Portfolio selection with GARCH volatility dynamics

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Abstract— This paper proposes and compares portfolio selection models under the assumption that the portfolios of returns follow a GARCH type process. We compute the price/return distribution at some future time approximating the GARCH process with a Markov chain. We consider either a GARCH(1,1) model or an asymmetric GARCH type model (E-GARCH, GJR-GARCH). We present an ex-post comparison of portfolio selection strategies applied to some assets of the US Market. Since the optimization problems present more local optima, we implement an heuristic algorithm for the global optimum in order to overcome the intrinsic computational complexity of the models.

Keywords— GARCH models, Portfolio selection, Performance strategies, Ex-post analysis, Heuristic, Global optimization, Markov chains.

I. INTRODUCTION

In this paper, we model the return portfolios with a Markov chain that account the GARCH evolution of the returns. In particular, we use the Duan and Simonato’s approximation of the returns evolution (Duan, Simonato 2001) in portfolios selection problems. Under this distributional hypothesis we compare the ex-post performance of some portfolio selection strategies.

There is a general consensus on the importance to model the time varying volatility (Engle [1982], Bollerslev [1986]) and the leverage effect (Black [1976]). Several empirical studies have showed that these statistical aspects serve to solve many biases between theoretical and empirical prices (see Bakshi, Cao and Chen [1997], Engle and Mustafa [1992], and Heston and Nandi [2000]). Because there is wide consensus that the variance of the financial asset returns is time variant, a great amount of efforts are directing to realize mathematical models which, by choosing the variance dynamics as the model corner-stone, should be effectively able to model financial prices. Surely the GARCH model is a reference instrument to study the volatility dynamics, and among its advantages there is its high flexibility to be suitable to capture the most important features of the financial variables. In this work we analyze the impact of choices based on the GARCH parametric characterization of financial asset series. It is to note that the passage from the GARCH parametric characterization of financial asset series to the computation of the price/return distribution at some future time is not immediate. In order to build portfolio wealth distribution we use Duan and Simonato’s GARCH approximation (Duan, Simonato 2001). Moreover, we extend the Duan and Simonato’s ideas to other possible GARCH type models (see Glosten, et al.(1993), Nelson (1991)). As these authors explain many GARCH models and in particular the GARCH(1,1) models can be represented as a bivariate Markovian system (i.e., the state of the process is uniquely represented by price and variance states). This feature allows to approximate GARCH models by a discrete Markov chain. The Markovian and semi-Markovian models has been used in different fields of the financial literature typically in option pricing and credit risk (see, among others, Duan and Simonato (2001); D’Amico and Di Biase, (2009), D’Amico et al. (2009, 2010)), and in portfolio theory (see Angelelli and Or托belli (2009), Iaquinta et al. (2010)). To build the transition matrixes we use the method discussed by Duan and Simonato (2001), Duan et al. (2003) for parametric Markovian processes.

With parametric portfolio selection models the transition matrix depends on the parameters of the underlying multivariate Markov process and the parameters are functions of the portfolio weights. Therefore we should check for a global optimum for most of the portfolio selection problems. In the paper we implement an optimization heuristic algorithm Angelelli and Or 托belli (2009) that reduces enormously the computational complexity with respect to
II. PORTFOLIO VALUE WITH GARCH VOLATILITY DYNAMICS

Let us consider a discrete-time economy and risky assets with log returns \( r_{t,i} = \ln \left( \frac{P_{t,i}}{P_{t-1,i}} \right) \). If we denote by \( x = [x_1, \ldots, x_n] \) the vector of the positions taken in the \( n \) risky assets, then the portfolio wealth at time \( t+1 \) is given by

\[
W_{t+1}(x) = \sum_{i=1}^{n} x_i e^{r_{t,i}}. \tag{1}
\]

In particular, we assume that investors want to maximize the performance of their choices at a given future date \( T \).

