 \\ 0.28 & -0.08 & 0.04 & -0.14 & 0.28 & -0.13 \\ 0.16 & -0.01 & -0.08 & -0.009 & 0.57 & -0.64 \\ 0.36 & 0.19 & -0.34 & -0.20 & 0.35 & 0.03 \\ 0.18 & 0.15 & 0.08 & -0.07 & -0.03 & -0.32 \\ 0.08 & 0.41 & -0.10 & -0.20 & 0.34 & -0.48 \end{bmatrix} \begin{bmatrix} \Delta A(-1) \\ \Delta BC(-1) \\ \Delta NB(-1) \\ \Delta NS(-1) \\ \Delta ON(-1) \\ \Delta Q(-1) \end{bmatrix} +$$

$$\begin{bmatrix} -0.84 & 0.21 & -0.11 & 0.11 & 0.13 & 0.34 \\ 0.21 & -0.43 & 0.14 & 0.06 & 0.30 & -0.22 \\ 0.27 & 0.13 & -0.54 & 0.25 & -0.27 & 0.16 \\ -0.27 & -0.23 & 0.43 & -0.46 & 0.53 & -0.05 \\ 0.05 & 0.13 & -0.02 & -0.10 & -0.03 & -0.02 \\ 0.23 & -0.32 & 0.21 & -0.11 & 0.41 & -0.34 \end{bmatrix} \begin{bmatrix} A(-1) \\ BC(-1) \\ NB(-1) \\ NS(-1) \\ ON(-1) \\ Q(-1) \end{bmatrix} + \begin{bmatrix} 12.10 \\ -4.97 \\ -1.12 \\ 4.56 \\ 0.10 \\ -5.85 \end{bmatrix}$$

These equations measure the long run relationship between the six times series throughout the period 1921 to 2009. Here $Z_{i,t}$ represents the stationary variable which quantifies the deviation from the equilibrium of the various life expectancies analyzed. Changes in provincial life expectancy are reflected in these three equations, which also involves change in trends of life expectancy.

The equation in matrix form for males representing the dynamic of life expectancy derived by the Vector Error Correction model is given below. The equations explain the variations of the improvements in mortality by patterns observed from other provinces at the first lag level. The variation in Alberta is explained by the other provinces in their first lag (coefficients are -0.84, 0.21,- 0.11 -0.11, 0.13 and 0.34) and also by the first difference in the mortality of each, as can be observed from the coefficients of the matrices (coefficients are 0.36, 0.09, 0.04, -0.40, 0.26 and -0.36). The same interpretation can be applied to the other provinces.

The three cointegrating relations with the criteria of the trace test are:

$Z_{1t-1} = A_{1t-1} - 12.21BC_{2t-1} + 13.74NB_{3t-1} - 6.08NS_{4t-1} + 12.97O_{5t-1} - 92.72 - 8.19Q_{6t-1}$

$Z_{2t-1} = -12.43A_{1t-1} + BC_{2t-1} + 0.72NB_{3t-1} + 1.20NS_{4t-1} + 3.16O_{5t-1} + 163.05 + 4.25Q_{6t-1}$

$Z_{3t-1} = -0.17A_{1t-1} + -0.14BC_{2t-1} + NB_{3t-1} + 1.08NS_{4t-1} + 1.013O_{5t-1} + 0.07 - 0.61Q_{6t-1}$

Diagnostic tests of residuals are conducted for both Portmanteau and Normality tests. The results provided by Table 3.6 show remaining autoro-correlation as the p-value is equal to 0.0018. However, they show evidence of normality on the residuals as p-value is equal to 0.0675. These results can be expected since we use only a few paramaters. Increasing the number of lags could improve the significance of the test of autocorrelation test. However, for the purpose of forecasting, it is better to use as little lag as possible.

Type of test	Autocorrelation	p-values
Autocorrelation	Portmanteau(4 lags)	0.0018
Normality	Both	0.0675
	Kurtosis	0.07
	Skewness	0.195

Table 3.6: The diagnostics tests of residuals of VECM

3.3 Forecasting procedure and backtesting of the various models

In this section, we fit data from six samples periods including 1921-2000, 1921-2001,1921-2002,1921-2003,1921-2004, and 1921-2005 with the three models analyzed and forecast life expectancy for the remaining part of each sample up to 2009. In this backtesting phase, we compute the Mean Absolute Percentage Error(MAPE) of the three models in six different sample periods 2001-2009, 2002-2009, 2003-2009, 2004-2009, 2005-2009 and 2006-2009. The results are presented in Table 3.5 and show that VAR(0.31%, 0.40%, 0.26% and so on) and VECM(0.29%, 0.27%,0.24% and so on) are reliable in being a good fit for the data as the errors are low for each sample. The ARIMA model presents poor results with the highest error performance(35.73%, 35.92%, 36.28% and so on). This illustrates the fact that forecasts from VECM and VAR are much closer to the historical data. In addition, we deduce that the VECM performed better than the VAR model in the quantification of residuals. This contrasts with the overall results obtained by Torri(2011). In the six regions as we can see(see Figures 3.3 to 3.8), the confidence intervals from the VECM performed better than the VAR and ARIMA models as we can see in Table 3.7. It allows one to account for more risk than other models. Consequently the VECM produces better results than the VAR model in terms of backtesting out-of-sample and quantification of future. The ad-

ditional variables(the first difference of mortality index with Γ coefficients) included in the VECM provide improvements over the VAR model in terms of confidence interval as well as future life improvements since the VAR model only considers variables in terms of their levels. Accordingly, a new approach based on the VECM explains time varying long-run relationship dependence between the various life expectancies of the Canadian regions considered .

Out-Of-Sample	VECM	VAR	ARIMA
h=2001-2009	0.29%	0.31%	35.73%
h=2002-2009	0.27%	0.40%	35.92%
h= 2003-2009	0.24%	0.26%	36.28%
h=2004-2009	0.28%	0.44%	33.73%
h=2005-2009	0.20%	0.23%	34.34%
h=2006-2009	0.28%	0.37%	35.24%

Table 3.7: The average MAPE for the ARIMA VAR and VECM models for the six provinces

Provinces	VECM	VAR	ARIMA
Alberta	(1.04-4.58)	(1.19-1.73)	(1.20-3.44)
British Columbia	(1.07-7.06)	(1.04-1.49)	1.34-2.32
New Brunswick	(1.05-6.52)	(1.18-2.20)	(1.36-5.65)
Nova Scotia	(1.11-6.73)	(1.27-2.09)	(1.32-6.21)
Ontario	(0.65-6.40)	(0.75-1.57)	(0.83-5.88)
Quebec	(1.08-6.33)	(1.24-2.64)	(1.30-6.07)

Table 3.8: The confidence interval of the VAR, VECM and ARIMA models for the six provinces derived from predictions 50 years forecasts

Table 3.8 reports the confidence intervals for each model presented in this work. Results from this analysis show that the VECM performs better than the others models. For example, Alberta's confidence interval length

is 3.58, which is equal to six times that of VAR(0.54) and 1.5 times that of ARIMA(2.24). Observing, British Columbia province shows the greatest interval confidence length. However, the confidence interval associated with ARIMA is smaller for the provinces Nova Scotia, Ontario and Quebec. We observe overall that the VECM is better than the ARIMA and VAR models in capturing the increasing level of life expectancy as well as in fitting historical data.

Here we show the computations of life expectancy at birth derived from the VECM which has proven to be the best model of the three models investigated. The results are exposed in this framework with a frequency of 10 years as in Table 3.9, where we can see future life expectancy from 2010, 2020, 2030, 2040 and 2059. Data from annual frequency results are also available on request. We observe that life expectancy at birth from 2010 to 2059 is close to 90 years particularly in Alberta, British Columbia, New Brunswick, Ontario, and Quebec. Only Nova Scotia shows a life expectancy level below 90.

Year	Alberta	British Columbia	New Brunswick	Nova Scotia	Ontario	Quebec
2010	79.28	78.12	78.02	79.74	79.74	79.30
2020	81.26	82.29	80.67	80.06	82.18	82.36
2030	83.57	84.71	83.23	82.41	84.72	85.59
2040	85.89	87.13	85.79	84.75	87.26	88.82
2050	88.21	89.55	88.35	87.10	89.79	92.05
2060	90.63	91.97	90.92	89.45	92.33	95.27

Table 3.9: Mean forecast of life expectancy with the VECM for the six provinces

3.4 Engineering of longevity bond

As several longevity products such EIB & BNP and Swiss Re bonds(see Blake, 2006 for a review on mortality securities have been proposed in literature, we suggest the regional life bond linked on stochastic value of future lifetime. Burrrows(2001) proposed for the the first survivor bond based on the life time of the last survivor into the portfolio reference. A pension fund can build also new product based on the dynamic of life expectancy from 6 regions within Canada. It consists of an initial payment of X with successive payment coupon C with frequency of 10 years with the maturity of 50 years to correspond to potential investors. These coupons payments depend the dynamic evolution of future life expectancy by province. We compute the variation of life expectancy of the considered regions between the period 2000-2009 which is equal to 2.2. Accordingly, if future life expectancy is greater than 2.2 then pension plan would pay a certain amount C to the investors. Otherwise investors in portfolio will not receive anything. Coupons are discounted at rate linked to Libor. We build a bond with maturity of 50 years since shorter period do not provide effective hedging(see Dowd et al., 2006b). Coupon offered periodically will be of course distributed with frequency of 10 years. In order to have a this product exchanged in financial market, we would assume that there will not credit risk from both part, No basis risk and the markets are liquid.

