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An axiomatic risk-reward framework for sustainable investing

Gabriele Torri[1] · Rosella Giacometti[1] · Darinka Dentcheva[2] · Svetlozar T. Rachev[3] · W. Brent Lindquist[3]

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Abstract
Continued interest in sustainable investing calls for an axiomatic approach to measures of risk and reward that focus not only on financial returns, but also on measures of environmental and social sustainability, i.e. environmental, social, and governance (ESG) scores. We propose axiomatic definitions for *ESG-coherent risk measures* and *ESG reward–risk ratios* based on functions of bivariate random variables that are applied to financial returns and real-time ESG scores, extending the traditional univariate measures to the ESG case. We provide examples, discuss the dual representation, and present an empirical analysis in which the ESG-coherent risk measures and ESG reward–risk ratios are used to rank stocks.

Keywords Risk measures · Multivariate risk · ESG · Sustainable investing

1 Introduction

ESG investing refers to the integration of environmental, social, and governance considerations into the asset allocation process. It has been one of the most significant trends in the asset management industry, due to continued focus on sustainability and to the growth of information related to non-financial impacts.

ESG investing encompasses a broad array of approaches, and its market practices are very heterogeneous, with different terminologies, definitions, and strategies. These practices vary due to the cultural and ideological diversity of investors (Sandberg et al. 2009; Widyawati 2020). According to Amel-Zadeh and Serafeim (2018), the most significant motivation for incorporating ESG factors is related to financial performance, as sustainability factors are perceived as relevant to investment returns. That is, investors

✉ Gabriele Torri
gabriele.torri@unibg.it

1 Department of Management, University of Bergamo, Bergamo 24127, Italy

2 School of Engineering and Science, Stevens Institute of Technology, Hoboken, NJ 07030, USA

3 Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409-1042, USA

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believe that ESG data can be used to identify potential risks and opportunities, and that such information is not yet fully incorporated into market prices. Hence, ESG information should help investors to control risk better and improve their financial performance. In line with Schanzenbach and Sitkoff (2020) we employ the expression *risk-return ESG* to refer to investment strategies that use ESG factors to improve returns while lessening risk. Academic evidence on the role of ESG in enhancing performance is inconclusive. The meta-analysis conducted by Revelli and Viviani (2015) shows that sustainable and responsible investing (SRI) is neither a weakness nor a strength compared with conventional investing. In contrast, Friede et al. (2015) combining the findings of approximately 2200 studies from 1970 to 2014 find that large majority of of the considered literature reports a positive relationship between ESG and corporate financial performance. More recently, Hornuf and Yüksel (2024) found that responsible investment neither outperforms nor underperforms the market portfolio.[1]

A second motivation that guides ESG strategies is the desire to improve the sustainable profile of the portfolios for ethical reasons or to improve investors' green image (Amel-Zadeh and Serafeim 2018).[2] Sustainability then becomes part of the investment goals, alongside monetary performance and the riskiness of the position. Schanzenbach and Sitkoff (2020) refer to investment strategies that incorporate ESG screenings for moral or ethical reasons as *collateral benefit ESG*, as they aim to provide benefits to a third party, rather than to improve risk-adjusted returns. We refer to investors who include sustainability considerations for ethical reasons as *ESG-oriented investors*, to distinguish them from *risk-return investors* who care exclusively about the financial performance of a position.

From a theoretical point of view, risk-return ESG does not pose any specific issue, as ESG is treated as any other information (e.g. balance sheet data, macroeconomic indicators, sentiment analysis, etc.) and is integrated in the investment process without affecting the main goal of the investor: improving the risk-adjusted performance. In contrast, the ethical motivations of ESG-oriented investors challenge many of the traditional assumptions adopted in finance theory, as the inclusion of other factors in the investment process (e.g. reduction of CO_2 emissions, respect of human right, exclusion of controversial sectors, etc.) are not necessarily motivated by the improvement of financial performance.

To illustrate how the ethical motivation breaks common investing principles, consider two stocks with the same reward–risk profile, but with different ESG scores. They may be equivalent for a risk-return investor but not for an ESG-oriented investor, as one of the two companies may be more environmentally sustainable or may adopt stricter human right policies. An ESG-oriented investor may decide to deliberately

[1] We note that, if ESG information is already incorporated in market prices, any strategy that restricts investment based on ESG criteria would result in sub-optimal allocation in terms of monetary performance. Moreover we underline that risk-return ESG strategies are not necessarily more sustainable than traditional ones. Indeed, an investor may implement a contrarian-ESG approach, investing in less sustainable assets if they are expected to outperform the market.

[2] Almel-Zadeh and Serafim reported additional motivations that push asset managers to propose ESG products, including growing client demand, the effectiveness of ESG investing in bringing about change in companies, and formal client mandates. For our purposes, these drivers can be attributed to either material or ethical motivations.

worsen their risk-adjusted performance to comply with non-negotiable principles, for instance by excluding certain sectors or companies, thus reducing the diversification of their portfolio.[3]

The problem of designing optimal portfolio strategies for ESG-oriented investors that jointly account for financial performance and sustainability considerations has been studied in the portfolio optimization literature. This line of research typically extends the classical reward-risk framework by introducing a third dimension – sustainability – proxied by ESG scores. The scores are incorporated into the portfolio selection problem either as explicit constraints or as additional terms in the objective function. Such an approach has been pursued by, among others, Utz et al. (2015), Gasser et al. (2017) and Cesarone et al. (2022) who studied the efficient frontier of non-dominated portfolios in the reward-risk-ESG space, extending the traditional optimal portfolio literature that started with Markowitz (1952). These works aim to find portfolios with the highest expected return for a given level of risk, where risk is measured using the portfolio variance, a coherent risk measure such as the average value at risk (AVaR)[4] (Rockafellar et al. 2000), or an asymmetric deviation measure (Giacometti et al. 2021). Pedersen et al. (2021) further contributes to the literature by studying the effect on the market equilibrium: they build a framework with three classes of investors (*ESG unaware*, that do not consider ESG scores; *ESG-aware* that use sustainability scores to update their views on risk and expected return; and *ESG motivated*, that have preferences for high ESG) and they consider an ESG-efficient frontier by considering the Sharpe ratio and the ESG score. The presence of sustainability-oriented investors in the market and they effect on asset prices has been studied also by Heinkel et al. (2001) that investigated the effects of exclusionary ethical investments on corporate behaviour, finding that if the percentage of ethical investors is sufficiently high, the cost of capital for polluting companies may increase. More recently, Pástor et al. (2021) built a two-factor model to study the market equilibrium in the presence of ESG investors, finding that, in equilibrium, sustainable assets have lower expected returns not only due to the fact that they can be used to hedge climate risk, but also because of the preference of ESG investors towards these assets.

One drawback shared by the modeling frameworks discussed so far is that they fail to take into account the stochasticity of ESG scores: using either the expected values of the scores or assuming that ESG scores can be treated as constants. Here, we aim to address this limitation and, more generally, to introduce a framework to jointly consider the financial and ESG dimensions.

In this work we target explicitly ESG-oriented investors, proposing measures that account for the joint distribution of financial returns and ESG scores. In particular, we propose axiomatic classes of measures that extend the concepts of coherent risk measures (Artzner et al. 1999), reward measures (Rachev et al. 2008), and reward-risk ratios (Cheridito and Kromer 2013). We underline that our framework treats ESG as an explicit argument of investor preferences rather than as a return-predictive signal. Still,

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[3] It is common practice for sustainable investment funds to exclude tobacco, weapons, fossil fuels, and other controversial sectors. The divestment campaign aimed at South Africa's apartheid regime in the 1980s was one of the key turning points in the history of responsible investing (Schanzenbach and Sitkoff 2020).

[4] The AVaR is also known in the literature as Conditional Value at Risk (CVaR) or Expected Shortfall (ES)

the approach can be integrated in asset allocation strategies that jointly incorporate ESG-based preferences and ESG-driven information for financial performance.

Our axiomatic approach is based on the idea that risk comes from two drivers: monetary performance (i.e., the financial returns of a position) and sustainability (represented by the ESG score of the company).[5] These quantities are random and not necessarily independent, and we can represent them as a bivariate random variable. Hence, it is natural to refer to the rich literature on multivariate risk measures (Jouini et al. 2004; Hamel 2009; Wei and Hu 2014; Ekeland et al. 2012). Such measures have been developed to study portfolios of non-perfectly fungible assets (e.g., assets valued in multiple currencies) or assets that are difficult to price. We propose using a bivariate risk measure to deal with a single asset that can be evaluated over two dimensions: the monetary returns and the sustainability (represented by ESG scores). We then define *ESG-coherent risk measures* as an extension of the *coherent risk measures* introduced by Artzner et al. (1999).

Since different investors may have different attitudes towards ESG, the proposed measures are parametrized using a value $\lambda \in [0, 1]$ to explicitly take into account the subjective trade-off between sustainability and financial performance: when $\lambda = 0$ an investor cares exclusively about financial risk, when $\lambda = 1$ the investor cares only about ESG. In addition to ESG-coherent risk measures, we define *ESG-coherent reward measures* and *ESG reward–risk ratios*.

Although the classical interpretation of a risk measure as the minimum capital required to make a position acceptable does not directly extend to our bivariate framework, since ESG scores are not expressed in monetary units, the proposed measures can still be interpreted as ordinal criteria for ranking financial positions. Moreover, an analogous interpretation can be recovered in terms of an ESG safe asset, that is, an asset characterized by both a deterministic monetary payoff and a deterministic ESG outcome. A natural field for the application of our measures is portfolio optimization where ESG-coherent risk or reward measures be directly embedded in optimal strategies (in the objective function or as constraints), with minimal modifications compared to the univariate case. Unlike ESG constraints or screening rules, ESG enters through the risk functional itself, preserving convexity and dual representations. Portfolio choice is therefore formulated as a well-defined optimization problem based on the joint distribution of financial returns and ESG outcomes.

From an applied perspective, the analysis is motivated by market participants' increasing focus on sustainability and by the emergence of high-frequency ESG measures, such as the *Truvalue scores* issued by FactSet, as opposed to the largely annual disclosure cycle of conventional ESG data.

Section 2 discusses our interpretation of ESG scores and outlines how we propose to use them. Section 3 introduces ESG-coherent risk measures and provides examples, while Sect. 4 introduces ESG reward–risk ratios. Section 5 presents an empirical application using the equity returns and real-time ESG data of the constituents of the Dow-Jones Industrial Average (DJIA) financial index. Section 6 discusses policies and practical applications. Section 7 highlights some conclusions.

[5] For convenience, in the rest of this paper, we refer to these two dimensions as the *monetary* and *ESG* components of risk.

2 Measuring ESG and financial performance

The first step in defining ESG risk measures is to clarify how we characterize and measure sustainability. Following an approach common in the literature, we use the ESG scores of a company as a proxy for sustainability (see, e.g., (Pedersen et al. 2021; Utz et al. 2015; Cesarone et al. 2022)). Here, we clarify how to treat such a variable from both practical and theoretical points of view, establishing the basis for measuring ESG scores in a way that allows us to consistently use them alongside monetary values for risk measurement. We emphasize that, at this stage, we keep the discussion on an abstract level, without reference to specific data providers or methodologies. This allows us not to be limited by the current state-of-the-art market ESG scores, which are still far from being standardized and comparable across data providers (see, e.g., (Billio et al. 2021; Berg et al. 2022; Lauria et al. 2026)).

Several approaches appear in the literature for modeling the random variable r_T used to compute risk. Artzner et al. (1999) define coherent risk measures for a random variable that represents net worth by following the principle that "bygones are bygones," meaning that future net worth is the only thing that matters. In an alternative approach, the random variable used to compute monetary risk represents the return of a financial position. This latter approach, which is often used in practical applications for the measurement of the risk of equity portfolios, can be interpreted as the payoff of a unitary investment in a risky position, and it introduces some differences in the interpretation of the axioms (see (Rachev et al. 2011), Chapter 6).[6]

Similarly, we need to establish the quantity measured by the random variable ESG_T and the sign convention used. In particular, one must consider whether it is a *stock variable* (measured at a specific point in time) or a *flow variable* (measured over an interval of time, as some sort of *sustainability return*). Based on how they are computed, we argue that ESG scores belong to the second category: indeed, they represent the current level of sustainability of the production and commercial practices of a company, which directly affects the impact of that company on the world.[7] Unlike static ESG ratings, our interpretation treats sustainability as an accumulated flow of externalities (positive or negative) that are generated over time. In this sense, the ESG score of a company is related to the rate at which it accumulates non-monetary "satisfaction" for the investor. The total non-monetary satisfaction for an investor is proportional to the holding time of the investment.

Our approach is thus to consider a stochastic process esg_t, that describes the instantaneous *sustainability flow*.[8] For a given time horizon, the satisfaction of the investor

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[6] Alternatively, some authors prefer to define risk measures computed on a variable that represents losses, thus assuming that lower values of the variable are preferred by an investor. Such an assumption does not significantly alter the analysis: it simply changes some signs in the definition of risk (see, e.g., (Rockafellar and Uryasev 2013)).

[7] The scores are typically computed as functions of several indicators related to the production methods, the supply chain management, the industry in which the company operates, the transparency of its governance, the presence of specific policies on human rights violations, etc.

[8] This quantity, although related to externalities, does not necessarily need to be expressed in monetary terms, as the value of this component is different for each investor. A monetary market price of sustainability may exist, but each investor may assign a subjective value to it.

depends on the amount of sustainability accumulated over time, which, for the interval of time from 0 to T, is defined by

$$ESG_T := \int_0^T \mathrm{esg}_t dt. \tag{1}$$

The variable ESG_T aggregates sustainability performance over the holding period. We can then interpret the ESG scores assigned by market providers with periodicity T (e.g. a day or a year), as (rescaled) realizations of the random variables ESG_T.

Finally, for each asset we define a bivariate random variable X_T used to compute ESG risk and reward measures, as:

$$X_T = \begin{bmatrix} r_T \\ ESG_T \end{bmatrix}, \tag{2}$$

where ESG_T is the sustainability dimension, and r_T is the monetary dimension, computed as the cumulative log-return over a static-horizon of length T.

Formally, our framework works under the assumption that the bivariate random vector X_T is measurable on the same probability space $(\Omega, \mathcal{F}, P)$. We acknowledge that in practical settings the ESG score process may depend on external information, or that sustainability related information may affect the dynamic of financial returns. We point out that the theoretical axiomatic setup does not impose structural assumptions beyond measurability and joint distribution of X_T.

The stochasticity of the process esg_t can be traced to two sources: the first is the uncertainty in the measurement of its value, and the second is the uncertainty in the evolution of the sustainability policies within the company, which is driven by the choices made by the management, by market conditions, and by the regulatory framework. We also expect a correlation structure between ESG_T and financial returns r_T due to the presence of common driving factors related, for instance, to sector-wide or country-wide dynamics.

In the rest of the work we use the following scaling convention for the random variable ESG_T: assuming that the ESG scores are bounded, we rescale the values of daily ESG scores such that10 $ESG_T \in [-1/c, 1/c]$, where T is equal to 1 trading day, and $c = 252$. As discussed later, when applied to the Factset Truvalue ESG scores this standardization leads to the "typical magnitude" of ESG_T for a daily time period to be comparable to the "typical magnitude" of the financial cumulative returns measured over the same time period.

The periodic return r_T may be viewed as the payoff of a unit investment in a risky position. Consequently, ESG_T is taken to refer to the same unit investment, so that the variable is proportional to the initial size of the position. With respect to sign conventions, it is natural to define ESG scores so that higher values are preferable, in line with the standard preference for higher returns. These conventions are adopted for comparability and do not restrict the generality of the axiomatic results.