Now we introduce the alternative GARCH volatility dynamics models implemented in this work.

For convention we consider the initial portfolio wealth equal to \( 1 \) (i.e., \( W_0 = 1 \)). In this work we use the standard GARCH(1,1) (see Bollerslev, T. (1986)) and some well-known extension of the standard GARCH(1,1). Each model can be represented as:

\[
r_{t,i} = \ln \left( \frac{W_{t+1}(x)}{W_t(x)} \right) = \mu + \sigma_{t,i} e_{t,i} \tag{2}
\]

where \( W_t(x) \) is the portfolio value at time \( t \), with asset position \( x \), and \( e_{t,i} \mid \phi \sim N(0,1) \) under \( P \).

For convention we consider the initial portfolio wealth equal to \( 1 \) (i.e., \( W_0 = 1 \)). In this work we use the standard GARCH(1,1) (see Bollerslev, T. (1986)) and some well-known extension of the standard GARCH(1,1). Each model can be represented as:

\[
\sigma_{t,i} = f(\rho, \sigma, \epsilon) \tag{3}
\]

where the relation expresses that the conditional variance at time \( t+1 \) is a function of the lagged value of the variance (\( \sigma_t \)), the lagged shock (\( \epsilon_t \)) and a set of parameters (\( \rho \)).

The variance dynamics models we consider are:

**Model I**: GARCH(1,1) (G11) (see Bollerslev, T. (1986))

\[
\sigma^2_{t,i} = \omega + \beta \sigma^2_{t-1,i} + \alpha \epsilon^2_{t-1,i} \tag{4}
\]

where \( \omega, \alpha, \beta > 0 \) and \( 0 < \beta < 1 \).

**Model II**: GJR-GARCH (GJR-G): (see Glosten, et al. (1993))

\[
\sigma^2_{t,i} = \omega + \beta \sigma^2_{t-1,i} + \alpha \epsilon^2_{t-1,i} + \gamma \epsilon_{t} e_{t-1,i} \tag{5}
\]

where \( \omega, \alpha, \beta, \gamma > 0 \) and \( 0 < \beta < 1 \), \( \epsilon_t \) is the lagged shock (\( \epsilon_t \)).

**Model III**: E-GARCH (E-G): (see Nelson (1991))

\[
\ln(\sigma^2_{t,i}) = \omega + \beta \ln(\sigma^2_{t-1,i}) + \alpha (\epsilon_t, -\gamma \epsilon_t) \tag{6}
\]

where \( \alpha > 0 \) and \( 0 < \beta < 1 \).

The parameters of the models are \( \theta = (\mu, \rho) \) where \( \mu \) is the constant drift term and \( \rho = (\omega, \alpha, \beta, \gamma) \) is the parameter vector related to the variance dynamics.

In the model II and III the parameter \( \gamma \) allows to model the asymmetric behavior of the variance, sometimes called Black's effect. It consists of a greater response of the variance when the news arrived in the market are negative (\( \epsilon_t < 0 \)) than when the news are positive (\( \epsilon_t > 0 \)).

All conditions on the parameters \( \omega, \alpha \) are used to avoid theoretical inconsistence on the variance value (i.e., \( \omega < 0 \) should mean a potential negative value for the variance, while \( \alpha < 0 \) should mean that greater shock movements induce a decreasing variance), while the conditions on \( \beta \) allow the variance process to be covariance-stationary. Let us consider the value of the stationary variance level in each model, supposing the weakly stationarity on \( \sigma^2_t \) then we obtain:

\[
\begin{align*}
\mu' &= E(\sigma^2_t) = \frac{\omega}{1 - \beta - \alpha E^\theta(\epsilon^2_t)} & \text{in the G11} \\
\mu' &= E(\sigma^2_t) = \frac{\omega}{1 - \beta - \alpha E^\theta(\epsilon^2_t) - \gamma E^\theta(\epsilon^2_t) I} & \text{in the GJR} \\
\mu' &= E(\ln(\sigma^2_t)) = \frac{\omega - \alpha E^\theta(\epsilon^2_t)}{1 - \beta} & \text{in the E-G}
\end{align*}
\]