3.5 Conclusions

In this paper we have investigated the forecasting scenario of multi-population life expectancy for provinces in Canada. We have presented three principal econometric models ARIMA, VAR and the VECM which have appeared recently in the literature. The VECM presents a better performance than ARIMA and VAR models in terms of backtesting, goodness of fit, and future

trend uncertainty quantification as shown by the confidence interval measured here. Furthermore, VECM highlights improvements in understanding the dynamics of life expectancy patterns over time as it captures common trends and also the correlation structure from the provinces monitored. We also illustrate the values of forecasts of life expectancy in the six provinces and found that it will surpass 90 years in the next 50 years except in Nova Scotia. The results from these analysis aim to help social security and insurance companies improve the quantification of future life expectancy and thus price pensions fairly.

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Life expectancy	Constant	lags	Dw stat	constant and trend	Lags	Dw stat
Alberta	0.62	1	2.06	0.027	0	2.05
$\Delta Alberta$	0	0	1.94	0	0	2.06
British Columbia	0.91	2	2.03	0	0	1.56
$\Delta BritishColumbia$	0	0	1.87	0	1	2.04
New Brunswick	0.64	1	2.08	0.42	1	2.08
$\Delta New_{B}runswick$	0.0063	4	1.94	0	0	2.08
$Nova_{S}cotia$	0.83	0	2.42	0.07	0	2.18
$\Delta Nova_{S}cotia$	0	0	2	0	0	2.08
Ontario	0.67	0	2.11	0.1	0	2
$\Delta Ontario$	0	0	1.93	0	0	1.93
Quebec	0.6	1	2.08	0.47	1	2.07
$\Delta Quebec$	0	0	2.08	0	0	2.09

Table 3.10: Unit root(Augmented Dickey Fuller) testing for nine Canadian provinces

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All the graphs fron the current chapter are inserted at the end of the thesis for reasons of spaces.

Life expectancy	Constant	lags	Dw stat	constant and trend	Lags	Dw stat
Alberta	0.72	13	2.32	0.027	0	2.05
$\Delta Alberta$	0	9	2.05	0	10	2.06
British Columbia	0.99	11	2.02	0	2	1.56
$\Delta BritishColumbia$	0	7	2.05	0	7	2.05
New Brunswick	0.35	17	2.71	0.03	0	2.45
$\Delta New_{B}runswick$	0	8	2.08	0	8	2.06
$Nova_{S}cotia$	0.85	3	2.42	0.098	3	2.18
$\Delta Nova_{S}cotia$	0	3	2.08	0	3	2.08
Ontario	0.66	1	2.11	0.1	2	2
$\Delta Ontario$	0	2	1.93	0	2	1.93
Quebec	0.53	10	2.52	0.44	2	2.38
$\Delta Quebec$	0	5	2.08	0	6	2.09

Table 3.11: Unit root(Phillips Perron) testing for nine Canadian provinces

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Life expectancy	Constant	lags	Dw stat	constant and trend	Lags	Dw stat
Alberta	1.19	7	0.01	0.21	6	0.52
$\Delta Alberta$	0.15	13	2.32	0.1	14	2.33
British Columbia	1.2	7	6.43	0.09	5	0.94
$\Delta BritishColumbia$	0.16	11	2.02	0.07	11	2.03
New Brunswick	1.18	7	0.01	0.21	6	0.41
$\Delta New_Brunswick$	0.27	19	2.69	0.11	22	2.72
$Nova_Scotia$	1.18	7	0.006	0.15	6	0.44
$\Delta Nova_Scotia$	0.059	4	2.43	0.053	4	2.43
Ontario	1.19	7	0.006	0.15	6	0.25
$\Delta Ontario$	0.12	0	2.1	0.08	1	2.11
Quebec	1.18	7	0.025	0.18	6	2.52
$\Delta Quebec$	0.23	9	2.51	0.11	10	2.52

Table 3.12: Unit root(KPSS) testing for nine Canadian provinces

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Critical values	Constant	$constant - trend$
Levels	0.463	0.146
Differential variable	0.463	0.146

Table 3.13: Table of unit root(KPSS) of critical values at 5%

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Chapter 4

Projecting the long run relationship of life expectancy between races in USA

4.1 Introduction

Life expectancy as well as mortality rates are getting improved, across developped countries, as several studies such as Tulgapurkar et. al(2007) and Oeppen(2002) have shown. The United States of America are not an exception since they have highlighted also signs of improvements within their national population as well as in different races groups living in the country. To illustrate in 1970, life expectancy of male people was respectively 68 for white race and 60 years old for black. In 2010, there are 76.5 and 71.8. Same results are observed for females as in 1970 white female life expectancy was 75.6 and black 68.3. But, in 2010 there are 81.3 and 78 for respectively white and black. The historical census of the USA population take on mainly population of 2 group races: White and Black people. However in last recent years, there are many other groups races of people who have immigrated in that country such as Latinos and Asian and their incidence is becoming

visible in the amercan society. Most work done on this topic have focused on predicting the pair black-white death rates such as Rives 1977; NCHS (1975), Manton(1980, 1982), Manton et. al(1979) Philipps and Burch(1960), Woodbury et. al(1981), Manton et. al(1979). We will focus not only on two but on more races groups which include life expectancy from asian and latino americans. Furthermore, in this we diverge to Carter(2010) who studied only the dynamic of death rates on the 2 predominant groups in the country.

In 1610, 100% of inhabitants in the USA were white and 10 years later 0.9% of newcommers were black(see table 4.19 retrieved from U.S Bureau Census 1975). In 1780 while 79.9% of population were white, 20.7% the rest was black(see Humes(2001,2010)). From 1850, hispanic population was added in the statistics in the USA census as 0.5% of population. After 10 years, the official census investigates asian newcomers as 0.1% of population and indians as 0.1%. In 1970 people of hispanic origin were 4.4% of the american population. The latter group increase in share to become the second largest comunity of the USA. In 2010, there were 4 communities in the USA including white, hispanic, black as well as asian. During 5 centuries these two racial group have been the two main habitants of the USA as we can see statistics here from table 4.19. The old pair is not anymore the major groups as black people have been surpassed by Latinos recently(see table 4.20). There is also a growing new population mainly with asian origin(chinese, corean, indians). Since, the population in the USA is not anymore a mixture of 2 groups, it arises issues on life expectancy of population by race with additional new comers. We aim at analysing crossover of 6 different groups of people living in the USA including white, black and others races(latinos and asian) with both females and males

Life expectancy (figure 4.1) shows an increasing trend for all the races during the period which spans from 1975 to 2010. We observe that Black

male life expectancy show the lowest level and white females the highest. Three gender's life expectancy including Black females, White males as well as males of other sexes show almost the same level particularly from 2000 onward. Furthermore, white females and females from other sexes present almost the same level from 1975 to 2010. Overall, life expectancy from different races in the USA show an increasing trend and also some convergence point between some of the component under study. The statistics of life expectancy show disadvantages for African American males over the other groups of residents. The general consensus is confirmed here as female's life expectancy is greater than for males in each type of race as it can be seen from figure 4.1.

Life expectancy data were provided by the National Vital Statistics Reports available at www.cdc.gov. It is managed by the Government of the United States of America. This database was created in order to provide information on mortality data, life expectancy, infant mortality and trends by characteristics such as age, sex, race, cause of death in USA to anyone interested in the dynamics of the evolution of demography. It supplies the data which has a frequency of 1 year on life expectancy from 1940 to 2010 for white, black and other races. In this reference all races group refer to other races than white and black. The cds's statistician regroup life expectancy of hispanic and asian into the same pool as other races. Statistics are available for Non-Hispanic separately are available only from 2006. We will work with data from other races as they can give us indication on future life expectancy for other races in both sexes. We first present the literature on principal component and then ARIMA before looking at the cointegration approach.
The methodology, which applies the cointegration, includes several steps as follows:
- The determination of order of integration for each of the 6 life expectancy using the Augmented Dickey Fuller, Philips-Perron as well as KPSS Test;
- The computation of the optimal value of lag of the vector autoregressive

model;
- the Johansen cointegration test which investigates the cointegration rank and specify which variable will enter in the cointegrated equations and in the Vector Error Correction model;
- the estimation of VECM and VAR and the forecasting of derived models.
- furthermore, third part will be reserved to the comparison between the ARIMA, VAR model forecasting and the VECM in terms of goodness of fit and robustness. Steps of cointegration analysis are described by Juselius(2006) and Harris and Sollis(2003). The empirical analysis will be done by using R statistical software package developed by Pfaff(2008).