Concerning empirical applications, we acknowledge that the quality and standardization of ESG scores across data providers is a major issue. At the time of writing, most of the available ESG scores are largely based on balance sheet items and selfreported information, and are updated annually. This makes it difficult to obtain high frequency and forward-looking information regarding the sustainability of a company. One notable exception is the *Truvalue scores* framework introduced by Factset, that are computed with daily frequency. Such scores will be used in the empirical analysis in Sect. 5. With the growing use of real-time data, the relevance of properly modelling the time-series dynamics of ESG, and the integration in quantitative risk and portfolio management will grow, making it relevant to develop a solid theoretical framework. Thus, further research should focus on the modeling and estimation of esg_t, considering both the time series evolution of reported ESG scores, and possibly the dispersion of ratings from different providers.

3 ESG-coherent risk measures

We recall the axioms that define a coherent risk measure before introducing the bivariate framework (Artzner et al. 1999). These axioms make it possible to identify measures with desirable properties, assisting both investors and regulators. Consider a convex set $\mathcal{X} \subseteq \mathcal{L}_p(\Omega, \mathcal{F}, P)$ of real-valued random variables r_T which are defined on a probability space $(\Omega, \mathcal{F}, P)$, have finite p-moments ($p \geq 1$), and are indistinguishable up to a set of P-measure zero. We assume that the random variables represent the cumulative return over time T or the payoff at time T of an asset. The functional $\rho : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is a coherent risk measure if it satisfies the following properties:

(SUB) sub-additivity: if $r_{1,T}, r_{2,T} \in \mathcal{X}$, then $\rho(r_{1,T} + r_{2,T}) \leq \rho(r_{1,T}) + \rho(r_{2,T})$;
(PH) positive homogeneity: if $\alpha \in \mathbb{R}_+$ and $r_T \in \mathcal{X}$, then $\rho(\alpha r_T) = \alpha\rho(r_T)$;
(TI) translation invariance: if a is deterministic, $\rho(r_T + a) = \rho(r_T) - a$;
(MO) monotonicity: if $r_{1,T}, r_{2,T} \in \mathcal{X}$ and $r_{1,T} \leq r_{2,T}$ a.s., then $\rho(r_{1,T}) \geq \rho(r_{2,T})$.

Examples of coherent risk measures are AVaR and expectile; but VaR and standard deviation are not coherent risk measures.[9]

The definition of a coherent risk measure is based upon a univariate random variable, and on the idea that the risk is a function only of the returns of the asset (consistent with the assumption that an investor is only interested in the monetary outcome of their position). In some contexts, the monetary outcome does not fully characterize the risk. Examples include: portfolios in two countries having floating exchange rates whose payoffs in the different currencies are not perfectly substitutable (the Siegel paradox, see (Black 1989)); the fact that various maturities for interest rate products are not perfect substitutes (failure of the pure expectation hypothesis); and cases in which it is difficult to attribute a monetary equivalent to various dimensions of risk, such as environmental or health risks. The same principle can be used for the ESG-oriented investor, whose risk depends on both the monetary return of a financial position and its sustainability, represented in our analysis by its ESG score, that is not priced in monetary units. Such an approach allows an investor to deal with trade-offs between

[9] Alternative axiomizations of risk measures, such as convex risk measures (Föllmer and Schied 2002) and regular risk measures (Rockafellar and Uryasev 2013) have been proposed since the work of (Artzner et al. 1999).

different goals that, although they are very different in nature, have to be taken into account in the investment process.

3.1 ESG-coherent risk measures — axiomatic definition

We now extend the classical axioms of coherent risk to a bivariate setting, where the two components represent monetary and sustainability dimensions of the same position.[10] Consider a convex set $\mathcal{X}_2$ of random vectors $X_T = [r_T,\ ESG_T]'$, defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^2$. We use the short-hand notation $\mathcal{X}_2 = \mathcal{L}_p(\Omega, \mathcal{F}, P; \mathbb{R}^2)$ for the space of random vectors with two components and finite p-th moments, which are indistinguishable on sets of P-measure zero. Here $p \in [1, \infty]$. Since individual investors may have different attitude towards ESG, to highlight the trade-off between the monetary and sustainability risk components, we use the parameter $\lambda \in [0, 1]$, a scalar which represents an investor preference for the relative weighing between the monetary and ESG components of risk. Such parameter is investor-specific but fixed at evaluation time.

An ESG risk measure is then a functional of the form $\boldsymbol{\rho}_\lambda : \mathcal{X}_2 \to \mathbb{R} \cup \{+\infty\}$. As in the univariate case, it is possible to axiomatically characterize a set of measures that have desirable properties. These axioms are:

- (SUB-M) sub-additivity: if $X_{1,T}, X_{2,T} \in \mathcal{X}_2$, then $\boldsymbol{\rho}_\lambda(X_{1,T} + X_{2,T}) \leq \boldsymbol{\rho}_\lambda(X_{1,T}) + \boldsymbol{\rho}_\lambda(X_{2,T})$;
- (PH-M) positive homogeneity: if $\beta \in \mathbb{R}_+$ and $X_T \in \mathcal{X}_2$, then $\boldsymbol{\rho}_\lambda(\beta X_T) = \beta \boldsymbol{\rho}_\lambda(X_T)$;
- (MO-M) monotonicity: if $X_{1,T}, X_{2,T} \in \mathcal{X}_2$ and $(r_{1,T} \leq r_{2,T} \wedge ESG_{1,T} \leq ESG_{2,T})$ a.s., then $\boldsymbol{\rho}_\lambda(X_{1,T}) \geq \boldsymbol{\rho}_\lambda(X_{2,T})$;
- (LH-M) lambda homogeneity: if $a = [a_1,\ a_2]' \in \mathbb{R}^2$ is deterministic, then $\boldsymbol{\rho}_\lambda(a) = -\big((1-\lambda)a_1 + \lambda a_2\big)$.

The first three axioms directly extend the univariate case, while LH-M axiom specifies the risk of deterministic ESG-monetary bundles and encodes investor preferences. We then provide the following definition:

Definition 1 *(ESG-coherent risk measure)* Consider a probability space $(\Omega, \mathcal{F}, P)$, a parameter $\lambda \in [0, 1]$, and $X_T = [r_T,\ ESG_T]'$ belonging to a set of bivariate random variables $\mathcal{X}_2$ where r_T measures the cumulative returns of a position or portfolio over a period T, and ESG_T measures the cumulative sustainability flow. We define an ESG-coherent risk measure as any functional $\boldsymbol{\rho}_\lambda : \mathcal{X}_2 \to \mathbb{R} \cup \{+\infty\}$ that satisfies the four axioms SUB-M through LH-M.

It follows from the axioms that any ESG-coherent risk measure satisfies the following property (TI-M), which generalizes translation invariance to the bivariate case.

[10] As discussed in Sect. 2, we use periodic returns for the monetary component and ESG scores (which represent the accumulated sustainability of the investment) for the sustainability component. The definition could be extended to alternative specifications that can be represented using two random variables with a joint distribution such that an investor has a preference for both higher r and higher ESG values (e.g., using the final net worth and the accumulated sustainability multiplied by the initial value of the position).

Proposition 1 *An ESG-coherent risk measure satisfies the translation invariance property TI-M:*

(TI-M) translation invariance: if $a = [a_1,\ a_2]' \in \mathbb{R}^2$ *is deterministic,* $\boldsymbol{\rho}_\lambda(X_T + a) = \boldsymbol{\rho}_\lambda(X_T) - \big((1-\lambda)a_1 + \lambda a_2\big)$*;*

Proof for SUB-M we know that, for $a = [a_1,\ a_2]' \in \mathbb{R}^2$

$$\boldsymbol{\rho}_\lambda(X_T + a) \leq \boldsymbol{\rho}_\lambda(X_T) + \boldsymbol{\rho}_\lambda(a). \tag{3}$$

Using SUB-M and noting that due to LH-M we have $\boldsymbol{\rho}_\lambda(-a) = -\boldsymbol{\rho}_\lambda(a)$, we show that

$$\boldsymbol{\rho}_\lambda(X_T) = \boldsymbol{\rho}_\lambda(X_T + a - a) \leq \boldsymbol{\rho}_\lambda(X_T + a) - \boldsymbol{\rho}_\lambda(a)$$

that implies

$$\boldsymbol{\rho}_\lambda(X_T + a) \geq \boldsymbol{\rho}_\lambda(X_T) + \boldsymbol{\rho}_\lambda(a). \tag{4}$$

Considering axiom LH-M and equations (3) and (4), we have:

$$\boldsymbol{\rho}_\lambda(X_T + a) = \boldsymbol{\rho}_\lambda(X_T) - \big((1-\lambda)a_1 + \lambda a_2\big).$$

□

We note the following.

- Axioms SUB-M and PH-M are straightforward multivariate generalization of the axioms SUB and PH in the univariate definition of coherent risk measures.
- Axiom MO-M generalizes MO. Specifically, we impose a monotonicity condition for which zeroth-order stochastic dominance implies the a.s. ordering of risks. Alternative approaches may be based on first- and second-order stochastic dominance; we leave such an analysis for future studies and maintain the most general (weakest) condition. We emphasize that the ordering induced by the rule in MO-M is partial.
- The last axiom LH-M defines the value of the risk measure for a constant quantity and introduces the parameter λ that accounts for the subjective preference of each investor between financial and ESG dimensions.
- The axiom LH-M enables the characterization of the risk of an *ESG safe asset* (SA), i.e. a position having constant values for both monetary and ESG components. This specification ensures that $\boldsymbol{\rho}_\lambda(\mathbf{1}) = -1,\ \forall\lambda \in [0, 1]$, where $\mathbf{1}$ is the vector $[1, 1]'$.
- Property TI-M extends the univariate translation invariance axiom TI.

This set of axioms is related to the work of Rüschendorf (2006), Wei and Hu (2014), and Chen and Hu (2020), which define convex risk for portfolio vectors using scalar-valued functions. A key difference from these papers, is the introduction of the parameter λ in the axiom LH-M.

The class of ESG-coherent risk measures extends coherent measures to the multivariate setting and provides a way to control the trade-off between the two sources of risk. This trade-off depends on the preferences of the individual investor expressed by λ. Axioms SUB-M and PH-M guarantee that an ESG-coherent risk measure is convex. This allows an investor to diversify not only by creating portfolios of multiple assets, but also to diversify between monetary risk and ESG risk, as highlighted by the following remarks.

Remark Given $X_T = [r_T, \ ESG_T]' \in \mathcal{X}_2$ and an ESG risk measure $\boldsymbol{\rho}_\lambda(X_T)$, for an investor with a given λ the *pure monetary risk* and *pure ESG risk* are defined by $\boldsymbol{\rho}_\lambda([r_T, \ 0]')$ and $\boldsymbol{\rho}_\lambda([0, \ ESG_T]')$, respectively.

Remark *(Diversification between monetary and ESG risk)* If $\boldsymbol{\rho}_\lambda(X_T)$ is an ESG-coherent risk measure, from SUB-M we observe that

$$\boldsymbol{\rho}_\lambda(X_T) \leq \boldsymbol{\rho}_\lambda([r_T, \ 0]') + \boldsymbol{\rho}_\lambda([0, \ ESG_T]'). \tag{5}$$

That is, the risk of a position is always less than or equal to the sum of the pure ESG risk and the pure monetary risk (the investor diversifies between the ESG risk and monetary risk).

Further insights can be gathered by considering the acceptance set characterization of univariate risk measures: as discussed in Artzner et al. (1999), in the univariate case risk can be interpreted as the minimum amount of cash that has to be added to a position to make it acceptable (i.e. to have risk smaller or equal than zero). Extending to a multivariate setting, we no longer have a single safe asset (cash), but rather several instruments with deterministic payoffs. More specifically, in the context of ESG investing, one can postulate the presence of multiple such instruments, each characterized by a different combination of return and ESG attributes. In this case, it is possible to make a risky position acceptable by adding specific amounts of the available safe assets. Indeed, an alternative set-valued multivariate extension discussed in the literature defines risk as the set of all the deterministic capital positions that added to a risky position make it acceptable. The advantages of the set-valued approach is to provide a more complete assessment of risk in relation to the multiple drivers, and to provide a general mathematical formulation, but at the cost of greater complexity and the inability to directly rank positions. We limit our analysis to the scalar-valued bivariate risk measures illustrated above, and we refer to Jouini et al. (2004); Hamel (2009); Hamel et al. (2013); Feinstein and Rudloff (2013) for a broader discussion on set-valued multivariate risk. The practical implication for ESG-oriented investors of the presence of multiple safe assets in the market, are further discussed in Sect. 3.2.

We note that the definition of an ESG-coherent risk measure remains agnostic concerning the measurement of either the financial performance or the ESG score; the former can be measured in terms of the final wealth, profit and loss, or periodic returns (Artzner et al. 1999), and the latter can be computed according to multiple methodologies and aggregated over time following several approaches. The only requirement is that the investor must have a preference for both higher financial gain and higher ESG scores (hence, the monetary part must be expressed in terms such that gains are positive).

In an analogous manner, we can define ESG-coherent reward measures that extend the work of Rachev et al. (2008) (see Appendix A).

3.1.1 Dual representation

It is well known that coherent risk measures have a dual representation – the supremum of a certain expected value over a risk envelope (Ruszczyński and Shapiro 2006; Ang et al. 2018; Dentcheva and Ruszczyński 2024). For ESG-coherent risk measures, the dual representation is introduced in Proposition 2.

Proposition 2 *Let $\mathcal{X}_2 = L_p(\Omega, \mathcal{F}, P; \mathbb{R}^2)$, $p \in [1, \infty]$, and let $\boldsymbol{\rho}_\lambda : \mathcal{X}_2 \to \mathbb{R} \cup \{+\infty\}$ be a proper ESG-coherent risk measure satisfying axioms SUB-M through LH-M and lower semicontinuous.*[11]

Then, for every $X_T = [r_T,\ ESG_T]' \in \mathcal{X}_2$, the dual representation of the risk measure is

$$\boldsymbol{\rho}_\lambda(X_T) = \sup_{\zeta \in \mathcal{A}_{\rho_\lambda}} \left\{ -\int_\Omega [\zeta_1(\omega) r_T(\omega) + \zeta_2(\omega) ESG_T(\omega)]\, P(d\omega) \right\}, \tag{6}$$

where $\mathcal{A}_{\rho_\lambda}$ consists of all non-negative functions $\zeta = (\zeta_1, \zeta_2) \in L_q(\Omega, \mathcal{F}, P; \mathbb{R}^2)$ such that $\mathbb{E}[\zeta] = [1 - \lambda,\ \lambda]'$.

Furthermore, $\mathcal{A}_{\rho_\lambda}$ coincides with the convex subdifferential of $\boldsymbol{\rho}_\lambda$ at the origin.

The proof of Proposition 2 is provided in Appendix B. Using a more compact notation, (6) can be written as

$$\boldsymbol{\rho}_\lambda(X_T) = \sup_{\zeta \in \mathcal{A}_{\rho_\lambda}} \{-\mathbb{E}[\zeta_1 r_T + \zeta_2 ESG_T]\}.$$

Proposition 3 addresses the marginal ESG-coherent risk measure when $\lambda = 1$ or $\lambda = 0$.