Since we have to avoid the asymptotic divergence and the negativity of the variance process we need the following additional conditions:

- \( \beta + \alpha E^\theta(\epsilon^2_t) < 1 \) in the G11
- \( \beta + \alpha E^\theta(\epsilon^2_t) + \gamma E^\theta(\epsilon^2_t) I < 1 \) in the GJR

Note that in the normal innovation case (i.e., \( \epsilon_{t,i} \mid \phi \sim N(0,1) \))

\[\]
\[ h' = \frac{\omega}{1 - \beta - \alpha} \] for G11, \[ h' = \frac{\omega}{1 - \beta - \alpha - \gamma / 2} \] in GJR, and the conditions are: \( \beta + \alpha < 1 \) in the G11, \( \alpha + \beta + \gamma / 2 < 1 \) in the GJR-G. In the E-GARCH model the value \( h' \) is approximated with the exponential of the expected log volatility. The GARCH(1,1) model is a benchmark model and it is used for model comparisons. The main characteristics of other GARCH models used can be summarized in:

1. Parsimonious model, only 4 parameters to model the variance dynamics
2. Two state variables: price and variance
3. Time varying variance: GARCH models drive the variance process.
4. Models are potentially able to explain the well-known stylised-facts as the "Leverage effect" (\( \gamma \) parameter) and the "Clustering effect" in the stochastic volatility (\( \beta \) parameter)

III. PORTFOLIO VALUE DYNAMICS AS MARKOVIAN EVOLUTION PROCESS

Duan and Simonato have shown that the GARCH(1,1) model can be represented as a bivariate Markovian system (i.e., the state of the process is uniquely represented by \((W_t, \sigma^2_{t-1})\) so the process is markovian of the first order). This feature allows to approximate GARCH models by a discrete Markov chain. Duan and Simonato’s analysis can be extended to GJR-GARCH and E-GARCH models as we show here in the following. In particular, we present the Markov chain approximation of a GARCH (1,1) process (Duan, Simonato 2001) adapted to the work models.

Let us consider an underlying portfolio log-return modeled by the equation (2) or equivalently let us consider
\[ \ln W_t(x) = \ln W_{t-1}(x) + \mu + \sigma z, \] where \( W_t \) denote the portfolio value at day \( t \). Let \( Q \) be some probability measure and \( \sigma \) be the variance modeled by G11, GJR or E-G. Let \( z \) a standardized random variable independently distributed with respect to the information up to time \( t-1 \), i.e., \( z_t \sim \mathcal{N}(0,1) \).

Following Duan and Simonato’s suggestions, we form the partitions by using the logarithm of adjusted wealth and log variance for the two state variables considered. The adjusted wealth is used to reduce the dimension of the transition matrix by a weight conversion. The logarithms of the values used are justified mainly for its better convergence behavior.

The adjusted wealth is computed by \( W_t' = e^{\mu} W_t \) where and \( \mu = \mu - h' / 2 \) is the stationary variance, the pre-adjusted wealth can be easily recover later. Also the unconditional variance can be computed in all the GARCH model mentioned.

Note that in term of log-adjusted wealth the log return dynamics becomes:
\[ \ln \left( \frac{W_t'}{W_{t-1}'} \right) = \ln \left( \frac{W_t}{W_{t-1}} \right) - \tilde{\mu} = \frac{1}{2} (h' - \sigma^2_t) + \sigma \epsilon_t. \]

The unconditional expectation of the continuously compounded return on the adjusted wealth is zero, since \( \mathbb{E}^Q (\sigma^2_t) = h' + \mathbb{E} (\sigma^2_t) = 0 \).