4.2 Principal component analysis

The principal component analysis (Joliffe, 2002) helps to extract component from a multivariate dataset and aimed to reduce to some factors principal components which must be uncorrelated. It explains variance of dataset while keeping as much as possible the relevant information. Principal component analysis(PCA) is a tool that is increasingly used to model mortality rates and longevity risk. Yang and Yue(2008) find that the principal component are better in forecasting mortality rates than the standard models such as Lee Carter. In order to visualize the effects of components, there are standards procedures which include:
- the first step consists of calculating the mean of the variables;
- second, the matrix variance-covariance are computed for the multivariate dataset;
- thirdly the eigenvalues and the eigenvectors are derived. Using larger number of principal component does not matter since it can worse the predictive power and the accuracy of the model.

The excessive number of components can lead to distortions in the model. The corresponding PCA can also give the results of score(values associate to every single point) as well as loadings.

Let consider a multivariate dataset such that $y = (y_1, y_2, y_3, ...y_T)$. Following the PCA scheme, we calculate the mean of the data with the following formula in (4.1) as:

$$y = \frac{\sum_{t=1}^{T} y}{T} \tag{4.1}$$

Then we can compute the covariance af these data according to (4.2):

$$\sigma_{i,j} = cov(y_i, y_j) = E[(y_i - \hat{y})(y_j - \hat{y})^T) \tag{4.2}$$

where $i = 1, 2, ...N$ $j = 1, 2, ...N$ if $i = jcov(y_i, y_j) = \sigma_i^2 = variance$

From the variance-covariance matrix, we compute the corresponding eigenvalues λ_i and the eigenvectors such that $\sigma_{ij} u_i = \lambda_i u_i$ with $i = 1, ...N$. The solution of this equation can be found by solving the following equation in (4.3):

$$det|sigma_{ij} - \lambda I| = 0 \tag{4.3}$$

where I is the identity matrix which has the same rank as σ_{ij}. The principal component are computed as the vectors u_i and the associated λ_i, the eigenvalues describe the variance associated to the eigenvectors. The largest λ_i is associated to the largest variance of the data. The first principal component is associated to the largest eigenvalue , the second is associated to the second largest eigenvalues and so on.

The Lee-Carter model(1992) can be identified as model with 1 principal component. However most of research papers results show more components are used in explaining mortality rates(see Hyndman et.al(2002)). Indeed, Heligman and Pollard(1980) use a 3 principal component model to capture

mortality data in Australia while Yang et al(2009) proposes a 2 pc model to forecast model mortality rates and show better predictive power than Lee Carter.

Regarding life expectancy by races, it is the first work which principal components analysis are used in literature to explain the dynamics of life expectancy. Earlier analysis of correlation show positive relationships between the various life expectancy by race under study as we can see from figure 4.1. In order to explain the 99.33%, the analysis of the principal components show that there are 2 components(see figure 4.2 and table 4.1). In addition these analysis show that 3 components are sufficient to explain the 99.99%. Furthermore the volatility of data are mainly driven by the first of the three components as volatility is 2.40 for 96% and for the third component volatility it is 0.18 to explain 99.99%. These results are suggesting potential common trends among life expectancy by race as you can see from table 4.1.

Standard deviation	Percentage of variance	Cumulative percentage
2.40	0.96	0.96
0.39	0.025	0.9933
0.18	0.005	0.999
0.069	0.00079	0.99980
0.029	0.0014	0.999
0.0172	0.00495	1

Table 4.1: The percentage of the observed variance on the 6 life expectancy by races explained by principal component analysis

4.3 Models and methodologies

In order to perfom the analysis from the three models, we first visualize(see figure 4.1) that life expectancy at birth are non stationary. We compute the unit root test from the life expectancy data of each race through the ADF test(see Dickey and Fuller(1979)), (see Philipps and Perron(1988)) as well as the KPSS(Kwiatkowski et.al(1992)). As for unit root tests(table 4.12, 4.13, 4.14) we compute the results from ADF tests. The test statistics are Φ_3 as we can see with trend criterion. It shows that life expectancy is integrated of order 1. The results under drift criterion also strenghtens that life expectancy from different races are integrated of order 1. These results are confirmed also by both results obtained by PP-tests(table 4.15 and 4.16) as well as KPSS tests(table 4.17 and 4.18).

The life expectancy is modelled as stochastic processes. The methodology consists of three phases including identification, estimation and diagnostics. The three steps are all described in Box and Jenkins(1976) or Hyndman(2013) who explain the process to choose an appropriate ARIMA(p,d,q) for a variable. The goal is to identify the correct model that fits well time series under study. There are two options used by existing literature to select the best model to use: selection of model by the user(Jenkins, 1976) or the automatic arima(see Hyndman, 2013). In general ARIMA is described in (4.4) as follows:

$$L_t = A_0 + A_1 L_{t-1} + e_t \tag{4.4}$$

where A_0 is the drift term , A_1, L_{t-1} is the lagged time series, and e_t is the error term.

The principal steps of the procedure are the following:

Identification of the model: consists to plot data and identify the pattern of the time series. As we can observe in figure 4.1, life expectancy presents a

positive trend with drift.The basics analysis consist also of differencing data until they appear to be stationary. The unit root tests, Augmented Dickey Fuller(ADF), Phillips-Perron(PP) and Kwiatkowski-Phillips-Schmidt-Shin(KPSS), tests are useful to determine the level of stationarity. The order of integration derived from the 3 unit roots corresponds to the value of parameter d. The successive path consists of choosing the best model corresponding to the lowest Akaike information criterion(AIC).

Identification of the order of the model: After derivation of the order of stationarity, one should experiment various combinations of p,d and q with p the number of autoregressive parameters d drift, q, the moving average parameters (q) to produce the best model. A Box-Cox transformation could be used to stabilize the variance if necessary.

As for the third step: the researcher, in a standard procedure, experiments various combinations of p and q with the number of autoregressive parameter d(derived in the first phase) q the moving average parameter (q) to produce the best model. It is recommended at this stage to examine the autocorrelation(ACF), the partial autocorrelation(PACF) and the diagnostic of residuals graphs to aid at choosing of the appropriate model. Hyndman and Athanasopoulos(2013) develop an automated algorithm which consists of the inclusion of a constant. Also, Box and Jenkins(1976) advised relying on the AIC and SIC(Schwarz criterion) to choose the best model.

Fourth phase regards the model validation which checks the diagnostics of residuals from the choosen models by plotting the Autocorrelation residuals and conducting a Portmanteau test of the residuals. The residuals are then checked to see whether they are white noise. The procedure is completed by computing the forecasts through the choice of the best fitting model. Table 4.10 displays the best numerical results of the procedure described here above.The Portmanteau test(see table 4.11) indicates significant residual autocorrelation with 4, 10, 15, 20 lags for each of the provinces. These results suggest that ARIMA appears to well behave with white noise disturbances

to forecast future life expectancy.

4.3.1 Cointegration methodology and forecasting future life expectancy

The Vector Autoregressive model(VAR) can be described as follows in (4.5):

$$L_t = \theta + \eta_1 L_{t-1} + \eta_1 L_{t-2} + ...\eta_p L_{t-p} + e_t \tag{4.5}$$

where $L_t = (L_{1,t}, L_{2,t}, ...L_{N,t})^{'}$, is a N-dimentional time series, η_i are matrix with coefficient $(K * K)$, $\nu = (\nu_1, \nu_2, ...\nu_p)^{'}$ is the intercept term, ζ_t is white noise, $t = 0, 1, ...T$.