Proposition 3 *If $\boldsymbol{\rho}_\lambda$ is an ESG-coherent risk measure, then*

$$\boldsymbol{\rho}_0([r_T,\ ESG_T]') = \boldsymbol{\rho}_0([r_T,\ 0]'),$$

$$\boldsymbol{\rho}_1([r_T,\ ESG_T]') = \boldsymbol{\rho}_1([0,\ ESG_T]').$$

Proof For $\lambda = 0$, we know from the dual representation that $\zeta_2 = 0$ a.s., since it has zero expected value and is non-negative. Hence,

$$\boldsymbol{\rho}_0([r_T,\ ESG_T]') = \sup_{\zeta \in \mathcal{A}_0} \left\{ -\int_\Omega \zeta_1(\omega) r_T(\omega) P(d\omega) \right\} = \boldsymbol{\rho}_0([r_T,\ 0]').$$

Analogously, for $\lambda = 1$ we have

[11] Lower semicontinuity is understood with respect to the topology associated with the pairing (L_p, L_q), $1/p + 1/q = 1$. For $p = \infty$, we use the dual pairing (L_∞, L_1) and the weak* topology $\sigma(L_\infty, L_1)$, under which $X_n \to X$ whenever $\mathbb{E}[X_n Y] \to \mathbb{E}[XY]$ for all $Y \in L_1$.

$$\boldsymbol{\rho}_1([r_T,\ ESG_T]') = \sup_{\zeta\in\mathcal{A}_1}\left\{-\int_\Omega \zeta_2(\omega)ESG_T(\omega)P(d\omega)\right\} = \boldsymbol{\rho}_1([0,\ ESG_T]').$$

□

In other words, Proposition 3 states that the risk for an investor with $\lambda = 0$ is not affected by the ESG score of the asset, while the risk for an investor with $\lambda = 1$ is not affected by the monetary returns.

3.2 Hedging risk by investing in ESG safe assets

To clarify both the structure and the practical use of ESG-coherent risk measures, we analyze how ESG-oriented investors may hedge a risky position by allocating capital to an ESG safe asset. Since ESG outcomes are not expressed in monetary units, the classical interpretation of a risk measure as the minimum capital required to make a position acceptable does not directly apply in the bivariate setting. Nevertheless, the introduction of an ESG safe asset provides a natural analogue, allowing us to reinterpret the translation invariance property and to characterize risk mitigation in terms of adjustments to both the monetary and ESG components of a position.

We will consider the case of an investor or an asset manager that is subject to the budget constraint, and thus to invest in an ESG safe asset needs to disinvest an equal amount from the risky asset, following standard treatments in the risk-measure literature (see, e.g. (Rachev et al. 2011), Chapter 6.4.4). In a traditional univariate framework, a safe asset by definition has a deterministic payoff;[12] its return is a constant $\mathrm{RF}^r_T \in \mathbb{R}$. If we define the risk on a univariate random variable that represents returns, axioms TI and PO state that the risk of a portfolio composed of the safe asset and a risky position r_T is

$$\rho((1-w)r_T + w\mathrm{RF}^r_T) = (1-w)\rho(r_T) - w\mathrm{RF}^r_T, \tag{7}$$

where $w \in [0, 1]$ is the weight of the safe asset in the portfolio. More formally, we address the problem of an investor who is willing to reduce the risk of a position to an acceptable level κ by creating a portfolio consisting of the risky position, and of the smallest possible amount of the safe asset. The motivation is, for instance, to satisfy requirements imposed by regulators or by the institutional mandate. Formally the problem is

$$\begin{aligned} w^* = \arg\min_w(w) & \\ \text{s.t. } & \rho((1-w)r_T + w\mathrm{RF}^r_T) \le \kappa, \\ & 0 \le w \le 1. \end{aligned} \tag{8}$$

[12] Note that a position can have a risk equal to zero and not be a safe asset. Similarly, an asset with a deterministic payoff (i.e., a safe asset) can have a risk that is different from zero. We can understand this point better by considering a position with a return r_T and risk $\rho(r_T) = m$. If $\rho(\cdot)$ is a coherent risk measure, by axiom TI we have $\rho(r_T + m) = 0$. That is, a position with a return $r'_T = r_T + m$ has zero risk (but its returns are not necessarily constant).

We assume that $-\mathrm{RF}_T^r < \kappa \leq \rho(r_T)$. Since the risk of a portfolio is an affine function of w as shown in (7), the solution of (8) is:

$$w^* = \frac{\rho(r_T) - \kappa}{\rho(r_T) + \mathrm{RF}_T^r}. \tag{9}$$

That is, the risky position can be hedged by constructing a portfolio that contains the safe asset having weight w^*.[13]

In the ESG-oriented investing framework, an ESG safe asset is a position having constant values for both monetary and ESG components. For illustrative purposes, we can postulate the existence of several types of such ESG safe assets, distinguished by different combinations of constant ESG and monetary return.

Consider first the case in which only one type of ESG safe asset is available in the market. The variable $\mathrm{SA}_T \in \mathbb{R}^2$ denotes the constant return and constant ESG for the period T of the safe asset:

$$\mathrm{SA}_T := \begin{bmatrix} \mathrm{RF}_T^r \\ \mathrm{RF}_T^{ESG} \end{bmatrix}.$$

We know that by axiom LH-M, for an ESG-coherent risk measure, the risk of this ESG safe asset is $\boldsymbol{\rho}_\lambda(\mathrm{SA}_T) = -((1-\lambda)\mathrm{RF}_T^r + \lambda \mathrm{RF}_T^{ESG})$. The problem of hedging the risk of an asset with bivariate return X_T is analogous to the univariate case: an investor with a given λ wants to construct a portfolio with ESG-risk smaller or equal than κ by creating a portfolio with the risky asset, and the smallest possible amount of the ESG safe asset (i.e. minimizing its weight in the portfolio):

$$\begin{aligned} w_\lambda^* = \arg\min_w&(w) \\ &\text{s.t. } \boldsymbol{\rho}_\lambda((1-w)X_T + w\mathrm{SA}_T) \leq \kappa, \\ &\quad 0 \leq w \leq 1. \end{aligned} \tag{10}$$

We assume that $-(1-\lambda)\mathrm{RF}_T^r - \lambda \mathrm{RF}_T^{ESG} < \kappa \leq \boldsymbol{\rho}_\lambda(X_T)$. The solution is

$$w_\lambda^* = \frac{\boldsymbol{\rho}_\lambda(X_T) - \kappa}{(1-\lambda)\mathrm{RF}_T^r + \lambda \mathrm{RF}_T^{ESG} + \boldsymbol{\rho}_\lambda(X_T)}. \tag{11}$$

We underline that w_λ^* is unique for each investor and its value varies with the parameter λ. In practice, such an ESG safe asset could be achieved by the investor making a guaranteed loan to an institution (either a for-profit company, a government, or a non-profit institution) that has a positive and stable environmental or social impact, which generates an interest RF_T^r for the investor.

[13] If the risk measure were defined in terms of final net worth rather than returns, to hedge a risky position with risk m, it would be necessary to add a cash position. For a broader discussion of the interpretation of the axioms expressed in terms of returns rather than the final net worth, see Rachev et al. (2011, Chapter6).

We discuss three special cases characterized by specific ESG safe assets. With the exception of case 3, we assume that the ESG safe assets have non-negative return and ESG.

1. **The only ESG safe asset is a *pure monetary safe asset***, that has the following bivariate return:

$$\mathrm{SA}_{\mathrm{CASH},T} := \begin{bmatrix} \mathrm{RF}_T^r \\ 0 \end{bmatrix},$$

where $\mathrm{RF}_T^r \geq 0$. An example of this is a risk-free, zero-coupon bond issued by a governmental institution not associated with a specific ESG profile.[14] An ESG investor can hedge the risk of a position X_T by constructing a portfolio composed of the *pure monetary safe asset* in proportion w_λ^* and investing the rest in the risky asset, with

$$w_\lambda^* = \frac{\boldsymbol{\rho}_\lambda(X_T) - \kappa}{(1-\lambda)\mathrm{RF}_T^r + \boldsymbol{\rho}_\lambda(X_T)}.$$

We note that if $\lambda = 1$ (i.e., an investor cares exclusively about the ESG component) $\boldsymbol{\rho}_1(\mathrm{SA}_{\mathrm{CASH},T}) = 0$.

2. **The only ESG safe asset is a *pure ESG safe asset*** with bivariate return

$$\mathrm{SA}_{ESG,T} := \begin{bmatrix} 0 \\ \mathrm{RF}_T^{ESG} \end{bmatrix}.$$

The analysis of this case is symmetrical to that for the pure monetary safe asset.

3. **There exists an ESG safe asset having a monetary return of –100%.** We consider the special case of an ESG safe asset represented by

$$\mathrm{SA}_{GRANT,T} := \begin{bmatrix} -1 \\ \mathrm{RF}_T^{ESG} \end{bmatrix}.$$

The financial component corresponds to a deterministic loss of the entire investment, while the ESG component is positive and constant. As a conceptual benchmark, this position may be interpreted as a grant to a charitable organization generating a fixed ESG contribution. Such an asset is clearly not a relevant investment opportunity for a risk-return investor with $\lambda = 0$.[15] On the other hand, for an ESG-oriented investor with $\lambda > 0$ it could be rational to invest in such asset. The measured risk of such an asset is $\boldsymbol{\rho}_\lambda(\mathrm{SA}_{GRANT,T}) = (1-\lambda) - \lambda\mathrm{RF}_T^{ESG}$; that is, for an investor with a given λ the risk is negative if $\lambda > \tilde{\lambda} = 1/(1+\mathrm{RF}_T^{ESG})$. For a "$\lambda < \tilde{\lambda}$" investor, such an asset provides no opportunity to hedge a risky position; however for a "$\lambda > \tilde{\lambda}$" investor, such an ESG safe asset provides a meaningful

[14] Rating agencies are starting to compute ESG scores for countries as well, although the criteria are different from those used to calculate companies' scores. The identification of pure monetary and pure ESG safe assets will be a significant challenge for practitioners and scholars.

[15] Assuming the absence of tax benefits associated to the donation, or other forms of monetary gains.

hedging tool through the donation of a wealth fraction:

$$w_\lambda^* = \frac{\boldsymbol{\rho}_\lambda(X_T) - \kappa}{\lambda - 1 + \lambda \mathrm{RF}_T^{ESG} + \boldsymbol{\rho}_\lambda(X_T)}$$

to a project with a positive environmental or social impact.

We can also study the case with multiple ESG safe assets available in the market. In such case, while it appears that an investor could choose multiple ESG safe assets, a simple analysis shows that each investor will choose only a single ESG safe asset among the available ones. If n ESG safe assets with returns $\mathrm{SA}_{i,T},\ i = 1, \ldots, n$, are available, an investor can hedge a risky position with bivariate return X_T by creating a portfolio with weight w_0 for the risky asset and $w_i,\ i = 1, \ldots, n$ for the safe assets, with $\sum_{i=0}^n w_i = 1$ such that the risk is equal to κ. We can formulate an optimization problem analogous to 10, with the difference that the objective function to maximize is the weight of the risky asset w_0 (that is equivalent to minimizing the sum of the portfolio weights invested in ESG safe assets). Formally:

$$\begin{aligned} \max_{w_0, w_1, w_2, \ldots, w_n} \quad & (w_0) \\ \text{s.t.} \quad & \boldsymbol{\rho}_\lambda(w_0 X_T + w_1 \mathrm{SA}_{1,T} + w_2 \mathrm{SA}_{2,T} + \cdots + w_n \mathrm{SA}_{n,T}) \leq \kappa, \\ & \sum_{i=1}^{n+1} w_i = 1, \quad 0 \leq w_i \leq 1, \quad i = 1, \ldots, n+1. \end{aligned} \tag{12}$$

The constraint set defines a convex feasible subset of $\mathbb{R}^n$ for the investment allocations, w_i. The solution is not a diversified portfolio; only one ESG safe asset is selected. This follows from the fact that, due to LH-M and TI-M, the risk of a sum of ESG safe assets is the sum of their risk: $\boldsymbol{\rho}_\lambda(\delta \mathrm{SA}_{i,T} + (1-\delta)\mathrm{SA}_{j,T}) = \delta \boldsymbol{\rho}_\lambda(\mathrm{SA}_{i,T}) + (1-\delta)\boldsymbol{\rho}_\lambda(\mathrm{SA}_{j,T}),\ \ \delta \in [0, 1]$. For any pair of ESG safe assets i and j, the risk of their convex combination is always greater or equal than $\min(\boldsymbol{\rho}_\lambda(\mathrm{SA}_i), \boldsymbol{\rho}_\lambda(\mathrm{SA}_j))$. Hence, risk minimization is achieved by selecting the ESG safe asset with minimal risk, and not by forming portfolios of safe assets. This uniqueness property, that is a direct consequence of ESG-coherence of $\boldsymbol{\rho}_\lambda$, parallels the univariate setting, where, in the presence of multiple safe assets, the optimal choice is the cheapest one.[16] Once the ESG safe asset with the lowest risk is identified, the problem then is the same as (10) where only one ESG safe asset was available. Note however that the choice of ESG safe asset is different for each investor, as the optimization is dependent on the investor's value for λ.

Remark In general, for risk minimization under a budget constraint, the choice of the ESG safe asset is not influenced by the characteristics of the risk of the position, it is only influenced by the availability and price of ESG safe assets and the λ preference of the investor.

[16] An exception is when two or more ESG safe assets with exactly the same risk are available. In such a case, they are indistinguishable to the investor in terms of risk, and either can be chosen.

3.3 Examples of ESG-coherent risk measures

After discussing ESG-coherent risk measures in general, we present two example approaches for extending univariate risk measures to a bivariate setting. In particular, starting from a univariate coherent risk measure $\rho(r_T)$, we identify the ESG-coherent risk measures ESG-$\boldsymbol{\rho}_\lambda(X_T)$, and ESG-$\boldsymbol{\rho}^l_\lambda(X_T)$. The measures encompass $\lambda \in [0, 1]$ as a parameter, thus they are families of bivariate risk measures suitable for investors having differing ESG "inclinations". In Sect. 3.3.1 we apply these approaches to the well-known coherent risk measure, average value at risk (AVaR), resulting in two versions of coherent ESG-AVaR risk measures. These examples extend naturally to the ESG formulation without changing their convex-analytic structure. In Sect. 3.3.2 we apply these approaches to two non-coherent risk measures, variance and volatility, producing ESG extensions that are not ESG-coherent.

Our first approach to generalizing a univariate risk measure $\rho(r_T)$ utilizes a linear combination of r_T and ESG_T. For $X_T = [r_T,\ ESG_T]'$

$$\text{ESG-}\boldsymbol{\rho}_\lambda\,(X_T) := \rho\big((1-\lambda)r_T + \lambda ESG_T\big). \tag{13}$$

Proposition 4 *If $\rho(\cdot)$ is a univariate coherent risk measure, then ESG-$\boldsymbol{\rho}_\lambda(\cdot)$ is an ESG-coherent risk measure.*

Proof Since the right-hand-side of (13) involves a direct application of $\rho(\cdot)$, the extended function ESG-$\boldsymbol{\rho}_\lambda(\cdot)$ inherits the properties SUB-M and PH-M. It is straightforward to show that axiom MO of the univariate function implies MO-M. Consider two vectors $X_{1,T}, X_{2,T} \in \mathcal{X}_2$. If $(r_{1,T} \le r_{2,T} \wedge ESG_{1,T} \le ESG_{2,T})$ a.s., then from the monotonicity of $\rho(\cdot)$, we have

$$\begin{aligned}\text{ESG-}\boldsymbol{\rho}_\lambda(X_{1,T}) &= \rho((1-\lambda)r_{1,T} + \lambda ESG_{1,T}) \ge \rho((1-\lambda)r_{2,T} + \lambda ESG_{2,T})\\ &= \text{ESG-}\boldsymbol{\rho}_\lambda(X_{2,T}).\end{aligned}$$

Finally, using the translation invariance of $\rho(\cdot)$, we can prove LH-M. □

We note the following properties of ESG-$\boldsymbol{\rho}_\lambda(\cdot)$.