Let \( p_t \) and \( q_t \) be the logarithm of the adjusted wealth (let us say log wealth) and the logarithm of the variance respectively (i.e., \( p_t = \ln(W_t') \) and \( q_t = \ln(\sigma^2_t) \)) then the models can be rewritten with:
\[ p_t = p_{t-1} + \frac{1}{2} (h' - \epsilon_t^2) + \sqrt{\sigma^2_t} \epsilon_t \]
\[ q_t = \ln(\omega + \beta \epsilon^2_t + \alpha \epsilon^3_t \gamma \epsilon_t I_t) \]
\[ q_t = \ln(\omega + \beta \epsilon^2_t + \alpha \epsilon^3_t \gamma \epsilon_t I_t) \]
\[ q_t = \ln(\omega + \beta \epsilon^2_t + \alpha \epsilon^3_t \gamma \epsilon_t I_t) \]

To find a states partition to approximate the GARCH process we use:

1) A log wealth partition centered on the logaritmic of the initial portfolio wealth: \( \{ p_t = I_{s-1} p_0 + I_s \} \), where \( I_s \) is determined by studying the conditional behavior of the logaritmic of the adjusted portfolio wealth over the investor time horizon \( T \):
\[ I_s = \delta^*_s(m) \sqrt{\sum_{t=1}^T \mathbb{E}^Q (\sigma^2_t | \phi_t) } \]

2) An analytical formula of the conditional variance of the log wealth can be derived for many GARCH processes.

3) Log variance partition: to form the partition we would study the conditional behavior of the logarithm of the variance \( q_t = \ln(\sigma^2_t) \). From the GARCH process features we know that there are two notable values of the variance:

a) the initial variance, which the process starts from,

b) the unconditional variance (\( h' \)) to whom the process asymptotically is attracted.

Both these values have to be considered in the variance partition, but the second has increasing importance as we are far from the begin instant. The partition center can be computed as: \( q^*_s = \ln \left( \frac{\tau - \min(\tau, T)}{\tau} \sigma^2_i + \min(\tau, T) h' \right) \). The value of \( \tau \) is a temporal index used to form the weights. As it increases as the relative weight of the unconditional variance respect to the initial variance increases. Then in the study of long-term horizon \( \tau \) has to be small. Anyway it is important to ensure that \( q^*_s \) belongs to the partition. The log variance partition is \( \{ q_t = I_{s-1} q^*_s + I_s \} \). In order to compute the width \( I_s \) of the partition it should be enough to study \( \text{Var}^Q(q_t | \phi_t) \), but in G11 and GJR-G it could result analytically complex.

We know by the Jensen inequality that so Duan and
Simonato propose to use a width 
\[ \text{Var}^\text{GARCH}(q_i | \phi_i) \leq \ln \left( \text{Var}^\text{GARCH}(\sigma_i^2 | \phi_i) \right) \]

Only in the E-G case we have to note that the log variance partition can be constructed directly by the E-GARCH equation, because it expresses the variance in logarithmic terms: 
\[ q_i' = I_s + q_i + I_I \]

In the E-G the sum of the conditional variance up to \( T \) is given by:
\[ \sum_{s = 0}^{T} \text{Var}^\text{GARCH}(q_i | \phi_i) = T \alpha T \left( 1 - e^{-2} \right) \]

Duan and Simonato showed that \( \delta_{\text{E}}(m) \rightarrow 0 \) are sufficient partition conditions for the approximating Markov chain to converge to its target GARCH process.

The logarithmic adjusted wealth partition and the logarithmic variance partition are equally divided in and odd parts respectively in order to determine the state of the bivariate process:
- \( \pi(i) = p_o + \frac{2i-1-m}{m-1} I_s \) and the corresponding cells are 
\[ C(i) = (c(i), c(i+1)) \] for \( i = 1, \ldots, m \), where \( c(1) = -\infty \), 
\[ c(i) = \frac{\pi(i-1) + \pi(i)}{2} \] for \( i = 2, \ldots, m \) and \( c(m+1) = +\infty \)
- \( \pi(j) = q_j + \frac{2j-1-n}{n-1} I_s \) and the corresponding cells are 
\[ D(j) = (d(j), d(j+1)) \] for \( j = 1, \ldots, n \), where \( d(1) = -\infty \), 
\[ d(j) = \frac{\pi(j-1) + \pi(j)}{2} \] for \( j = 2, \ldots, n \) and \( d(n+1) = +\infty \).