If we estimate the variables included in a VAR model for predictions in the future, we select the lag p in order to get the best predictive model with least error. The optimal lag length of variables in VAR and VECM models are derived by choosing the lag p which minimizes the value of information criteria model such as Akaike(AIC), the HQ(Hannan-Quinn), the Schwarz(SC) as well as Final Prediction criteria(FPE). The optimal lag is obtained by minimizing one of the following criteria stated below in (4.6), (4.7), (4.8) as well as (4.9):

$$AIC(p) = log\ det(\sum(p)) + \frac{2}{T}pK^2 \tag{4.6}$$

$$HQ(p) = log\ det(\sum(p)) + \frac{2logT}{T}pK^2 \tag{4.7}$$

$$SC(p) = log\ det \sum(p) + \frac{logT}{T}pK^2 \tag{4.8}$$

$$FPE(p) = \frac{T + p^*}{T - p^*})(\frac{T + p^*}{T - p^*})^K det\ \sum(p) \tag{4.9}$$

Where $\sum p$ is estimated by $T^{-1} \sum_{i=1}^{n} u_t u_t^{'}$, p^* is the total number of parameters in each equation of the model when p is the lag order of the endegeneous variables. In many cases, Information criteria may identify the same lag or

different levels. However, Lutkepohl(2005) suggests that in case the criteria choose different lags, SC criteria is the more parsimonious in choosing the good forecasting models than others because it produces the lowest forecast error variance. And p_* is the local number of the parameters in each equation and p assigns the lag order VAR(p) is said to be stationary if and only if : $det(I_k - A_1 z - A_2 z - A_p z^p) \neq 0$ with $-1 \leq z \leq 1$. If $z = 1$, at least one of these variables inserted into VAR(p) are integrated of order 1. Therefore cointegration does exist between variables. This conclusion introduces the Vector Correction Model(VECM). According to Pfaff(2008) and Engle(1987), the VAR (p) in the equation (4) can be converted into VECM. It is converted into the short run term and the long run. The latter form of VECM, which will be used in our case studies is defined, in (4.10) as follows:

$$\Delta L_t = \Gamma_1 \Delta L_{t-1} + \Gamma_2 \Delta L_{t-2} + ... + \Gamma_{p-1} \Delta L_{t-p+1} + \Pi L_{t-p} + \nu + \varepsilon_t \quad (4.10)$$

where $\Gamma_i = -(I - \eta_1 + ... - \eta_i)$, for $i = 1, ...p-1$ and $\Pi = -(I - \eta_1 - ..... - \eta_p)$.

The forecasting of mortality indices is operated through the Johansen Maximum Likelihood methodology. It is based on the Vector Error Correction model.

If $r = K$, the number of cointegrated variables r which are stationary equals the rank(K) of π then the model will be estimated by using the standard statistical model.

If $r = 0$ this means that there is no cointegrated relationships between the variables. The variables are stationary if we take the differences of variables above.

If $0 < r < K$ there exists 2 matrices α and β such that $\Gamma = \alpha\beta'$ and there will be r cointegrating relationship or $n - r$ common trends. Variables into the VECM are all stationary. The test of cointegration is reduced to the two following hypothesis:

The rank test which is specified in the following form as in (4.11):

$$H_0 : rank(\Pi) = r, H_1 : rank(\Pi) > r \tag{4.11}$$

The Likelihood Ratio statistic is explained in (4.12) and (4.13):

$$LR(r) = -(T-p)\sum(1-\lambda_i) \tag{4.12}$$

$$LR(r) = -(T-p)\sum(1-\lambda_i) LR(r) = -(T-p)\sum(1-\lambda_i) \tag{4.13}$$

$r = number\ of\ cointegrated\ relationship$, λ is the eigenvalue associated to the linear relationship. The decision of the cointegration rank is based on the trace test and the maximum eigenvalue test of Johansen(1988 and 1991). In addition, the test on Maximum eigenvalue test is specified as follows in (4.14):

$$H_0 : rank(\Pi) = r, H_1 : rank(\Pi) = m + 1 \tag{4.14}$$

The related statistic value is in (4.15):

$$LR(r) = -(T-p)\sum ln(1-\hat{\lambda}_{m+1}) \tag{4.15}$$

The first step path followed to derive the cointegration approach consists of computing the optimal lag which drives the dynamic of life expectancy.

Information criteria	Life expectancy by race
AIC	3
HQ	2
SC	1
FPE	2

Table 4.2: Optimal lag length for both females and males

We compute the optimal lag length of the VAR model. The Information criteria(see table 4.2) shows the following results: AIC choose 3 optimal lags, HQ and FPE the value of 2 while SC only 1. Since they differ each from others, Lutkepohl(2005) shows that the preference will be given to SC. Consequently, the lag lenth is 1. Denoting by A= all races males, B=All races Females, C= White Males, D=White Females, E= Black Males, F=Black Females, The VAR model is described empirically with the following equations:

$$L_{a,t} = -0.34L_{a,t-1} - 1.04L_{b,t-1} + 1.26L_{c,t-1} - 0.03L_{d,t-1} - 0.041L_{e,t-1} + 0.54L_{f,t-1} + 0.06\lambda + 51.42$$

$$L_{b,t} = -0.33L_{a,t-1} - 0.53L_{b,t-1} + 0.33L_{c,t-1} + 0.17L_{d,t-1} - 0.17L_{e,t-1} + 0.70L_{f,t-1} + 0.07\lambda + 65.38$$

$$L_{c,t} = -0.27L_{a,t-1} - 0.84L_{b,t-1} + 1.03L_{c,t-1} + 0.14d, t-1 - 0.079L_{e,t-1} + 0.34L_{f,t-1} + 0.085\lambda + 50.42$$

$$L_{d,t} = -0.43L_{a,t-1} - 0.39L_{b,t-1} + 0.32L_{c,t-1} + 0.31L_{d,t-1} - 0.14L_{e,t-1} + 0.49Lf, t-1 + 0.084\lambda + 63.65$$

$$L_{e,t-1} = -0.60L_{a,t-1} - 3.11L_{b,t-1} + 1.18L_{c,t-1} + 0.67L_{d,t-1} + 0.59e, t-1 + 1.28f, t-1 + 0.06\lambda + 80.067$$

$$L_{f,t-1} = 0.18L_{a,t-1} - 1.32L_{b,t-1} - 0.51L_{c,t-1} + 0.46L_{d,t-1} - 0.064L_{e,t-1} + 1.36L_{f,t-1} + 0.12\lambda + 67.35$$

The diagnostic test on residuals in table 4.3 show evidence of normality (with p-value=0.77) as well as non autocorrelation of residuals (with p-value=0.91). Since normality as well as non autocorrelation of residuals are held, we conclude that this model, which shows white noise on residuals,

well behaves to forecast future life expectancy from different races in the USA.

Type of test	Autocorrelation	p-values
Autocorrelation	Portmanteau(4 lags)	0.91
Normality	Both	0.77
	Kurtosis	0.55
	Skewness	0.78

Table 4.3: The diagnostics tests of residuals of VAR

4.3.2 Vector Error Correction model

After obtaining the VAR, we can derive easily Vector Error Correction Model. It is fitted for the 6 life expectancy subdivided by race groups which span the period from 1975 to 2010. We performed the trace test and the Eigenvalue maximum for the cointegrated equations. We test the Johansen cointegration on the life expectancy from different life expectancy by races which are all integrated of order 1. They are estimated using the maximum likelihood estimators and the results are presented here below for both eigen and trace test values as follows:

The Johansen cointegration tests results are shown here(see tables 4.4 and 4.5). Careful analysis shows 1 cointegrated equation for the 6 various groups in both cases of trace and eigenvalue tests driven by 5 common trends. Both the trace and the eigenvalue tests have shown that there is 1 cointegration relation as follows:

$Z_{1t-1} = A_{1t-1}0.72B_{2t-1} - 1.02C_{3t-1} - 0.31D_{4t-1} + E_{5t-1} - 0.24F_{6t-1}$

Here $Z_{i;t}$ represents the stationary variable which quantify the deviation from the equilibrium of the various life expectancies analysed.
The cointegration equations shows the dependence of each race life expectancy

Cointegrating relationship	critical values	5pct	1pct
5	0.64	8.18	11.65
4	8.02	14.90	19.19
3	13.19	21.07	25.75
2	19.65	27.14	32.14
1	23.58	33.32	38.78
0	57.79	39.43	46.82

Table 4.4: The cointegration relations under Trace test

Cointegrating relationship	critical values	5pct	1pct
5	0.64	8.18	11.65
4	8.65	17.95	23.52
3	21.84	31.52	37.22
2	41.49	48.28	55.43
1	65.07	70.60	78.87
0	122.85	90.39	104.20

Table 4.5: The cointegration relations under Trace test

group with others. In these equations, we see that there is an interdependence between various races in the United States of America. Since this study is carried on life expectancy of different races, it also implies that there exist an interdependence between life expectancy(from which the mortality indices are derived). The life expectancy improvement in one group race is not only influenced by the lagged life expectancy of the same race but also influenced by improvement from another group.