- If $\rho(r_T)$ is convex, then ESG-$\boldsymbol{\rho}_\lambda([r_T, ESG_T]')$ is a convex function of λ.
- For $\lambda = 0$ and $\lambda = 1$, ESG-$\boldsymbol{\rho}_\lambda([r_T, ESG_T]')$ is equal to the univariate $\rho(r_T)$ computed on returns alone or $\rho(ESG_T)$ computed on ESG scores alone, respectively.

Our second approach utilizes a linear combination of univariate risk measures,

$$\text{ESG-}\boldsymbol{\rho}^l_\lambda\left(\begin{bmatrix} r_T \\ ESG_T \end{bmatrix}\right) := (1-\lambda)\rho(r_T) + \lambda\rho(ESG_T). \tag{14}$$

It is straightforward to show that ESG-$\boldsymbol{\rho}^l_\lambda(\cdot)$ is an ESG-coherent risk measure if $\rho(\cdot)$ is coherent because axioms SUB-M, PH-M, and MO-M follow from the respective univariate axioms, while LH-M follows from TI.

We note the following properties of ESG-$\boldsymbol{\rho}^l_\lambda(\cdot)$.

- The measure ESG-$\boldsymbol{\rho}^l_\lambda([r_T, ESG_T]')$ is more conservative than ESG-$\boldsymbol{\rho}_\lambda([r_T, ESG_T]')$ as it is linear in λ and, hence, always greater than or equal to ESG-$\boldsymbol{\rho}_\lambda([r_T, ESG_T]')$.
- ESG-$\boldsymbol{\rho}^l_\lambda([r_T, ESG_T]')$ is equivalent to ESG-$\boldsymbol{\rho}_\lambda([r_T, ESG_T]')$ for the case of perfect co-monotonicity between r_T and ESG_T (i.e. when no diversification between the two is possible). In this sense, we can consider it a worst-case measure of ESG-$\boldsymbol{\rho}_\lambda([r_T, ESG_T]')$.
- For the limiting cases $\lambda = 0$ and $\lambda = 1$, we have ESG-$\boldsymbol{\rho}^l_\lambda([r_T, ESG_T]') =$ ESG-$\boldsymbol{\rho}_\lambda([r_T, ESG_T]')$.
- For $\lambda = 0$ or $\lambda = 1$, ESG-$\boldsymbol{\rho}^l_\lambda([r_T, ESG_T]')$ corresponds to the univariate risk measure $\rho(X_T)$ computed on just the monetary part or just the ESG part, respectively.

3.3.1 ESG-AVaR

We demonstrate the two approaches presented in equations (13) and (14) to develop ESG-coherent risk measures based on AVaR. Given a random bivariate vector $X_T = [r_T,\ ESG_T]' \in \mathcal{X}_2 = \mathcal{L}_1(\Omega, \mathcal{F}, P; \mathbb{R}^2)$, the first measure, given by (13), is[17]

$$\begin{aligned} \text{ESG-AVaR}_{\lambda,\tau}\left(\begin{bmatrix} r_T \\ ESG_T \end{bmatrix}\right) &:= \text{AVaR}_\tau\left((1-\lambda)r_T + \lambda ESG_T\right) \\ &= \inf_{\beta\in\mathbb{R}}\left\{\frac{1}{1-\tau}\mathbb{E}\left[(\beta - ((1-\lambda)r_T + \lambda ESG_T))^+\right] - \beta\right\}, \end{aligned} \tag{15}$$

where $(a)^+$ denotes $\max(a, 0)$. Following the discussion on (13), we conclude that ESG-AVaR$_{\lambda,\tau}$ is coherent. It is similar to the multivariate expected shortfall presented by Ekeland et al. (2012) (although their measure lacks the parametrization using λ allowing investor-specific ESG preferences). For $\lambda = 0$ the measure reduces to the univariate AVaR. Since ESG-AVaR$_{\lambda,\tau}(\cdot)$ is computed on univariate data (i.e., as a linear combination of r and ESG), numerical applications using ESG-AVaR$_{\lambda,\tau}(\cdot)$ do not present any particular challenge; it is possible to fully utilize existing procedures developed for AVaR for risk estimation, portfolio optimization, and risk management. (See, e.g. (Shapiro et al. 2021)).

Proposition 5 *The dual representation of ESG-AVaR$_{\lambda,\tau}(X_T)$, defined as is Equation (15), is*

$$\textit{ESG-AVaR}_{\lambda,\tau}\left(\begin{bmatrix} r_T \\ ESG_T \end{bmatrix}\right) = \sup_{[\zeta_1,\ \zeta_2]'\in\mathcal{A}_{\textit{ESG-AVaR}_{\lambda,\tau}(X_T)}} \left(-\mathbb{E}[r_T\zeta_1 + ESG_T\zeta_2]\right), \tag{16}$$

[17] See Ogryczak and Ruszczyński (2002); Rockafellar and Uryasev (2002) for the extremal representation.

where

$$\mathcal{A}_{ESG\text{-}AVaR_{\lambda,\tau}} = \Big\{ [\zeta_1,\ \zeta_2]' \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P; \mathbb{R}^2) :\ [\zeta_1,\ \zeta_2]' = \xi[1-\lambda,\ \lambda]'; \\ 0 \le \xi \le \frac{1}{1-\tau}\ a.s.\ ; \mathbb{E}[\xi] = 1 \Big\}. \tag{17}$$

The proof of Proposition 5 is given in Appendix C.
The second measure, obtained using (14), is

$$\text{ESG-AVaR}^{l}_{\lambda,\tau}\left(\begin{bmatrix} r_T \\ ESG_T \end{bmatrix}\right) := (1-\lambda)\text{AVaR}_\tau(r_T) + \lambda\text{AVaR}_\tau(ESG_T). \tag{18}$$

ESG-AVaR$^{l}_{\lambda,\tau}(\cdot)$ is also ESG-coherent. It can be viewed as the limit of ESG-AVaR$_{\lambda,\tau}(\cdot)$ in the case of an asset for which it is not possible to diversify between the monetary and ESG components as they are comonotone. From an economic perspective, ESG-AVaR$^{l}_{\lambda,\tau}(\cdot)$ is significant for investors who consider the worst-case scenario in terms of the dependency structure.

Proposition 6 *The dual representation of ESG-AVaR$^{l}_{\lambda,\tau}(X_T)$ is*

$$ESG\text{-}AVaR^{l}_{\lambda,\tau}\left(\begin{bmatrix} r_T \\ ESG_T \end{bmatrix}\right) = \sup_{[\zeta_1,\ \zeta_2]' \in \mathcal{A}_{ESG\text{-}AVaR^{l}_{\lambda,\tau}(X_T)}} -\mathbb{E}[r_T\zeta_1 + ESG_T\zeta_2], \tag{19}$$

where

$$\mathcal{A}_{ESG\text{-}AVaR^{l}_{\lambda,\tau}} = \Big\{ [\zeta_1,\ \zeta_2]' \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P; \mathbb{R}^2) : \mathbb{E}[\zeta_1] = 1-\lambda; \mathbb{E}[\zeta_2] = \lambda; \\ \zeta_1, \zeta_2 \ge 0; \zeta_1 \le \frac{1-\lambda}{1-\tau}; \zeta_2 \le \frac{\lambda}{1-\tau} \Big\}. \tag{20}$$

The proof of Proposition 6 is also given in Appendix C.

3.3.2 Non-ESG-coherent measure examples

In order to better delineate the boundaries of the axiomatic framework, we discuss here examples of measures that are not ESG-coherent. It is well known that the standard deviation σ (the volatility) and the variance $\mathbb{V}$ are not coherent risk measures. We consider the application of (13) and (14) to σ and $\mathbb{V}$ and show that, in all cases, the result is an ESG measure that does not satisfy the ESG-coherency axioms.

Given a vector $X_T = [r_T,\ ESG_T]' \in \mathcal{X}_2 = \mathcal{L}_2(\Omega, \mathcal{F}, P; \mathbb{R}^2)$, from (13) the ESG variance and ESG volatility are

$$\text{ESG-}\mathbb{V}_\lambda(X_T) := \mathbb{V}[(1-\lambda)r_T + \lambda ESG_T], \tag{21}$$

$$\text{ESG-}\sigma_\lambda(X_T) := \sqrt{\text{ESG-}\mathbb{V}_\lambda(X_T)}. \tag{22}$$

Table 1 Summary of ESG risk measures and their properties. A checkmark indicates that the corresponding property is satisfied

Risk measure	SUB-M	PH-M	MO-M	LH-M
ESG-AVaR$_{\lambda,\tau}$	✓	✓	✓	✓
ESG-AVaR$^{l}_{\lambda,\tau}$	✓	✓	✓	✓
ESG-$\mathbb{V}_\lambda$				
ESG-σ_λ	✓	✓		
ESG-$\mathbb{V}^{l}_\lambda$				
ESG-σ^{l}_λ	✓	✓		

ESG-$\mathbb{V}_\lambda(\cdot)$ is not ESG-coherent, as it does not satisfy PH-M, MO-M, SUB-M, and LH-M. ESG-$\sigma_\lambda(\cdot)$ is not ESG-coherent, as it does not satisfy MO-M and LH-M.

Using (14), the corresponding risk measures are

$$\text{ESG-}\mathbb{V}^{l}_{\lambda}(X_T) := (1-\lambda)\mathbb{V}[r_T] + \lambda\mathbb{V}[ESG_T], \tag{23}$$

$$\text{ESG-}\sigma^{l}_{\lambda}(X_T) := \sqrt{\text{ESG-}\mathbb{V}^{l}_{\lambda}(X_T)}. \tag{24}$$

The former does not satisfy PH-M, MO-M, SUB-M, and LH-M, and the latter does not satisfy MO-M and LH-M.

A summary of the properties of the examples considered in Sects. 3.3.1 and 3.3.2 is given in Table 1.

4 ESG reward–risk ratios

It is natural to extend the ESG framework to reward–risk ratios (RRRs), used to measure risk-adjusted performance of an investment. Following Cheridito and Kromer (2013), a reward–risk ratio $\alpha(r)$ in a univariate setting is

$$\alpha(r_T) := \frac{\theta(r_T)^+}{\rho(r_T)^+}, \tag{25}$$

where $\theta : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$ and $\rho : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ are reward and risk measures, respectively. Cheridito and Kromer (2013) identified four conditions desirable for RRRs:

- (MO-R) monotonicity: if $r_{1,T}, r_{2,T} \in \mathcal{X}$ and $r_{1,T} \leq r_{2,T}$ a.s., then $\alpha(r_{1,T}) \geq \alpha(r_{2,T})$;
- (QC-R) quasi-concavity: if $r_{1,T}, r_{2,T} \in \mathcal{X}$ and $\delta \in [0, 1]$, then $\alpha(\delta r_{1,T} + (1-\delta)r_{2,T}) \geq \min(\alpha(r_{1,T}), \alpha(r_{2,T}))$;
- (SI-R) scale invariance: if $r_T \in \mathcal{X}$ and $\delta > 0$ s.t. $\delta r_T \in \mathcal{X}$, then $\alpha(\delta r_T) = \alpha(r_T)$;
- (DB-R) distribution-based: $\alpha(r_T)$ only depends on the distribution of r_T under P.

Following the approach used for risk and reward measures, we introduce ESG reward–risk ratios (ESG-RRRs). Let $X_T = [r_T,\ ESG_T]'$. We define an ESG-RRR $\boldsymbol{\alpha}_\lambda : \mathcal{X}_2 \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\boldsymbol{\alpha}_\lambda(X_T) := \frac{\boldsymbol{\theta}_\lambda(X_T)^+}{\boldsymbol{\rho}_\lambda(X_T)^+}, \tag{26}$$

where $\boldsymbol{\theta}_\lambda(X_T)$ and $\boldsymbol{\rho}_\lambda(X_T)$ are ESG reward and risk measures as defined in Sect. 3 and Appendix A using the same preference parameter λ.[18] The extension of the Cheridito-Kromer conditions for ESG-RRRs are:

(MO-RM) monotonicity: if $X_{1,T}, X_{2,T} \in \mathcal{X}_2$ and $(r_{1,T} \le r_{2,T} \wedge ESG_{1,T} \le ESG_{2,T})$ a.s., then $\boldsymbol{\alpha}_\lambda(X_{1,T}) \le \boldsymbol{\alpha}_\lambda(X_{2,T})$;

(QC-RM) quasi-concavity: if $X_{1,T}, X_{2,T} \in \mathcal{X}_2$ and $\delta \in [0,1]$, then $\boldsymbol{\alpha}_\lambda(\delta X_{1,T} + (1-\delta)X_{2,T}) \ge \min(\boldsymbol{\alpha}_\lambda(X_{1,T}), \boldsymbol{\alpha}_\lambda(X_{2,T}))$;

(SI-RM) scale invariance: if $X_T \in \mathcal{X}_2$ and $\delta > 0$ s.t. $\delta X_T \in \mathcal{X}_2$, then $\boldsymbol{\alpha}_\lambda(\delta X_T) = \boldsymbol{\alpha}_\lambda(X_T)$;

(DB-RM) distribution-based: $\boldsymbol{\alpha}_\lambda(X_T)$ only depends on the distribution of X_T under P.

Verification of DB-RM depends on the bivariate distribution of $X_T \in \mathcal{X}_2$ and not on the univariate distribution of the returns.

Proposition 7 *The ESG-RRR (26), where $\boldsymbol{\theta}_\lambda(X_T)$ is an ESG-coherent risk measure and $\boldsymbol{\rho}_\lambda(X_T)$ is an ESG-coherent reward measure (Appendix A) satisfies conditions MO-RM, QC-RM and SI-RM.*

Proof MO-RM follows from the monotonicity of $\boldsymbol{\rho}_\lambda(X_T)$ and $\boldsymbol{\theta}_\lambda(X_T)$. QC-RM follows from the convexity of $\boldsymbol{\rho}_\lambda(X_T)$ and the concavity of $\boldsymbol{\theta}_\lambda(X_T)$. SI-RM follows from the corresponding properties of $\boldsymbol{\rho}_\lambda(X_T)$ and $\boldsymbol{\theta}_\lambda(X_T)$. □

Remark Proposition 7 does not guarantee that the DB-RM property is satisfied since, as discussed in (Cheridito and Kromer 2013), risk and reward measures may not depend on the distribution of returns and ESG under a single probability measure, and in such case the distribution-based property is not satisfied. This is for example the case of distributionally robust reward–risk ratios, which consider the fact that agents do not know with certainty the distribution of the random variables, and consider a whole family of distributions (see e.g. (Liu et al. 2017)).

4.1 Examples of ESG reward–risk ratios

We present six examples of ESG reward–risk ratios derived from RRRs commonly used in the literature. The ratios are obtained by generalizing the univariate reward–risk ratios using the approach described by (13). Note that, as in the case of risk measures, the definition of a reward–risk ratio can be based on several alternative specifications of the random variable $X_T = [r_T, \ ESG_T]'$. In particular, the ratios can be computed using the returns, the excess returns over a risk-free rate, the final wealth, profit and/or losses, etc. The same logic applies to the ESG component. Here, we only provide illustrative guidance concerning which approach to use in practice;

[18] In principle, it is possible for the risk and reward measures to have different values of λ. We consider the case of values of lambda for both the numerator and the denominator for conciseness.

the choice depends on the specific needs of the practitioner or regulator who uses these measures.

- **ESG Sharpe ratio**. The Sharpe ratio is the ratio between the excess return of an asset and its standard deviation over a period of time. The ESG Sharpe ratio is

$$\text{ESG-SR}_\lambda(X_T) := \frac{\text{ESG-}\mu_\lambda(X_T - \text{SA}_T)}{\text{ESG-}\sigma_\lambda(X_T)}, \tag{27}$$

where

$$\text{ESG-}\mu_\lambda(X_T) := (1-\lambda)\mathbb{E}[r_T] + \lambda\mathbb{E}[ESG_T], \tag{28}$$

the ESG standard deviation $\text{ESG-}\sigma_\lambda(X_T)$ is given by (22), and $\text{SA}_T = [\text{RF}_T^r,\ \text{RF}_T^{ESG}]'$ is the bivariate return of an ESG safe asset. $\text{ESG-}\mu_\lambda$ is an ESG-coherent reward measure, while $\text{ESG-}\sigma_\lambda$ is not an ESG-coherent risk measure. The ESG Sharpe ratio satisfies QC-RM, SI-RM, and DB-RM, but it does not satisfy condition MO-RM.[19]
- **ESG Rachev ratio**. The univariate Rachev ratio is defined as (Biglova et al. 2004)

$$\text{RR}_{\beta,\gamma}(r_T) := \frac{\text{AVaR}_\beta(-r_T)}{\text{AVaR}_\gamma(r_T)}. \tag{29}$$

We generalize to an ESG-RR by replacing AVaR with ESG-CVaR. Let $X_T \in \mathcal{X}_2$. Then

$$\text{ESG-RR}_{\beta,\gamma,\lambda}(X_T) := \frac{\text{ESG-AVaR}_{\lambda,\beta}(-X_T)}{\text{ESG-AVaR}_{\lambda,\gamma}(X_T)}. \tag{30}$$

Note that ESG-RR satisfies MO-RM, SI-RM and DB-RM but, as in the univariate case, fails to satisfy QC-RM since the numerator is not concave.
- **ESG STAR ratio**. When $\beta = 0$, the ESG Rachev ratio becomes an ESG generalization of the stable tail-adjusted return ratio,

$$\text{ESG-STARR}_{\alpha,\lambda}(X_T) := \frac{\text{ESG-}\mu_\lambda(X_T)}{\text{SG-AVaR}_{\lambda,\alpha}(X_T)}. \tag{31}$$

As a special case of the ESG Rachev ratio, $\text{ESG-STARR}_{\alpha,\lambda}$ satisfies conditions MO-RM, SI-RM and DB-RM; as the numerator is linear, it also satisfies QC-RM.
- **ESG Sortino–Satchell ratio**. The univariate Sortino–Satchell ratio (Sortino and Satchell 2001) is defined as $\mathbb{E}[r_T]^+/||r_T^-||_p$, where $||r_T||_p = \left(\int_{-\infty}^{\infty} |x|^p f_r(x)dx\right)^{1/p}$

[19] The SI-RM property applies if $\text{SA} = [0,\ 0]'$ or if X is intended to represent a vector of excess returns over the risk-free rate.

and $r_T \sim f_r(x)$. We extend this measure to the bivariate ESG setting by[20]

$$\text{ESG-SSR}_\lambda(X_T) = \frac{(\text{ESG-}\mu_\lambda(X_T))^+}{||Y_T^-||_p}, \tag{32}$$

where $Y_T = (1-\lambda)r_T + \lambda ESG_T$ and $Y_T \sim f(y)$. This measure satisfies all four ESG-RRR conditions.

The proposed formulation assumes a required rate of return and ESG target of zero. We can introduce a non-zero target by subtracting a bivariate vector (e.g. the bivariate return of an ESG safe asset $\text{SA}_T = [\text{RF}_T^r, \ \text{RF}_T^{ESG}]'$) from the numerator before applying the positive operator and using $\tilde{Y}_T = (1-\lambda)(r_T - \text{RF}_T^r) + \lambda(ESG_T - \text{RF}_T^{ESG})$ in the denominator in place of Y_T. In such a case, this measure no longer satisfies SI-RM.

- **ESG Omega ratio**. Defining $Y_T = (1-\lambda)r_T + \lambda ESG_T$, and $F(y)$ as the cumulative distribution function of Y_T, the ESG version of the Omega ratio (Keating and Shadwick 2002) is

$$\text{ESG-OR}_\lambda(X_T) = \frac{\int_\tau^\infty [1-F(y)]dy^+}{\int_{-\infty}^\tau F(y)dy}. \tag{33}$$

  Equivalently (see (Farinelli and Tibiletti 2008)),

$$\text{ESG-OR}_\lambda(X_T) = \frac{\mathbb{E}[(Y_T-\tau)^+]}{\mathbb{E}[(Y_T-\tau)^-]}. \tag{34}$$

  This measure satisfies MO-RM and DB-RM. It does not satisfy QC-RM, and SI-RM is only satisfied if $\tau = 0$.
- **ESG Farinelli–Tibiletti ratio**. This ratio aims to take into account the asymmetry in the return distribution; it generalizes the Omega ratio. By defining $Y_T = (1-\lambda)r_T + \lambda ESG_T$, the ESG version of the ratio can be defined as

$$\text{ESG-FTR}_{\lambda,m,n,p,q}(X_T) = \frac{||(Y_T-m)^+||_p}{||(n-Y_T)^+||_q}, \tag{35}$$

  where $m, n \in \mathbb{R}$ and $p, q > 0$. This ratio satisfies MO-RM and DB-RM. It also satisfies SI-RM if $n = m = 0$. Mirroring the univariate case, it does not satisfy QC-RM (see the example in (Cheridito and Kromer 2013), Sect. 4.3).

Table 2 summarizes the properties satisfied by each of these six ratios.

5 Empirical analysis

We present an empirical analysis in which we estimate a set of daily ESG risk measures and ESG reward–risk ratios. We use such measures to rank equity assets from

[20] The definition provided here assumes a target return of 0, similarly to Cheridito and Kromer (2013).

Table 2 Summary of ESG reward–risk ratios and their properties. A checkmark indicates that the corresponding property is satisfied

Ratio	MO-RM	QC-RM	SI-RM	DB-RM
ESG Sharpe		✓	✓	✓
ESG Rachev	✓		✓	✓
ESG STAR	✓	✓	✓	✓
ESG Sortino–Satchell	✓	✓	✓	✓
ESG Omega	✓		✓	✓
ESG Farinelli–Tibiletti	✓		✓	✓

the Dow-Jones Industrial Average (DJIA) index, and the main goal is to illustrate how rankings vary with the preference parameter λ. We use the log-returns computed on adjusted close prices downloaded from Factset, and for the ESG component we consider the Factset Truvalue scores. These ESG scores, in contrast to scores by other data providers, are not computed on the basis of balance sheet items and metrics reported by companies, but instead they are obtained by analysing news and documents, performing sentiment analyses on news and event tracking on sustainability-related events with artificial intelligence techniques. Such scores are based on a large number of sources, and reflect the public perception of the investors on the sustainability of companies. One unique feature of these scores is that they are updated with daily frequency, allowing the dynamics of the sustainability of each company to be tracked. In contrast, other data providers update ESG scores with quarterly or yearly frequency. The Truvalue scores represent therefore a desirable dataset for showcasing ESG-coherent risk measures and reward ratios. Truvalue scores are provided in two variants: *Pulse Scores* and *Insight Scores*, where the former focus more on near-term performance changes, and the latter considers the longer-term track record. In this analysis we consider Pulse Scores, as they provide a more dynamic measure of the sustainability profile of the companies. The high-frequency of the Truvalue dataset however comes at the cost of a less transparent scoring methodology compared to competitors due to the reliance to AI techniques.

We consider 28 constituents of the DJIA index as of December 31, 2023. We use the daily time series of log-returns (r_T) and Truvalue Pulse Scores (ESG_T) in the period from January 1, 2020 to December 31, 2023. The daily Pulse Score is normalized between $-1/c$ and $1/c$, where $c = 252$ (the number of trading days in a year). The normalization leads to a broadly similar scale of average returns and average ESG (see Table 3) The scaling of the ESG score, as long as it is consistent across assets, does not affect the generality of the results, as it can be compensated by changing the scaling of the subjective parameter λ. One concern with the dataset is that, when ESG scores are strictly bounded, some axiomatic conditions may lose their standard economic interpretation. In applications where this issue is relevant, one may adopt the modeling convention of working with a monotone transformation of the scores that has unbounded support, thereby restoring a more formal setting without altering their ordinal content.

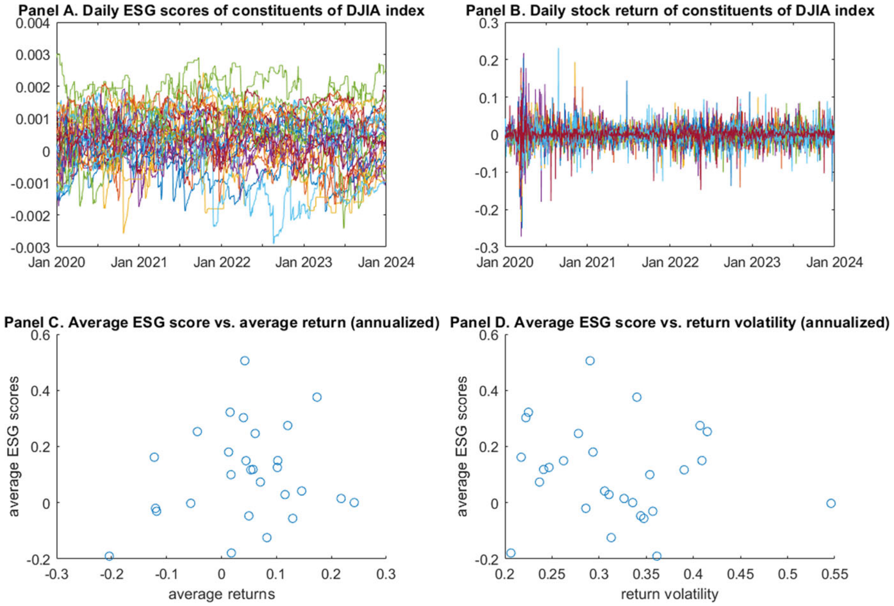

Fig. 1 Panel A: Evolution of the daily ESG score of the constituents of the index. Panel B: Evolution of stock return. Panel C: Scatterplot of average annualized daily returns and the average annualized ESG score for each asset. Panel D: Scatterplot of the annualized daily standard deviation of returns and the annualized average ESG score

Figure 1, illustrates the series of ESG scores for the constituents of the index (Panel A), and the log-return of the assets over time (Panel B). Panels C and D report the average ESG score across the entire period in relation to the average returns and the return volatility, respectively; looking at this aggregate data, the relation between the ESG score and the financial performance seems limited. Higher ESG scores seem to correlate slightly with higher returns, but the relation does not appear to be strong (correlation: 0.18). We see a small negative correlation between ESG scores and volatility (correlation: -0.12).

Table 3 reports, by company, the distribution mean and standard deviation of daily returns and ESG scores, as well as the correlation between returns and ESG scores. The dataset presents a wide cross-sectional variability in terms of the average return and risk, as well as in terms of the average ESG scores. ESG data show a smaller volatility compared to the returns despite a comparable cross-sectional dispersion of average returns and average ESG. This reflects the fact that the changes in the sustainability profile of a company happen at a much slower pace than the shifts seen in the monetary value of a stock. The correlation between the daily returns and ESG score is very close to zero for all the stocks. Figure 2 displays the correlation matrix between the returns and ESG of each stock (the first 28 rows and columns are the returns, the latter 28 are the ESG values). We see that the returns show consistently positive and strong correlation with each other. In contrast, ESG-to-ESG correlations have mixed signs. ESG-to-return correlations are very close to zero. In terms of risk management, the lack

Table 3 Company name, ticker symbol and GICS sector; annualized mean and standard deviation of daily returns ($\mathbb{E}[r], \sigma(r)$) and ESG scores ($\mathbb{E}[ESG], \sigma(ESG)$), correlation of daily returns and ESG score ($\rho(r, ESG)$) for the period 2020-2023

Name	GICS Sector	$\mathbb{E}[r]$	$\sigma(r)$	$\mathbb{E}[ESG]$	$\sigma(ESG)$	$\rho(r, ESG)$
Apple Inc.	IT	0.242	0.336	0.000	0.005	−0.025
Amgen Inc.	Health Care	0.045	0.262	0.149	0.012	−0.002
American Express Co.	Financials	0.102	0.409	0.150	0.010	−0.015
Boeing Company	Industrials	−0.056	0.546	−0.002	0.009	0.041
Caterpillar Inc.	Industrials	0.174	0.340	0.376	0.006	−0.019
Salesforce, Inc.	IT	0.121	0.407	0.275	0.007	−0.035
Cisco Systems, Inc.	IT	0.013	0.293	0.180	0.008	0.060
Chevron Corporation	Energy	0.053	0.390	0.117	0.007	0.018
Walt Disney Company	Comm. Services	−0.118	0.357	−0.030	0.008	0.035
Goldman Sachs Gr., Inc.	Financials	0.130	0.347	−0.056	0.010	−0.030
Home Depot, Inc.	Consumer Discr.	0.116	0.310	0.029	0.010	−0.008
Honeywell Int. Inc.	Industrials	0.042	0.290	0.505	0.007	0.047
IBM Corporation	IT	0.061	0.278	0.246	0.005	0.011
Intel Corporation	IT	−0.044	0.415	0.253	0.008	0.007
Johnson & Johnson	Health Care	0.018	0.206	−0.179	0.008	0.024
JPMorgan Chase & Co.	Financials	0.050	0.344	−0.047	0.009	−0.022

Table 3 continued

Name	GICS Sector	$\mathbb{E}[r]$	$\sigma(r)$	$\mathbb{E}[ESG]$	$\sigma(ESG)$	$\rho(r, ESG)$
Coca-Cola Company	Consumer Staples	0.016	0.225	0.322	0.004	0.037
McDonald's Corp.	Consumer Discr.	0.102	0.247	0.125	0.006	−0.010
3M Company	Industrials	−0.120	0.286	−0.020	0.013	0.010
Merck & Co., Inc.	Health Care	0.057	0.241	0.118	0.005	0.023
Microsoft Corporation	IT	0.218	0.326	0.015	0.006	0.011
NIKE, Inc. Class B	Consumer Discr.	0.017	0.354	0.100	0.008	−0.007
Procter & Gamble Co.	Consumer Staples	0.040	0.222	0.303	0.009	−0.024
Travelers Companies, Inc.	Financials	0.083	0.313	−0.124	0.014	−0.021
UnitedHealth Group Inc.	Health Care	0.146	0.306	0.042	0.008	0.016
Verizon Comm. Inc.	Comm. Services	−0.122	0.217	0.162	0.007	0.005
Walgreens Boots All., Inc.	Consumer Staples	−0.204	0.361	−0.190	0.013	0.017
Walmart Inc.	Consumer Staples	0.071	0.237	0.074	0.006	−0.006

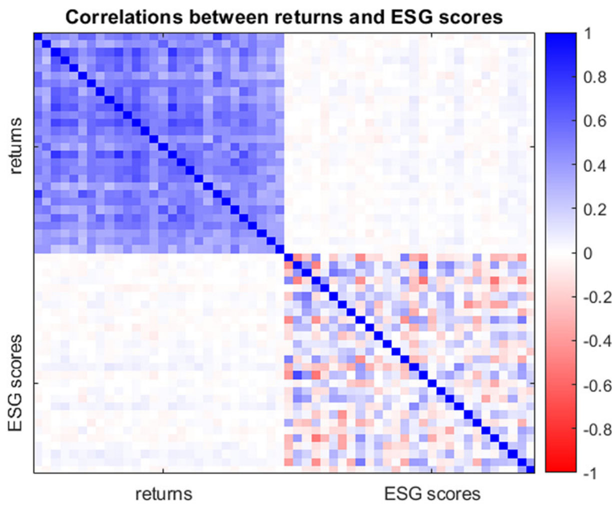

Fig. 2 Correlations between returns and ESG of the companies in the sample

of positive correlations among ESG and returns is beneficial to investors, as it allows diversification of risk between the two, supporting the motivation of the axiomatic framework.