$$\begin{bmatrix}\Delta a\\ \Delta b\\ \Delta c\\ \Delta d\\ \Delta e\\ \Delta f\end{bmatrix} = \begin{bmatrix} 0.32 & -1.01 & -0.007 & 0.69 & -0.026 & 0.17\\ -0.48 & -0.83 & 0.95 & 0.07 & -0.22 & 0.67\\ 0.062 & -0.99 & -0.003 & 0.53 & -0.03 & -0.42\\ -0.42 & -0.70 & 0.58 & 0.27 & -0.21 & 0.63\\ 1.82 & -1.35 & -1.11 & -0.34 & -0.35 & 1.50\\ 0.72 & -0.44 & -0.38 & -0.44 & -0.45 & 0.94\end{bmatrix} \begin{bmatrix}\Delta A(-1)\\ \Delta BC(-1)\\ \Delta NB(-1)\\ \Delta NS(-1)\\ \Delta ON(-1)\\ \Delta Q(-1)\end{bmatrix} +$$

$$\begin{bmatrix} -1.62 & -0.23 & 1.94 & -0.88 & -0.07 & 0.52\\ 0.43 & -0.54 & -0.16 & -0.06 & -0.15 & 0.29\\ -0.14 & 0.06 & 0.364 & -0.58 & -0.12 & 0.21\\ 0.04 & 0.45 & 0.22 & -1.01 & -0.13 & 0.17\\ -0.87 & -0.27 & 1.59 & -1.27 & -0.39 & 0.76\\ -0.31 & -0.08 & 0.41 & -0.06 & 0.05 & -0.03\end{bmatrix} \begin{bmatrix}A(-1)\\ BC(-1)\\ NB(-1)\\ NS(-1)\\ ON(-1)\\ Q(-1)\end{bmatrix} + \begin{bmatrix}31.31\\ 17.48\\ 17.65\\ 20.82\\ 39.08\\ 4.68\end{bmatrix}$$

Analysis on diagnostic residual test(see table 4.6) shows also that the residuals are normal (with p-value equals to 0.50) and are non autocorrelated(p= 0.98). The significance of these two tests show that VECM is appropriate to forecast future life expectancy.

Type of test	Autocorrelation	p-values
Autocorrelation	Portmanteau(4 lags)	0.98
Normality	Both	0.5076
	Kurtosis	0.5078
	Skewness	0.42

Table 4.6: The diagnostics tests of residuals of VECM

4.3.3 Forecasting procedure and Backtesting of the various models

In the phase of backtesting, we compute the MAPE each model(see table 4.7) from 6 different sample period 2000-2010, 2001-2010, 2003-2010, 2004-2010, 2005-2010 as well as 2006-2010, 2007-2010, 2008-2010. We oberve first that the VAR and VECM are reliable in fitting well data. The error are close to 0.50% for each sample in table 4.7. Second, ARIMA model present definetly results with higher error performance mean which is around 7%. The added components on the firsts two models have improved the performance of life expectancy prediction for each model. The VECM realized better performance than VAR.

Out-Of-Sample	VECM	VAR	ARIMA
h=2000-2010	0.5%	2.31%	5.1%
h=2001-2010	0.55%	2.3%	5.8%
h=2002-2010	0.4%	0.62%	6.2%
h= 2003-2010	1.02%	0.77%	6.41%
h=2004-2010	1.1%	0.60%	6.69%
h=2005-2010	1.39%	0.48%	7.37%
h=2006-2010	0.280%	0.62%	7.34%
h=2007-2010	0.29%	0.32%	7.9%
h=2008-2010	0.19%	0.42%	8.39%

Table 4.7: The average MAPE for models, ARIMA VAR and VECM for the 6 provinces

These findings are strengthened by the table 4.8 which show the confidence interval. In the 6 regions as you can see in figures 5 to 9, the confidence interval produced by VECM presents better performance than VAR. However, the VECM model present similar confidence interval for some races with ARIMA even though for Black race it is better definetely in explain-

ing uncertainty. It allows to account for more risk than other models. The error term is really higher for ARIMA. Consequently the VECM presents better results in terms of backtesting out-of-sample and of confidence interval. Accordingly, life expectancy by races must be explained by taking into consideration the long run relationships between races rather than exploring first the Lee Carter model parameters and the derived life expectancy.

Races	VECM	VAR	ARIMA
All sexes Males	(0.23-2.13)	(0.24-0.46)	(0.31-2.24)
All sexes Females	(0.23-1.82)	(0.26-0.72)	(0.35-1.89)
White females	(0.21-9.21)	(0.23-0.31)	(0.28-2.04)
White Males	(0.35-5.21)	(0.23-0.62)	(0.31-3.12)
Black Females	(0.35-7.66)	(0.80-2.17)	(0.9-6.35)
Black Males	(1.08-6.33)	(0.40-1.68)	(0.47-4.72)

Table 4.8: The Confidence interval of models VAR, VECM and ARIMA for the 6 provinces derived from predictions 50 years ahead

Year	All males	All races females	White Males	White females	Black Males	Black Fem
10	78.43	82.25	78.42	82.32	75.27	80.46
20	80.59	83.46	80.34	83.35	78.18	82.59
30	82.73	84.65	82.27	84.39	80.39	84.67
40	84.87	85.85	84.20	85.43	83.77	86.73
50	87.01	87.05	86.112	86.47	86.56	88.79

Table 4.9: Future forecast of life expectancy with model VECM for the 6 provinces

Models	All Sex Ma	All Sex Fe	White Ma	White Fe	Black Ma	Black Fe
ARIMA(p,d,q)	(0,1,0)	(1,1,0)	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)
ar1		-0.32	-			-
(se)		(0.15)	-	(0.10)	-	-
ma1					-	-
(se)		-	-	-	-	
ma2		-	-	-	-	-
(se)	-	-	-	-	-	-
drift	0.21	0.12	0.20	0.20	0.22	0.19
(se)	(0.02)	(0.02)	(0.025)	(0.02)	(0.077)	(0.04)

Table 4.10: The best ARIMA models from the analysis of life expectancy

lags	All Sex Ma	All Sex Fe	White Ma	White Fe	Black Ma	Black Fe
4 ags	0.63	0.77	0.09	0.53	0.63	0.57
10 lags	0.66	0.87	0.24	0.91	0.66	0.94
15 lags	0.10	0.66	0.08	0.45	0.10	0.93
20 lags	0.11	0.59	0.13	0.11	0.11	0.75

Table 4.11: P-values of the Portmanteau test resulted from ARIMA models over the period 1921-2009

4.4 Conclusion

Since the last century, life expectancy trend is increasing in developed countries. In this part of our thesis we have explored new developpement on forecasting life expectancy by race in the USA. We take into account the emergence of new ethnic groups such as hispanic and asians. Early analysis based on principal component analysis have shown potential existence of common trends. This has been confirmed by the cointegration analysis through Vector Error Correction model as the error term is more precise than

in ARIMA model. The cointegration analysis are applied on life expectancy by races through the period from 1975 to 2010. VAR and VECM have shown better performance than ARIMA model in backtesting samples as well as in the evaluation of error components. Furthermore the results show also that life expectancy will be improving in the future for all the group races in the future as well as their future trends. These results are helping and giving some insights to demographers on the performace of future life expectancy for each group of race living in the USA.

Races	1910	1950	1970	2000	2010
White	88.9%	89.5%	87.7%	75.1%	72.4%
Black	10.7%	10%	11.1%	12.3%	12.6%
$American/Indian$	-0.3%	0.2%	0.8%	3.8%	4.9%
Asian	0.2%	0.2%	0.8%	3.8%	4.9%
Hispanic	0.9%	0.8%	0.1%	12.5%	16.3%

Table 4.12: Statistics census of American population

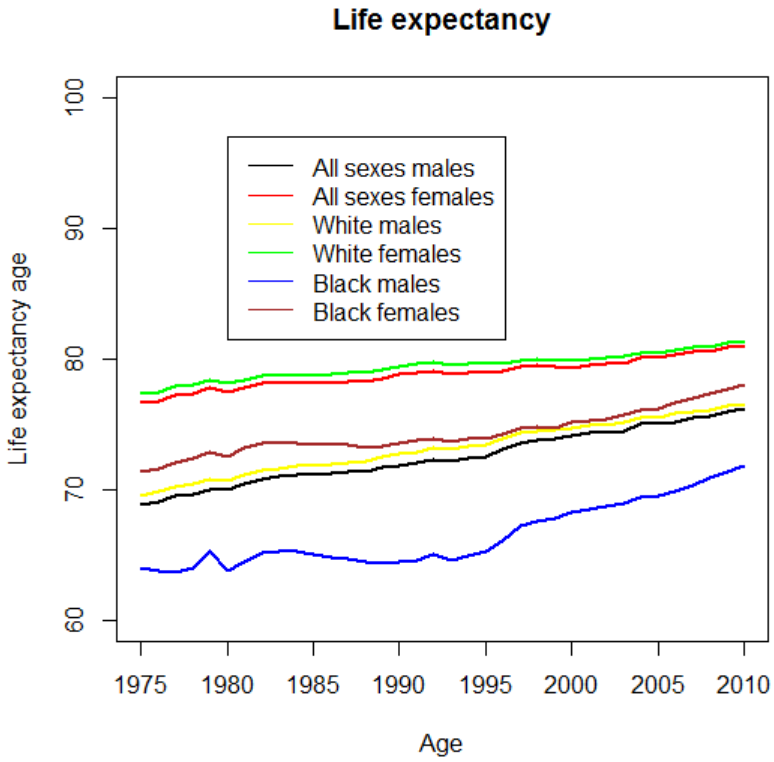

Figure 4.1: Life expectancy by races from all groups in the USA

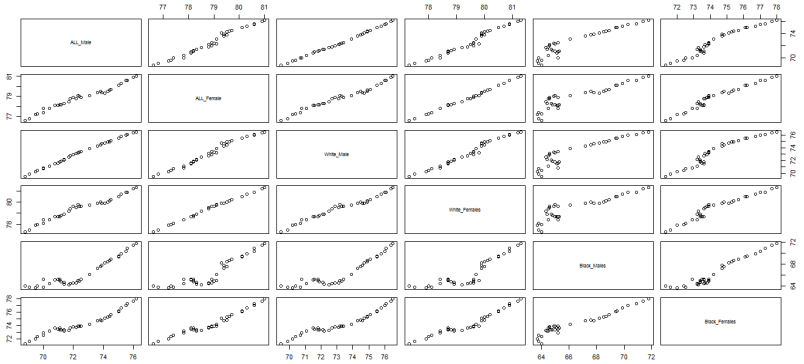

Figure 4.2: Correlation relations in the three groups in life expectancy subdivided by races