For each of the stocks, we compute a set of daily ESG risk measures and ESG reward–risk ratios, considering a range of values of λ. We then use the measures to rank the stocks in the dataset. In particular, we compute ESG-AVaR$_{\lambda,\tau}$ (with τ equal to 0.95 and 0.99), the ESG standard deviation (ESG-σ_λ), the ESG-mean (ESG-μ_λ), the ESG Rachev ratio (with τ equal to 0.95 and 0.99), the ESG STAR ratio (with τ equal to 0.95 and 0.99), and the ESG Sharpe ratio. The risk measures are computed using a historical simulation approach, employing all the observations in the period 2020-2023. The range of λ considered is intentionally extreme and serves as a stress test for the framework. In practice, investors would typically choose smaller values of λ, resulting in portfolios that remain closer to conventional market portfolios. Figure 3 displays the ordering of the assets: each company is color-coded according to the industrial sector in which it operates, and the companies in the upper part of the plot are those with the highest value for the indicator. On the horizontal axis, we vary the investor's λ from zero to one. Tables 4 and 5 in Appendix D report the names of the companies in the top and bottom five positions for a selected sample of indicators with $\lambda \in \{0, 0.25, 0.5, 0.75, 1\}$.[21]

We make the following observations:

- ESG-AVaR tends to give similar rankings for levels of λ up to 0.5; then, the ranking changes significantly and converges to a significantly different ranking for $\lambda = 1$ compared to $\lambda = 0$. We can explain this behavior by taking into consideration that the ESG scores are characterized by a smaller variability over time; hence the tails of the distributions of returns is the most relevant determinant of the ESG-AVaR

[21] For brevity we report only the measures for the 95th quantile. The complete rankings and the results for the 99th quantile are available upon request.

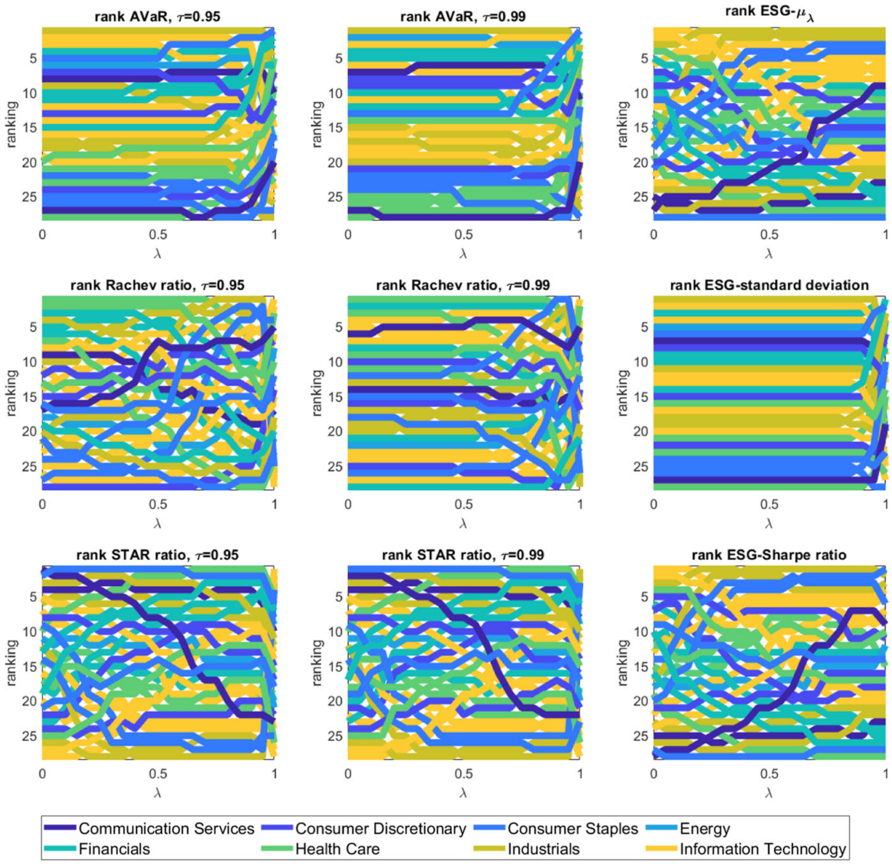

Fig. 3 Visual representation of the ranking of the assets in the DJIA index based on ESG-AVaR$_{\lambda,\tau}$ (τ equal to 0.95 and 0.99), ESG-μ_λ, the ESG Rachev ratio (τ equal to 0.95 and 0.99), the standard deviation, ESG-STARR$_{\alpha,\lambda}$ (the STAR ratio with p = 2), and the Sharpe ratio. Values of λ are arranged on the X axis from 0 (only the monetary component considered, left) to 1 (only the ESG component considered, right), and companies are color-coded by sector

(and thus the ranking) even for relatively large values of λ. For irrealistically large values of λ close to 1, the ESG component becomes dominant in the determination of the ranking. The results are similar for $\tau = 0.95$ and $\tau = 0.99$; the main difference is that for the highest value of τ, the monetary component remains dominant for higher values of λ.

- The ranking according to the Rachev ratio is also stable for λ smaller than 0.5. This is related to the fact that this ratio is computed as the ratio of the AVaRs for the top and bottom tails of the distribution.
- The ranking according to the STAR ratio shows significant changes when λ changes. This can be attributed in large part to the fact that the numerator of the ratio (ESG-μ_λ) by construction changes significantly with λ.
- The ranking according to the standard deviation is almost identical for all values of λ, except for values very close to 1. This is because, as stated previously,

the variability of ESG scores is small, and the ESG standard deviation is almost exclusively driven by monetary returns. Unlike the AVaR, the standard deviation is not influenced by parallel shifts of the distribution; hence, increasing the value of λ does not cause any changes in the ranking.
- ESG-μ_λ, the ESG Sharpe ratio, and the ESG Sortino-Satchell ratio rankings change significantly for different values of λ. These changes are driven by the significant differences in the rankings of the average returns and average sustainability, which is also suggested by Panel C in Fig. 1.

Overall, the example shows the structural flexibility and interpretability of the proposed framework, which can be used to develop empirical analyses both to answer theoretical questions and to implement viable investment strategies suitable for ESG-oriented investors. Further empirical research is required for more extensive results, in particular for the modelization of the joint evolution of returns and ESG using stochastic process and the extension to more assets and geographical areas. Another natural extension is a benchmarking of the results against more ESG data providers, especially considering that the ESG scores from different rating agencies are often inconsistent (Billio et al. 2021).

6 Policy and practical implications

Modeling investment outcomes as a bivariate random vector $X_T = [r_T,\ ESG_T]'$, with r_T denoting financial payoffs and ESG_T the sustainability dimension, extends the standard univariate risk and reward measure framework in a structurally meaningful way. ESG-coherent risk measures embed sustainability considerations directly into the risk functional through a preference parameter λ, allowing ESG to enter portfolio choice in the same formal manner as other preference-driven dimensions. This provides a unified setting in which financial and ESG components are evaluated jointly, rather than sequentially or through external constraints.

For optimal asset allocation, this formulation yields well-defined optimization problems in which portfolio choice depends on the joint distribution of X_t, with only minimal modifications relative to standard univariate setups. ESG preferences are reflected through λ, inducing smooth adjustments in optimal portfolios reflecting the subjective trade-off between the monetary and the sustainability dimensions. Unlike exclusion or screening approaches, this mechanism preserves diversification properties and allows for a systematic analysis of sensitivity with respect to ESG preferences, dependence between r_T and ESG_T, and joint tail behavior. From a theoretical perspective, it opens the way to studying ESG-aware allocation using familiar tools from coherent risk theory and convex optimization.

The framework is deliberately focused on ESG-oriented investing, in the sense that ESG enters the model as an argument of investor preferences rather than as a predictive signal for financial returns. This does not preclude the use of ESG information for improving financial performance, for example through forecasting, risk premia estimation, or alpha generation. Rather, the advantage of the proposed approach is to keep these two roles of ESG analytically distinct: ESG can be treated either as a component of the payoff vector X_T reflecting non-financial objectives, or as an informational variable influencing the distribution of r_T. By separating preference-driven valuation from return-predictive use, the framework avoids conflating ethical objectives with performance claims and clarifies the economic interpretation of ESG integration, while remaining flexible enough to accommodate their joint use within advanced asset allocation strategies designed to address multiple, potentially heterogeneous, investment objectives.

The stochastic treatment of ESG_T also has direct implications for ESG data provision. Meaningful ESG-coherent risk measurement requires time-series ESG data at frequencies compatible with financial modeling, creating incentives for data providers to move beyond static or purely cross-sectional scores. Higher-frequency and methodologically consistent ESG measures directly improve estimation of the joint law of $X_T = [r_T,\ ESG_T]'$, its dependence structure, and joint tail risk, rather than merely refining descriptive asset rankings.

In terms of transparency and reporting, ESG-coherent risk measures make the aggregation of financial and ESG outcomes explicit. The role of preferences is isolated in the parameter λ, which facilitates comparability across portfolios and products by separating differences due to investor objectives from differences due to underlying risk exposures. For institutional investors, this supports internal coherence between mandates, portfolio construction, and performance evaluation. For retail investors, it clarifies the trade-offs embedded in ESG-oriented products, reducing reliance on implicit or opaque screening rules.

Finally, the framework makes explicit several structural limitations that are often implicit in ESG integration. The choice of λ is normative and cannot be identified from data alone, and the quality of ESG-coherent risk estimates is constrained by data availability and by the modeling of dependence between r_T and ESG_T. Making these elements explicit shifts attention from the proliferation of ESG scores to the underlying statistical and preference-based structure of risk measurement for ESG-oriented investors, where further theoretical and empirical progress is most likely to be achieved.

7 Conclusions

Individuals and institutions are increasingly aware of the non-monetary impact of their investments, and many are willing to structure their portfolios considering not only monetary risk and gains, but also the environmental, social, and governance implications. These ESG-oriented investors, who are rational under a richer set of objectives, challenge the traditional financial literature and require the development of new analytical tools to describe their behavior. Our work contributes to the literature by introducing ESG-coherent risk measures, a framework for the measurement of risk for investors with both monetary and ESG goals that provides an axiomatic definition in which risk is measured as a function of a bivariate random variable. The investor in our approach is still rational but follows a multi-dimensional evaluation that considers not only the monetary part but also sustainability. This framework extends the traditional coherent risk measures approach of Artzner et al. (1999).

The measures we provide can be used in several contexts and, due to the introduction of a parameter λ, can be adapted to individuals with different relative preferences for the monetary and ESG components of risk. We also provide the dual representation for ESG-coherent risk measures, present several examples that generalize univariate risk measures, and introduce ESG-coherent reward measures and ESG reward–risk ratios.

We stress that the goal of the proposed approach is not to integrate ESG scores for improving monetary risk-adjusted performances, but to take into account the ethical preference of an investor for sustainable assets. This challenges the traditional assumptions of monetary profit-maximizing and risk-minimizing agents.

This paper is only an initial step in the development of a new financial theory that will be capable of describing the behavior of ESG-oriented investors. Future work will study optimal asset allocations, utility theory, and asset pricing for ESG-oriented investors.

Appendix A ESG-coherent reward measures

Similarly to how we defined ESG-coherent risk measures, we define ESG-coherent reward measures. As discussed by Rachev et al. (2008), in the univariate case, a reward measure can be defined as any functional defined on the space of the random variables of interest, that is, it should be isotonic with market preferences. Still, it is useful to formalize axiomatically the characteristics of reward measures. We now introduce axioms for multivariate reward measures inspired by the ones for the univariate case discussed in Rachev et al. (2008). Considering a probability space $(\Omega, \mathcal{F}, P)$ and a set of bivariate random variables $\mathcal{X}_2$, we can define an ESG-adjusted reward measure $\boldsymbol{\theta}_\lambda : \mathcal{X}_2 \to \mathbb{R} \cup \{\pm\infty\}$, where $\lambda \in [0, 1]$ is then ESG-coherent if the following axioms are satisfied:

(SUP-M+) Super-additivity: if $X_{1,T}, X_{2,T} \in \mathcal{X}_2$, then $\boldsymbol{\theta}_\lambda(X_{1,T} + X_{2,T}) \geq \boldsymbol{\theta}_\lambda(X_{1,T}) + \boldsymbol{\theta}_\lambda(X_{2,T})$;

(PH-M+) Positive Homogeneity: if $\beta \geq 0$ and $X_T \in \mathcal{X}_2$, then $\boldsymbol{\theta}_\lambda(\beta X_T) = \beta\boldsymbol{\theta}_\lambda(X_T)$;

(MO-M+) Monotonicity: if $X_{1,T}, X_{2,T} \in \mathcal{X}_2$ and $(r_{1,T} \leq r_{2,T} \wedge ESG_{1,T} \leq ESG_{2,T})$ a.s., then $\boldsymbol{\theta}_\lambda(X_{1,T}) \leq \boldsymbol{\theta}_\lambda(X_{2,T})$;

(LH-M+) Lambda Homogeneity: if $a = [a_1,\ a_2]' \in \mathbb{R}^2$, then $\boldsymbol{\theta}_\lambda(a) = (1-\lambda)a_1 + \lambda a_2$.

The translation invariance property follows directly from SUP-M+ and LH-M+, analogously to Proposition 1.

(TI-M+) Translation Invariance: if $a = [a_1,\ a_2]' \in \mathbb{R}^2$, $\boldsymbol{\theta}_\lambda(X_T + a) = \boldsymbol{\theta}_\lambda(X_T) + (1-\lambda)a_1 + \lambda a_2$.

Appendix B Proof of dual representation of ESG-coherent risk measures

For convenience define $Z := [Z_1,\ Z_2]' = -X_T$ where X_T is the bivariate vector of monetary returns and ESG. Consider the space $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P; \mathbb{R}^2)$, $p \in [1, \infty]$. When $p < \infty$, the dual space $\mathcal{Z}^*$ is isomorphic to $\mathcal{L}_q(\Omega, \mathcal{F}, P; \mathbb{R}^2)$, where $q \in (1, \infty]$ is such that $1/p + 1/q = 1$ with $q = \infty$ for $p = 1$.

Consider the bilinear form $\langle \cdot, \cdot \rangle$ on the product $\mathcal{Z} \times \mathcal{Z}^*$, which is defined as follows. For $Z \in \mathcal{Z}$ and $\zeta \in \mathcal{Z}^*$ the value of the bilinear form is given by

$$\langle \zeta, Z \rangle = \int_{\Omega} (\zeta_1(\omega) Z_1(\omega) + \zeta_2(\omega) Z_2(\omega))\, P(d\omega). \tag{B1}$$

The form provides the corresponding continuous linear functionals on $\mathcal{Z}$ and $\mathcal{Z}^*$ when equipped with appropriate topologies. For each fixed $\zeta \in \mathcal{Z}^*$, the mapping $Z \mapsto \langle \zeta, Z \rangle$ is a continuous linear functional on $\mathcal{Z}$, equipped with the norm topology. For $p \in (0, 1)$, all continuous linear functional on $\mathcal{Z}^*$ have this form. For $p = 1$, we equip Z^* with the weak* topology. For $p = \infty$, the dual space $\mathcal{Z}^*$ is formed by finitely-additive measures and it is inconvenient to work with. In this case, we pair $\mathcal{Z} = \mathcal{L}_\infty(\Omega, \mathcal{F}, P, \mathbb{R}^2)$ with $\mathcal{L}_1(\Omega, \mathcal{F}, P, \mathbb{R}^2)$ and equip the latter space with its norm topology and the former with its weak* topology. We use the bilinear form (B1) with $Z \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P, \mathbb{R}^2)$ and $\zeta \in \mathcal{L}_1(\Omega, \mathcal{F}, P, \mathbb{R}^2)$.

Let $\boldsymbol{\rho}_\lambda : \mathcal{Z} \to \mathbb{R} \cup \{+\infty\}$ be a lower-semi-continuous functional with non-empty domain. In the case of $p = \infty$ we make the additional assumption that $\boldsymbol{\rho}_\lambda$ is lower-semicontinuous with respect to its weak* topology.

The Fenchel conjugate function $\boldsymbol{\rho}_\lambda^* : \mathcal{Z}^* \to \overline{\mathbb{R}}$ of the risk measure $\boldsymbol{\rho}_\lambda$ is defined as

$$\boldsymbol{\rho}_\lambda^*(\zeta) = \sup_{Z \in \mathcal{Z}} \left\{ \langle \zeta, Z \rangle - \boldsymbol{\rho}_\lambda(Z) \right\},$$

and the conjugate of $\boldsymbol{\rho}_\lambda^*$ (the bi-conjugate function) is defined as

$$\boldsymbol{\rho}_\lambda^{**}(Z) = \sup_{\zeta \in \mathcal{Z}^*} \left\{ \langle \zeta, Z \rangle - \boldsymbol{\rho}_\lambda^*(\zeta) \right\}.$$

$\mathcal{A}_{\boldsymbol{\rho}_\lambda}$ denotes the domain of $\boldsymbol{\rho}_\lambda^*$. The Fenchel-Moreau Theorem, which is valid in paired spaces, implies that $\boldsymbol{\rho}_\lambda^{**}(Z) = \boldsymbol{\rho}_\lambda(Z)$ whenever $\boldsymbol{\rho}_\lambda$ is proper, convex, and lower-semicontinuous.

Claim: The MO-M property holds iff for all $\zeta \in \mathcal{A}_{\boldsymbol{\rho}_\lambda}$, $\zeta \geq 0$ a.s.

Assume that the opposite is true. This means that there exists a set $\Delta \in \mathcal{F}$ with $P(\Delta) > 0$ such that for $\omega \in \Delta$, we have $\zeta_i(\omega) < 0$ for $i = 1$ or $i = 2$. We define $\bar{Z}_i = \mathbb{1}_{\Delta \cap \zeta_i < 0}$, where $\mathbb{1}_B$ is the indicator function of the event B. Take any Z with support in Δ such that $\boldsymbol{\rho}_\lambda(Z)$ is finite and and define $Z_t := Z - t\bar{Z}$. Then for $t \geq 0$, we have that $Z_t \leq Z$ componentwise, and $\boldsymbol{\rho}_\lambda(Z_t) \leq \boldsymbol{\rho}_\lambda(Z)$ by monotonicity.

Consequently,

$$\rho_\lambda^*(\zeta) \geq \sup_{t\in\mathbb{R}_+} \left\{\langle \zeta, Z_t\rangle - \rho_\lambda(Z_t)\right\} \geq \sup_{t\in\mathbb{R}_+} \left\{\langle \zeta, Z\rangle - t\langle \zeta, \bar{Z}\rangle - \rho_\lambda(Z)\right\}.$$

On the right-hand side, $\langle \zeta, \bar{Z}\rangle < 0$ on Δ and zero otherwise, while the other terms under the supremum are fixed. Hence, the supremum is infinite and $\zeta \notin \mathcal{A}_{\rho_\lambda}$.

Conversely, suppose that every $\zeta \in \mathcal{A}_{\rho_\lambda}$ is nonnegative. Then for every $\zeta \in \mathcal{A}_{\rho_\lambda}$ and $Z \geq Z'$ componentwise, we have

$$\begin{aligned}\langle \zeta, Z'\rangle &= \int_\Omega \left(\zeta_1(\omega)Z_1'(\omega) + \zeta_2(\omega)Z_2'(\omega)\right) P(d\omega) \\ &\leq \int_\Omega \left(\zeta_1(\omega)Z_1(\omega) + \zeta_2(\omega)Z_2(\omega)\right) P(d\omega) = \langle \zeta, Z\rangle.\end{aligned}$$

Consequently

$$\rho_\lambda(Z) = \sup_{\zeta\in\mathcal{Z}^*} \left\{\langle \zeta, Z\rangle - \rho_\lambda^*(\zeta)\right\} \geq \sup_{\zeta\in\mathcal{Z}^*} \left\{\langle \zeta, Z'\rangle - \rho_\lambda^*(\zeta)\right\} = \rho_\lambda(Z').$$

<u>Claim:</u> The PH-M property holds iff ρ_λ is the support function of $\mathcal{A}_{\rho_\lambda}$.

Suppose that $\rho_\lambda(tZ) = t\rho_\lambda(Z)$ for all $Z \in \mathcal{Z}$. For all $t > 0$ and for all $Z \in \mathcal{Z}$

$$\rho_\lambda^*(\zeta) = \sup_{Z\in\mathcal{Z}} \left\{\langle \zeta, Z\rangle - \rho_\lambda(Z)\right\} \geq \langle \zeta, tZ\rangle - \rho_\lambda(tZ)$$

Thus for all $t > 0$

$$\rho_\lambda^*(\zeta) = \sup_{Z\in\mathcal{Z}} \left\{\langle \zeta, Z\rangle - \rho_\lambda(Z)\right\} \geq \sup_{Z\in\mathcal{Z}} \left\{\langle \zeta, tZ\rangle - t\rho_\lambda(Z)\right\} = t\rho_\lambda^*(\zeta).$$

Hence, if $\rho_\lambda^*(\zeta)$ is finite, then $\rho_\lambda^*(\zeta) = 0$ as claimed. Furthermore,

$$\rho_\lambda(0) = \sup_{\zeta\in\mathcal{Z}^*} \left\{\langle \zeta, 0\rangle - \rho_\lambda^*(\zeta)\right\} = 0.$$

For the converse, if $\rho_\lambda(Z) = \sup_{\zeta\in\mathcal{A}_{\rho_\lambda}} \langle \zeta, Z\rangle$, then ρ_λ is positively homogeneous as a support function of a convex set. Hence when the PH-M property holds, the conjugate function is the indicator function of convex analysis of the set $\mathcal{A}_{\rho_\lambda}$.

<u>Claim:</u> Assume ρ_λ is a proper, convex, and positively homogeneous risk functional. Then the risk measure is additive for any constant vectors $a, b \in \mathbb{R}$ (i.e. $\rho_\lambda(a+b) = \rho_\lambda(a) + \rho_\lambda(b)$) iff $\mu_\zeta = \int_\Omega \zeta(\omega)P(d\omega) = \mu$ and $\langle \mathbf{1}, \mu\rangle = \rho_\lambda(\mathbf{1})$, for all $\zeta \in \mathcal{A}$, where $\mathbf{1} = [1,\ 1]'$. Furthermore, for all $Z \in \mathcal{X}_2$ and $a \in \mathbb{R}^2$,

$$\rho_\lambda(Z + a) = \rho_\lambda(Z) + \rho_\lambda(a).$$

Let us denote $\int_{\Omega} \zeta(\omega) P(d\omega) = \mu_{\zeta}$. If $\zeta \in \mathcal{A}_{\boldsymbol{\rho}_{\lambda}}$ and SUB-M and PH-M hold, then for all $a \in \mathbb{R}^2$

$$\boldsymbol{\rho}_{\lambda}(a) = - \sup_{\zeta \in \mathcal{Z}^*} \langle a, \mu_{\zeta} \rangle \tag{B2}$$

and $\boldsymbol{\rho}_{\lambda}(\mathbf{0}) = 0$, where $\mathbf{0} = [0,\ 0]'$. If $\mu = \mu_{\zeta} = \int_{\Omega} \zeta(\omega) P(d\omega)$ for all $\zeta \in \mathcal{A}$, then equation B2 implies that the risk measure is additive for any constant vectors. Assume now that the measure is additive for constant vectors. Hence,

$$0 = \boldsymbol{\rho}_{\lambda}(\mathbf{0}) = \boldsymbol{\rho}_{\lambda}(a - a) = \boldsymbol{\rho}_{\lambda}(a) + \boldsymbol{\rho}_{\lambda}(-a).$$

Consequently $\boldsymbol{\rho}_{\lambda}$ is linear on constants and $\langle a, \mu_{\zeta} \rangle = \langle a, \mu \rangle$ with $\mu = \mu_{\zeta}$ for all $a \in \mathbb{R}^2$ and for all $\zeta \in \mathcal{A}_{\boldsymbol{\rho}_{\lambda}}$. Indeed,

$$\sup_{\zeta \in \mathcal{Z}^*} \langle a, \mu_{\zeta} \rangle = \sup_{\zeta \in \mathcal{Z}^*} \langle -a, \mu_{\zeta} \rangle = 0$$

or equivalently

$$\sup_{\zeta \in \mathcal{Z}^*} \langle a, \mu_{\zeta} \rangle = \inf_{\zeta \in \mathcal{Z}^*} \langle a, \mu_{\zeta} \rangle.$$

Hence $\langle a, \mu_{\zeta} \rangle$ is the same for all $\zeta \in \mathcal{A}_{\boldsymbol{\rho}_{\lambda}}$. Since $a \in \mathbb{R}^2$ is arbitrary, we conclude that $\mu = \mu_{\zeta}$. For any $\zeta \in \mathcal{A}_{\boldsymbol{\rho}_{\lambda}}$ we have

$$\int_{\Omega} \langle \mathbf{1}, \zeta(\omega) \rangle P(d\omega) = \langle \mathbf{1}, \mu_{\zeta} \rangle = \sup_{\zeta \in \mathcal{Z}^*} \langle \mathbf{1}, \mu_{\zeta} \rangle = -\boldsymbol{\rho}_{\lambda}(\mathbf{1}).$$

Let $Z \in \mathcal{Z}$, $X_T = -Z$, and $a \in \mathbb{R}$.

$$\begin{aligned} \boldsymbol{\rho}_{\lambda}(X_T + a) &= \sup_{\zeta \in \mathcal{Z}^*} \left\{ \int_{\Omega} \langle Z - a, \zeta(\omega) \rangle P(d\omega) \right\} \\ &= \sup_{\zeta \in \mathcal{Z}^*} \left\{ \int_{\Omega} \langle Z, \zeta(\omega) \rangle P(d\omega) - \langle a, \mu \rangle \right\} = \boldsymbol{\rho}_{\lambda}(X_T) + \boldsymbol{\rho}_{\lambda}(a). \end{aligned}$$

<u>Claim:</u> If additionally LH-M holds, then $\mu = \left[(1 - \lambda), \lambda\right]'$ and, hence, for all $\zeta \in \mathcal{A}_{\boldsymbol{\rho}_{\lambda}}$, we have

$$\int_{\Omega} \zeta_1(\omega) + \zeta_2(\omega) P(d\omega) = 1.$$

In summary, when axioms SUB-M through LH-M are satisfied, then the dual representation of the risk measure is

$$\begin{aligned} \boldsymbol{\rho}_{\lambda}(X_T) &= \sup_{\zeta \in \mathcal{A}_{\boldsymbol{\rho}_{\lambda}}} \left\{ - \int_{\Omega} \left(\zeta_1(\omega) r_T(\omega) + \zeta_2(\omega) ESG_T(\omega) \right) P(d\omega) \right\} \\ \boldsymbol{\rho}_{\lambda}(X_T) &= - \inf_{\zeta \in \mathcal{A}_{\boldsymbol{\rho}_{\lambda}}} \left\{ \int_{\Omega} \left(\zeta_1(\omega) r_T(\omega) + \zeta_2(\omega) ESG_T(\omega) \right) P(d\omega) \right\} \end{aligned} \tag{B3}$$

where $\mathcal{A}_{\rho_\lambda}$ contains non-negative functions $(\zeta_1(\omega), \zeta_2(\omega))$ on $\mathbb{R}^2$ whose expected value is $\left[(1-\lambda), \quad \lambda\right]'$. Furthermore, $\mathcal{A}_{\rho_\lambda}$ is equal to the convex subdifferential of $\boldsymbol{\rho}_\lambda([0, 0]')$. Note that in (B3) we adjusted the signs since we express the risk measure in terms of X_T and not $Z = -X_T$.

Appendix C Proofs of dual representation for ESG-AVaR$_{\lambda,\tau}$ and ESG-AVaR$^{l}_{\lambda,\tau}$

Let $X_T = [r_T, \ \ ESG_T]' \in \mathcal{X}_2 = \mathcal{L}_1(\Omega, \mathcal{F}, P; \mathbb{R}^2)$ be a bivariate random variable associated with an asset, and for convenience, define $Z := [Z_1, \ Z_2]' = -X_T$ (i.e., the corresponding vector with inverted signs).

Dual representation for ESG-AVaR$_{\lambda,\tau}$ **(Proposition 5)**

We show that that ESG-AVaR$_{\lambda,\tau}(X_T)$ has the following dual representation:

$$\text{ESG-AVaR}_{\lambda,\tau}(X_T) = \sup_{[\zeta_1 \ \zeta_2]' \in \mathcal{A}_{\text{ESG-AVaR}_{\lambda,\tau}}} \mathbb{E}[Z_1\zeta_1 + Z_2\zeta_2], \quad \text{with}$$

$$\mathcal{A}_{\text{ESG-AVaR}_{\lambda,\tau}} = \Bigg\{[\zeta_1, \ \zeta_2]' \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P; \mathbb{R}^2) : \ [\zeta_1, \ \zeta_2]' = \xi[1-\lambda, \ \lambda]'; \\ 0 \le \xi \le \frac{1}{1-\tau} \text{ a.s.}; \mathbb{E}[\xi] = 1\Bigg\}.$$

Proof We assume $\mathcal{X}_2 = \mathcal{L}_1(\Omega, \mathcal{F}, P; \mathbb{R}^2)$, which entails that the paired space is $\mathcal{L}_\infty(\Omega, \mathcal{F}, P; \mathbb{R}^2)$.

$$\begin{aligned}\text{ESG-AVaR}_{\lambda,\tau}(X_T) &= \min_{\beta\in\mathbb{R}}\left\{\frac{1}{1-\tau}\mathbb{E}\big[\big(\beta - ((1-\lambda)r + \lambda ESG)\big)_+\big] - \beta\right\}, \\ &= \min_{\beta\in\mathbb{R}}\left\{\beta + \frac{1}{1-\tau}\mathbb{E}\big[\big((1-\lambda)Z_1 + \lambda Z_2 - \beta\big)_+\big]\right\}, \quad \tau \in (0,1).\end{aligned}$$

Using the rules of subdifferential calculus and Strassen's theorem (Strassen 1965), we get

$$\partial\mathbb{E}\big[\big((1-\lambda)Z_1 + \lambda Z_2 - \beta\big)_+\big] = [1-\lambda, \ \lambda]'\xi,$$

where

$$\xi(\omega) = \begin{cases} 1 & \text{if } (1-\lambda)Z_1 + \lambda Z_2 > \beta \\ 0 & \text{if } (1-\lambda)Z_1 + \lambda Z_2 < \beta \\ [0,1] & \text{if } (1-\lambda)Z_1 + \lambda Z_2 = \beta. \end{cases}$$

Note that $\xi \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P)$ and $\xi \ge 0$. We define $\zeta \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P; \mathbb{R}^2)$ by setting $\zeta = [1-\lambda, \ \lambda]'\xi$ for any measurable selection $\xi \in \partial\mathbb{E}\big[\big((1-\lambda)r + \lambda ESG)\big)_+\big]$. Let $\tilde{\mathcal{A}}$ be the set containing all such elements ζ. Evidently, we have $\mathbf{0} \le \zeta \le [1-\lambda, \ \lambda]'$

a.s. for all. Using the subgradient inequality at $\bar{Z} = (\beta, \beta)$, we obtain for all Z

$$\mathbb{E}\Big[\big(((1-\lambda)Z_1 + \lambda Z_2) - \beta\big)_+\Big] \geq \langle \zeta, Z - \beta(1,1)\rangle = \langle \zeta, Z\rangle - \beta\mathbb{E}[\xi].$$