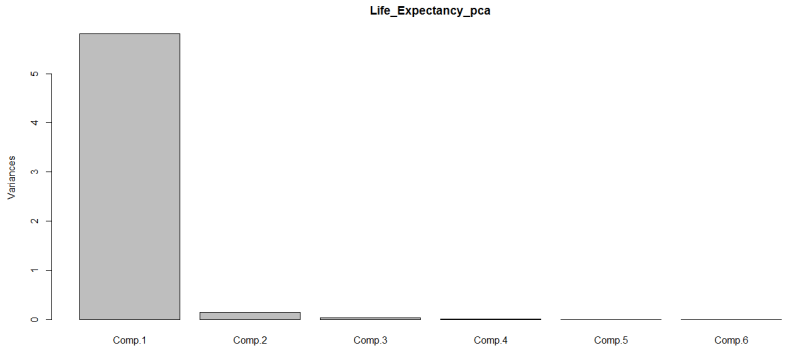

Figure 4.3: Principal Component analysis

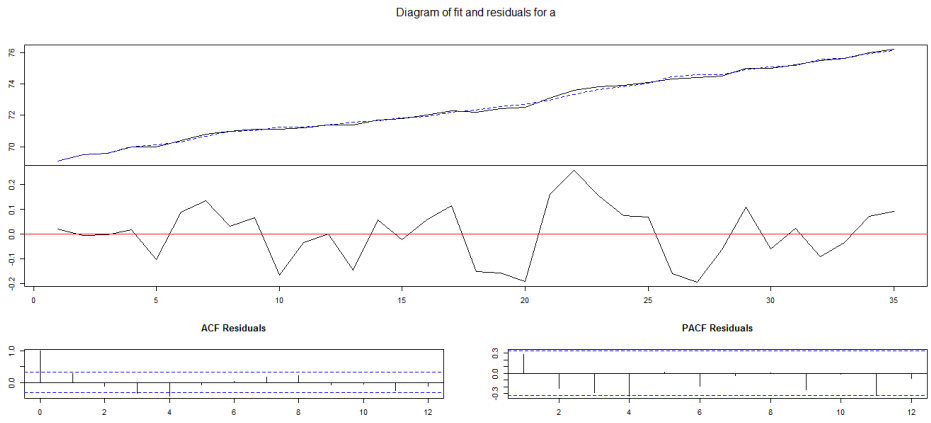

Figure 4.4: Diagnostics of residuals with reference to All sexes

Figure 4.4: Projections males life expectancy from all other races in the USA

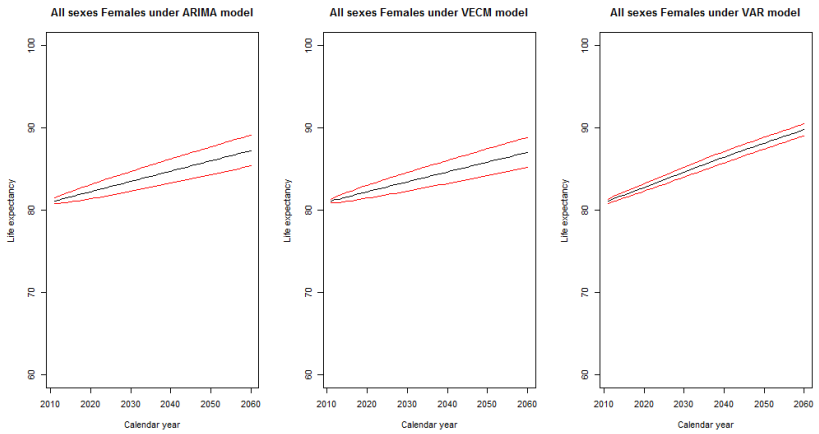

Figure 4.5: Projections females life expectancy from all other races in the USA

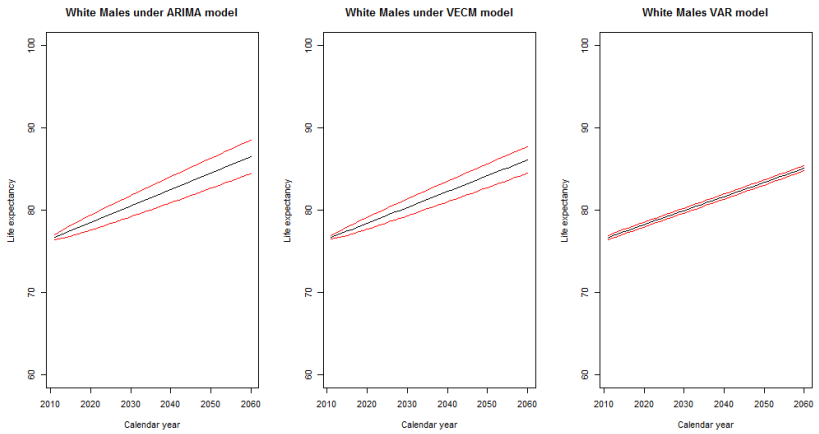

Figure 4.6: Projections life expectancy from white males in the USA

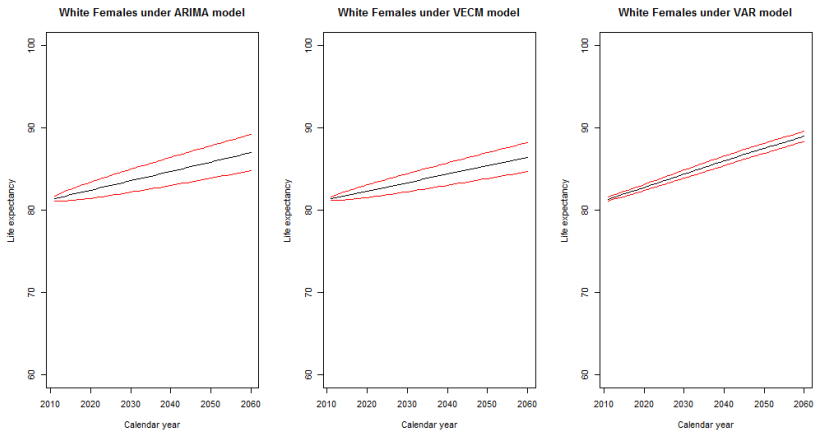

Figure 4.7: Projections life expectancy by races from white females in the USA

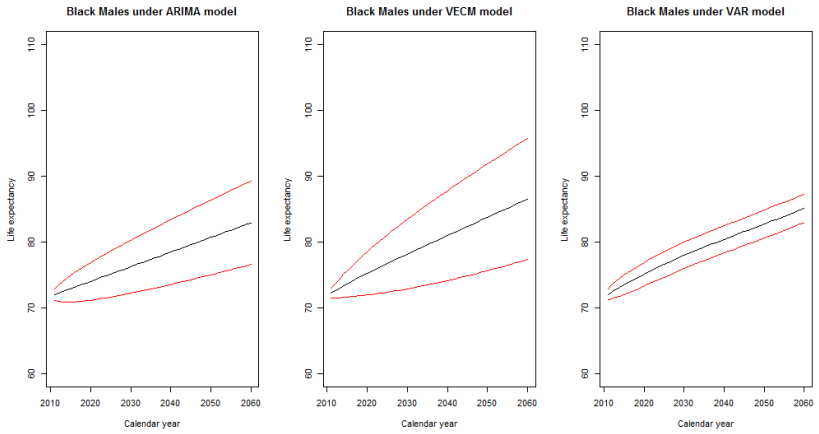

Figure 4.8: Projections life expectancy from black males in the USA

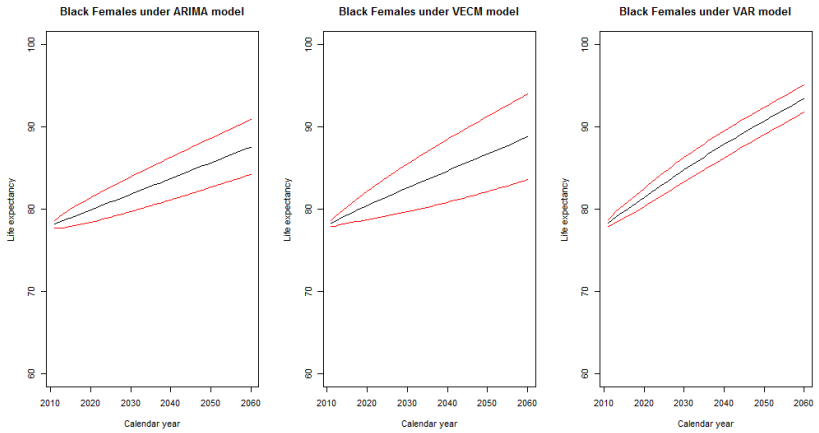

Figure 4.9: Projections life expectancy from black females the USA

Life expectancy	τ_2	ϕ_1	τ_3	ϕ_2	ϕ_3
ALL Males	0.14	4.82	-2.13	5.19	2.33
$\Delta ALL\ Males$	-3.75	7.06	-3.71	4.72	7.08
ALL Females	0.14	3.65	-2.70	9.69	3.67
$\Delta ALL\ Females$	-4.56	10.52	-4.53	7.03	10.52
White Males	-0.61	4.38	-3.05	12.23	4.94
$\Delta\ WhiteMales$	-2.49	3.13	-4.11	5.71	8.50
White Females	-0.085	3.67	-3.21	10.24	5.26
$\Delta White\ Females$	-4.48	10.12	-4.41	6.78	10.14
Black Females	0.92	2.26	4.14	0	2.11
$\Delta Black\ Females$	-1.83	1.68	-4.33	6.31	9.42
Black males	1.38	2.71	-0.56	4.31	0.64
$\Delta Black\ males$	-2.38	2.85	-4.04	5.581	8.37

Table 4.13: Unit root(Augmented Dickey Fuller) testing for 6 life expectancy by races in USA

.

Test statistics	1%	5%	10%
τ_3	-4.15	-3.50	-3.18
ϕ_2	7.02	5.13	4.31
τ_3	9.31	6.73	5.61

Table 4.14: Unit root cutoff of test statistics under trend criterion

.

Test statistics	1%	5%	10%
τ_2	-3.58	-2.93	-2.60
ϕ_1	7.06	4.86	3.94

Table 4.15: ADF test statistics under constant criterion

.

Life expectancy	$Z(mu)_constant$	$\alpha_constant$	$Z(\alpha)$	$Z(mu)$	ϕ_3
ALL Males	0.51	-0.24	2.82	2.19	2.53
$\Delta ALL\ Males$	5.27	-6.76	0.01	7.42	-6.65
ALL Females	0.80	-0.73	2.87	3.47	-3.29
$\Delta ALL\ Females$	4.45	-8.26	-0.20	4.86	-8.11
White Males	-1.79	-1.56	-0.45	6.12	-3.28
$\Delta\ WhiteMales$	5.16	-6.90	-0.91	8.90	6.85
White Females	1.15	-1.08	0.44	4.54	-3.24
$\Delta White\ Females$	4.15	-7.32	-0.48	5.12	-7.23
Black Females	-1.52	1.64	1.41	1.56	-0.95
$\Delta Black\ Females$	2.79	-6.11	1.93	3.791	-6.74
Black males	-0.34	0.42	2.69	-0.03	-1.15
$\Delta Black\ males$	3.52	-5.76	1.02	3.02	-5.74

Table 4.16: Unit root(Philips Perron) testing for 6 life expectancy by races in USA

.

Test statistics	1%	5%	10%
τ_2	-4.20	-3.54	-3.20

Table 4.17: PP test statistics under constant criterion

.

Life expectancy	P-value	KPSS Level	Truncation	P-value	KPSS Level	Truncation
ALL Males	0.01	1.86	1	0.02	0.17	1
$\Delta ALL\ Males$	0.1	0.072	1	0.1	0.07	1
ALL Females	0.01	1.80	1	0.04	0.15	1
$\Delta ALL\ Females$	0.1	0.11	1	0.1	0.10	1
White Males	0.01	1.87	1	0.1	0.07	1
$\Delta\ WhiteMales$	0.1	0.11	1	0.1	0.05	1
White Females	0.01	1.81	1	0.04	0.15	1
$\Delta White\ Females$	0.1	0.12	1	0.1	0.10	1
Black Females	0.01	1.63	1	0.01	1.63	1
$\Delta Black\ Females$	0.052	0.45	1	0.1	0.063	1
Black males	0.01	1.70	1	0.01	0.32	0.01
$\Delta Black\ males$	0.1	0.26	1	0.02	0.17	1

Table 4.18: Unit root(KPSS) testing for 6 life expectancy by races in USA

.

Test statistics	1%	5%	10%
τ_2	-3.63	-2.94	-2.61

Table 4.19: PP test statistics under constant criterion

.

Races	1610	1710	1800	1850	1900
White	100%	86.5%	81.1%	84.3%	87.9%
Black	0%	13.5%	18.9%	15.7%	11.6%
$American/Indian$	-	-	-	0.5%	0.3%
Asian	-	-	-	-	0.2%
Hispanic	-	-	-	0.50%	0.7%

Table 4.20: Statistics census of American population

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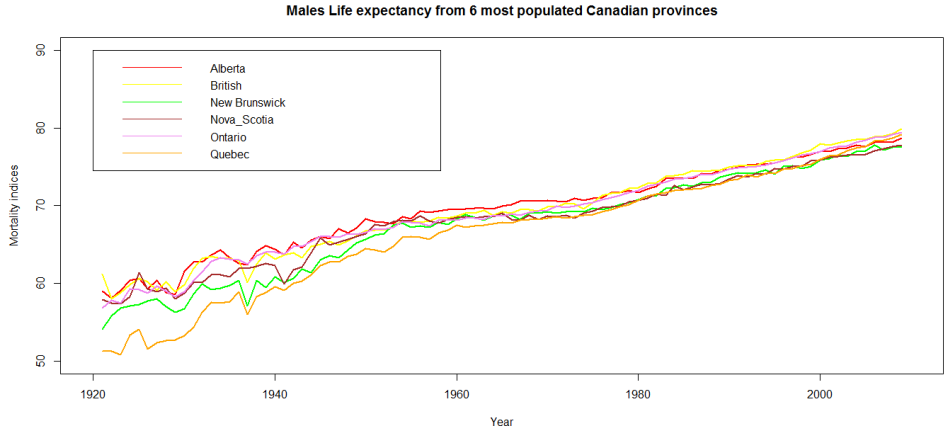

Figure 3.1: Male Life expectancy in the six Canadian provinces

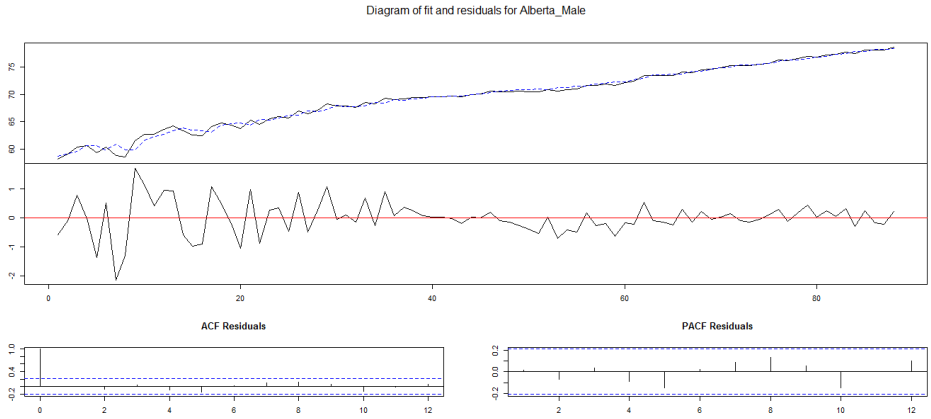

Figure 3.2: Diagnostics of residuals with reference to Alberta

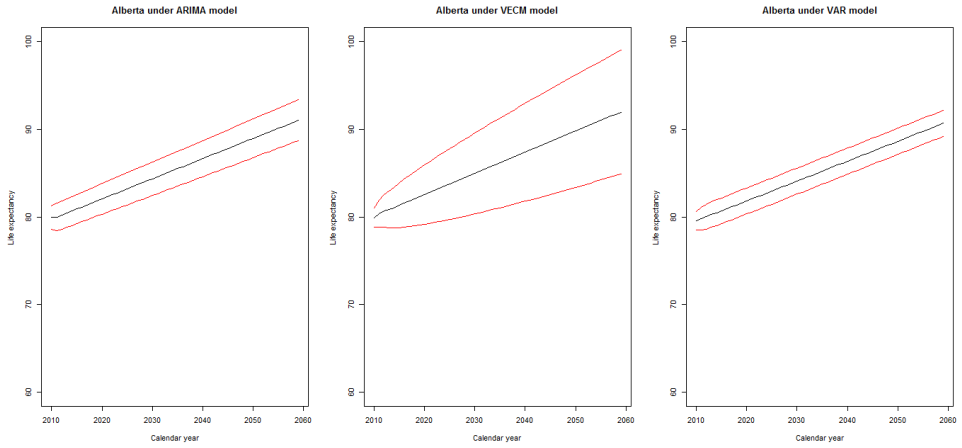

Figure 3.3: Life expectancy of Alberta: the red lines represent the lower and upper forecasting, the black line represents the point forecast