On the other hand, for any $\zeta = [1-\lambda,\ \lambda]'\xi$

$$\begin{aligned}&\langle \zeta, Z\rangle = \langle [1-\lambda,\ \lambda]'\xi, Z - \beta(1,1)\rangle + \beta\langle [1-\lambda,\ \lambda]'\xi, (1,1)\rangle \\ &= \langle \xi, ((1-\lambda)Z_1 + \lambda Z_2 - \beta)\rangle + \beta\mathbb{E}[\xi] \leq \mathbb{E}\Big[\big((1-\lambda)Z_1 + \lambda Z_2 - \beta\big)_+\Big] + \beta\mathbb{E}[\xi].\end{aligned}$$

Hence, we can represent

$$\mathbb{E}\Big[\big((1-\lambda)Z_1 + \lambda Z_2 - \beta\big)_+\Big] = \max_{\zeta\tilde{\mathcal{A}}} \Big(\langle \zeta, Z\rangle - \beta\mathbb{E}[\xi]\Big).$$

Now, we can express the risk measure as follows:

$$\begin{aligned}\boldsymbol{\rho}(Z) &= \min_{\beta\in\mathbb{R}} \left\{\beta + \frac{1}{1-\tau}\max_{\zeta\in\tilde{\mathcal{A}}}\Big(\langle\zeta, Z\rangle - \beta\mathbb{E}[\xi]\Big)\right\} \\ &= \max_{\zeta\in\tilde{\mathcal{A}}}\inf_{\beta\in\mathbb{R}}\left\{\beta + \frac{1}{1-\tau}\langle\zeta, Z\rangle - \frac{\beta}{1-\tau}\mathbb{E}[\xi]\right\} \\ &= \max_{\zeta\in\tilde{\mathcal{A}}}\inf_{\beta\in\mathbb{R}}\left\{\beta(1 - \frac{1}{1-\tau}\mathbb{E}[\xi]) + \frac{1}{1-\tau}\langle\zeta, Z\rangle\right\},\end{aligned}$$

where $\boldsymbol{\rho}(Z) = \text{ESG-AVaR}_{\lambda,\tau}(X_T)$. The exchange of the “min” and “max” operations is possible, because the function in braces is bilinear in (β, ζ), and the set $\tilde{\mathcal{A}}$ is compact. The inner minimization with respect to β yields $-\infty$, unless $\mathbb{E}[\xi] = 1-\tau$. Consequently,

$$\boldsymbol{\rho}(Z) = \max_{\substack{\zeta=\xi[1-\lambda,\lambda]'\in\mathcal{A} \\ \mathbb{E}[\xi]=1-\tau}} \frac{1}{1-\tau}\langle Z, \zeta\rangle.$$

Setting $\zeta' = \zeta/(1-\tau)$, we obtain the support set

$$\partial\boldsymbol{\rho}(0) = \left\{\zeta \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P; \mathbb{R}^2) : \zeta = \frac{\xi}{1-\tau}[1-\lambda, \lambda]';\ \xi \geq 0,\ \mathbb{E}[\xi] = 1-\tau,\ \|\xi\|_\infty \leq 1\right\}.$$

□

Dual representation for $\text{ESG-AVaR}^l_{\lambda,\tau}$ **(Proposition 6)**

We now prove that the dual representation of $\text{ESG-AVaR}^l_{\lambda,\tau}(X_T)$ is defined as follows:

$$\text{ESG-AVaR}^l_{\lambda,\tau}(X_T) = \sup_{[\zeta_1,\ \zeta_2]'\in\mathcal{A}_{\text{ESG-AVaR}_{\lambda,\tau}}} \mathbb{E}[Z_1\zeta_1 + Z_2\zeta_2], \tag{C4}$$

Proof with $Z := [Z_1,\ Z_2]' = -X_T$ and

$$\mathcal{A}_{\text{ESG-AVaR}^l_{\lambda,\tau}} = \Bigg\{[\zeta_1,\ \zeta_2]' \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P; \mathbb{R}^2) : \mathbb{E}[\zeta_1] = 1-\lambda; \mathbb{E}[\zeta_2] = \lambda; \\ \zeta_1, \zeta_2 \geq 0; \zeta_1 \leq \frac{1-\lambda}{1-\tau}; \zeta_2 \leq \frac{\lambda}{1-\tau}\Bigg\}. \tag{C5}$$

We use the known representation of Average Value at Risk for scalar random variables (cf. (Shapiro et al. 2021), Example 6.19 eq. 6.76). Denote the dual set in that representation by $\mathcal{A}'$, i.e.,

$$\mathcal{A}' = \left\{\xi \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P) :\ \mathbb{E}[\xi] = 1;\ 0 \leq \xi \leq \frac{1}{1-\tau} \text{ a.s.}\right\},$$

Hence

$$\begin{aligned}\text{ESG-AVaR}^l_{\lambda,\tau}(X) &= (1-\lambda) \sup_{\xi\in\mathcal{A}'} \mathbb{E}[\xi, Z_1] + \lambda \sup_{\xi\in\mathcal{A}'} \mathbb{E}[\xi, Z_2] \\ &= \sup_{\xi\in\mathcal{A}'} \mathbb{E}[(1-\lambda)\xi, Z_1] + \sup_{\xi\in\mathcal{A}'} \mathbb{E}[\lambda\xi, Z_2] \\ &= \sup_{[\xi_1,\xi_2]'\in\mathcal{A}'\times\mathcal{A}'} \Big(\mathbb{E}[(1-\lambda)\xi_1, Z_1] + \mathbb{E}[\lambda\xi_2, Z_2]\Big).\end{aligned} \tag{C6}$$

Now, we define the set $\mathcal{A}_{\text{ESG-AVaR}^l_{\lambda,\tau}} = (1-\lambda)\mathcal{A}' \times \lambda\mathcal{A}'$ and continue the last chain of equations as follows:

$$\text{ESG-AVaR}^l_{\lambda,\tau}(X_T) = \sup_{[\xi_1,\xi_2]'\in\mathcal{A}'\times\mathcal{A}'} \mathbb{E}[(1-\lambda)\xi, Z_1] + \mathbb{E}[\lambda\xi, Z_2] = \sup_{\zeta\in\mathcal{A}_{\text{ESG-AVaR}^l_{\lambda,\tau}}} \langle\zeta, Z\rangle.$$

This concludes the proof. □

Appendix D Company rankings based on ESG risk, reward, and RRR measures

See Tables 4 and 5.

Table 4 Top and bottom ranking according to several ESG risk and reward measures (ESG-AVaR$_{\lambda,0.95\%}$, ESG-standard deviation, ESG-mean) for selected values of λ

Ranking by ESG-AVaR$_{\lambda,0.95\%}$

Rank	$\lambda = 0$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
1	Boeing Company	Boeing Company	Boeing Company	Boeing Company	Walgreens Boots Alliance, Inc.
2	Intel Corporation	Intel Corporation	Intel Corporation	Intel Corporation	Travelers Companies, Inc.
3	Salesforce, Inc.	Salesforce, Inc.	Salesforce, Inc.	Walgreens Boots Alliance, Inc.	Goldman Sachs Group, Inc.
4	American Express Company	American Express Company	American Express Company	American Express Company	3M Company
5	Walgreens Boots Alliance, Inc.	Walgreens Boots Alliance, Inc.	Walgreens Boots Alliance, Inc.	Salesforce, Inc.	Johnson & Johnson
24	McDonald's Corporation	McDonald's Corporation	McDonald's Corporation	Coca-Cola Company	Salesforce, Inc.
25	Walmart Inc.	Walmart Inc.	Walmart Inc.	McDonald's Corporation	IBM Corporation
26	Procter & Gamble Company	Procter & Gamble Company	Procter & Gamble Company	Johnson & Johnson	Caterpillar Inc.
27	Verizon Communications Inc.	Verizon Communications Inc.	Verizon Communications Inc.	Procter & Gamble Company	Coca-Cola Company
28	Johnson & Johnson	Johnson & Johnson	Johnson & Johnson	Verizon Communications Inc.	Honeywell International Inc.

Ranking by ESG-standard deviation

Rank	$\lambda = 0$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
1	Boeing Company	Boeing Company	Boeing Company	Boeing Company	Travelers Companies, Inc.
2	Intel Corporation	Intel Corporation	Intel Corporation	Intel Corporation	Walgreens Boots Alliance, Inc.
3	American Express Company	American Express Company	American Express Company	American Express Company	3M Company
4	Salesforce, Inc.	Salesforce, Inc.	Salesforce, Inc.	Salesforce, Inc.	Amgen Inc.
5	Chevron Corporation	Chevron Corporation	Chevron Corporation	Chevron Corporation	American Express Company
24	Walmart Inc.	Walmart Inc.	Walmart Inc.	Walmart Inc.	Microsoft Corporation
25	Coca-Cola Company	Coca-Cola Company	Coca-Cola Company	Coca-Cola Company	Apple Inc.
26	Procter & Gamble Company	Procter & Gamble Company	Procter & Gamble Company	Procter & Gamble Company	Merck & Co., Inc.
27	Verizon Communications Inc.	Verizon Communications Inc.	Verizon Communications Inc.	Verizon Communications Inc.	IBM Corporation
28	Johnson & Johnson	Johnson & Johnson	Johnson & Johnson	Johnson & Johnson	Coca-Cola Company

Table 4 continued

Ranking by ESG-standard deviation					
Rank	$\lambda = 0$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
1	Apple Inc.	Caterpillar Inc.	Caterpillar Inc.	Honeywell International Inc.	Honeywell International Inc.
2	Microsoft Corporation	Apple Inc.	Honeywell International Inc.	Caterpillar Inc.	Caterpillar Inc.
3	Caterpillar Inc.	Microsoft Corporation	Salesforce, Inc.	Coca-Cola Company	Coca-Cola Company
4	UnitedHealth Group Inc.	Salesforce, Inc.	Procter & Gamble Company	Salesforce, Inc.	Procter & Gamble Company
5	Goldman Sachs Group, Inc.	Honeywell International Inc.	Coca-Cola Company	Procter & Gamble Company	Salesforce, Inc.
24	Boeing Company	Boeing Company	Boeing Company	3M Company	JPMorgan Chase & Co.
25	Walt Disney Company	Verizon Communications Inc.	Johnson & Johnson	Walt Disney Company	Goldman Sachs Group, Inc.
26	3M Company	3M Company	3M Company	Travelers Companies, Inc.	Travelers Companies, Inc.
27	Verizon Communications Inc.	Walt Disney Company	Walt Disney Company	Johnson & Johnson	Johnson & Johnson
28	Walgreens Boots Alliance, Inc.	Walgreens Boots Alliance, Inc.	Walgreens Boots Alliance, Inc.	Walgreens Boots Alliance, Inc.	Walgreens Boots Alliance, Inc.

Table 5 Top and bottom ranking according to several ESG-RRRs (ESG Sharpe ratio, ESG-STARR$_{\lambda,0.95\%}$, ESG-RR$_{\lambda,0.95\%}$) for selected values of λ

ranking by ESG Sharpe ratio					
Rank	$\lambda = 0$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
1	Apple Inc.	Caterpillar Inc.	Honeywell International Inc.	Honeywell International Inc.	Coca-Cola Company
2	Microsoft Corporation	Apple Inc.	Caterpillar Inc.	Coca-Cola Company	Honeywell International Inc.
3	Caterpillar Inc.	Microsoft Corporation	Procter & Gamble Company	Procter & Gamble Company	Caterpillar Inc.
4	UnitedHealth Group Inc.	Honeywell International Inc.	Coca-Cola Company	Caterpillar Inc.	IBM Corporation
5	McDonald's Corporation	McDonald's Corporation	IBM Corporation	IBM Corporation	Salesforce, Inc.
24	Intel Corporation	Johnson & Johnson	Boeing Company	Walt Disney Company	JPMorgan Chase & Co.
25	Walt Disney Company	Walt Disney Company	Walt Disney Company	3M Company	Goldman Sachs Group, Inc.
26	3M Company	Verizon Communications Inc.	3M Company	Travelers Companies, Inc.	Travelers Companies, Inc.
27	Walgreens Boots Alliance, Inc.	3M Company	Johnson & Johnson	Walgreens Boots Alliance, Inc.	Walgreens Boots Alliance, Inc.
28	Verizon Communications Inc.	Walgreens Boots Alliance, Inc.	Walgreens Boots Alliance, Inc.	Johnson & Johnson	Johnson & Johnson
ranking by ESG-STARR$_{\lambda,0.95\%}$					
Rank	$\lambda = 0$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
1	Verizon Communications Inc.	Walgreens Boots Alliance, Inc.	Walgreens Boots Alliance, Inc.	Johnson & Johnson	IBM Corporation
2	Walgreens Boots Alliance, Inc.	3M Company	Johnson & Johnson	Walgreens Boots Alliance, Inc.	Caterpillar Inc.
3	3M Company	Verizon Communications Inc.	3M Company	Travelers Companies, Inc.	Honeywell International Inc.
4	Walt Disney Company	Walt Disney Company	Walt Disney Company	3M Company	Coca-Cola Company
5	Intel Corporation	Johnson & Johnson	Boeing Company	Walt Disney Company	Johnson & Johnson
24	McDonald's Corporation	McDonald's Corporation	IBM Corporation	IBM Corporation	Cisco Systems, Inc.
25	UnitedHealth Group Inc.	Honeywell International Inc.	Coca-Cola Company	Caterpillar Inc.	Procter & Gamble Company
26	Caterpillar Inc.	Microsoft Corporation	Procter & Gamble Company	Coca-Cola Company	Merck & Co., Inc.
27	Microsoft Corporation	Apple Inc.	Caterpillar Inc.	Procter & Gamble Company	Intel Corporation
28	Apple Inc.	Caterpillar Inc.	Honeywell International Inc.	Honeywell International Inc.	Salesforce, Inc.

Table 5 continued

ranking by ESG-STARR$_{\lambda,0.95\%}$					
Rank	$\lambda = 0$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
1	Amgen Inc.	Amgen Inc.	Amgen Inc.	Amgen Inc.	Salesforce, Inc.
2	UnitedHealth Group Inc.	UnitedHealth Group Inc.	UnitedHealth Group Inc.	Honeywell International Inc.	Intel Corporation
3	JPMorgan Chase & Co.	JPMorgan Chase & Co.	Caterpillar Inc.	Caterpillar Inc.	Merck & Co., Inc.
4	Boeing Company	American Express Company	JPMorgan Chase & Co.	UnitedHealth Group Inc.	Procter & Gamble Company
5	American Express Company	Boeing Company	American Express Company	Procter & Gamble Company	Verizon Communications Inc.
24	Procter & Gamble Company	Walgreens Boots Alliance, Inc.	Coca-Cola Company	Cisco Systems, Inc.	Johnson & Johnson
25	IBM Corporation	IBM Corporation	3M Company	Johnson & Johnson	Coca-Cola Company
26	Cisco Systems, Inc.	Coca-Cola Company	Walgreens Boots Alliance, Inc.	3M Company	Honeywell International Inc.
27	Coca-Cola Company	Cisco Systems, Inc.	Cisco Systems, Inc.	Home Depot, Inc.	Caterpillar Inc.
28	Home Depot, Inc.	Home Depot, Inc.	Home Depot, Inc.	Walgreens Boots Alliance, Inc.	IBM Corporation

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Declarations

Competing interests The authors have no competing interests to declare.

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