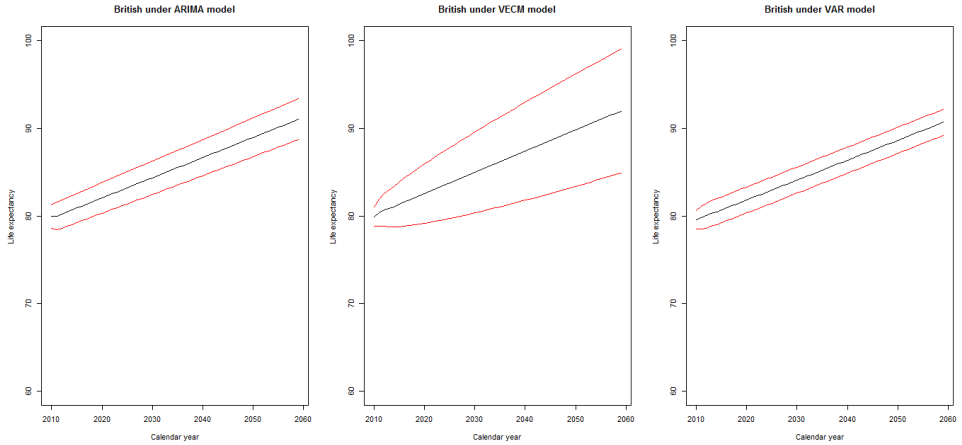

Figure 3.4: Life expectancy of British Columbia:the red lines represent the lower and upper forecasting, the black line represents the point forecast

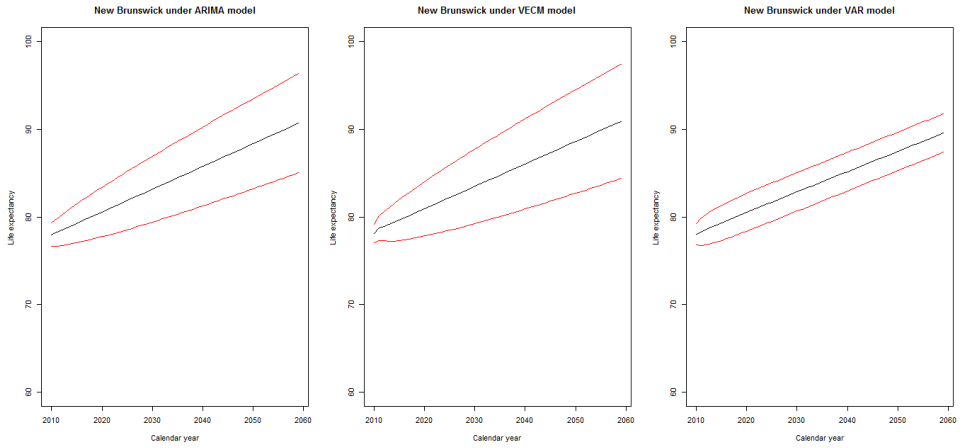

Figure 3.5: Life expectancy of New Brunswick: the red lines represent the lower and upper forecasting, the black line represents the point forecast

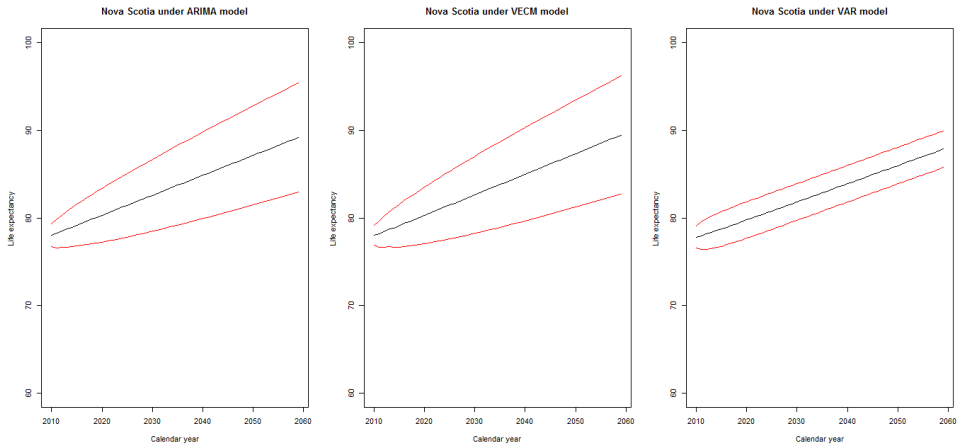

Figure 3.6: Life expectancy of Nova Scotia: the red lines represent the lower and upper forecasting, the black line represents the point forecast

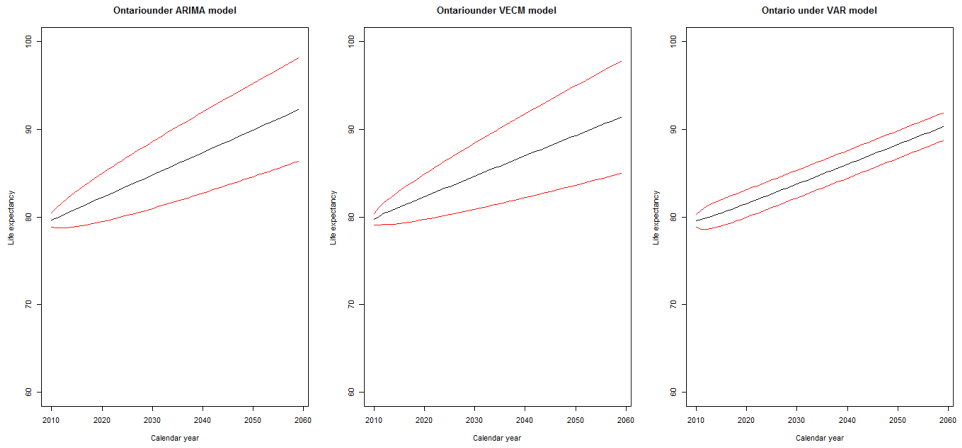

Figure 3.7: Life expectancy of Ontario: the red lines represent the lower and upper forecasting, the black line represents the point forecast

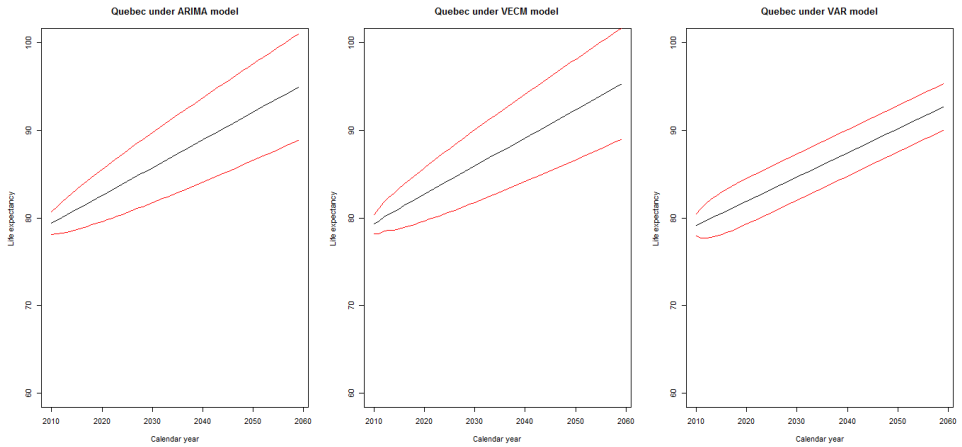

Figure 3.8: Life expectancy of Quebec: the red lines represent the lower and upper forecasting, the black line represents the point forecast