The Markov transition probability from state \((i, j)\) at time \(t\) to state \((k, l)\) at time \(t+1\) is defined as 
\[ \pi(i, j, k, l) = \Pr[p_{i,s} \in C(k), q_{i,s} \in D(l) | p_t = \pi(i), q_{i,s} = \pi(j)] \]
for \( t = 0, \ldots, T-1 \).

It is typical in the GARCH(1,1) models that the variance at time \(t+1\) is a deterministic function of the information set at time \(t\). In particular in the models investigated we can write the variance as function of its lagged value, and two lagged wealth, i.e. \( q_{i,s} = \Phi(q_{i,s}, p_{i,s}, p_t) \).

First we recover \( \varepsilon_{i,s} \) from the log price equation written one time forward:
\[ \varepsilon_{i,s} = \frac{p_{i,s} - p_i + \frac{1}{2} (e^{\varepsilon_{i,s}} - h^i)}{\sqrt{e^{\varepsilon_{i,s}}}} \]
and substituting in the log variance equation we obtain:
\[ \Phi^{E-G}(q_{i,s}, p_{i,s}, p_t) = \ln \left( \omega + \beta e^{\varepsilon_{i,s}} + \alpha \left( p_{i,s} - p_i + \frac{1}{2} (e^{\varepsilon_{i,s}} - h^i) \right) \right) \]
\[ \Phi^{E-G}(q_{i,s}, p_{i,s}, p_t) = \ln \left( \omega + \beta e^{\varepsilon_{i,s}} + \alpha \left( p_{i,s} - p_i + \frac{1}{2} (e^{\varepsilon_{i,s}} - h^i) \right) \right) \]

This implies a source of sparsity in the markovian transition matrix: for each combination of \((i, j, k)\) it exists only an index \(l\) where the transition probability can be non zero. Thus we can rewrite the Markov transition probability as:
\[ \pi(i, j, k, l) = \begin{cases} \Phi(\pi(j), \pi(k), \pi(i)) \in D(l) & \text{if } \Phi(\pi(j), \pi(k), \pi(i)) \in D(l) \\ 0, \text{otherwise} & \end{cases} \]

The conditional probability can be computed as:
\[ \Pr[p_{i,s} \in C(k) | p_t = \pi(i), q_{i,s} = \pi(j)] = \Pr[p_t, q_{i,s}^2 \in \pi(j)] = \]
\[ \frac{c(k) - \pi(i) - \frac{1}{2} (h^i - e^{\varepsilon_{i,s}})) + \sqrt{\pi(i)^2} \pi^i < c(k+1)}{\sqrt{\pi(i)^2} \pi^i} \leq \varepsilon_{i,s} < \frac{c(k+1) - \pi(i) - \frac{1}{2} (h^i - e^{\varepsilon_{i,s}})}{\sqrt{\pi(i)^2} \pi^i} \]

Clearly these transition probabilities can be easily computed for any classical distributional assumption on the innovations. In this paper we use Gaussian distributed innovations.

Once we have computed the transition matrix \( M \) we can obtain the wealth distribution at time \( T \) considering the power of the transition matrix \( M^T \). As a matter of fact, given the state \((i, j)\) of the bivariate process (return, variance) corresponding to the k-th raw of the transition matrix, the distribution of the bivariate process at time \( T \) conditioned to start by \((i, j)\) state is given by the k-th raw of the matrix \( M^T \). Thus, to get the probability the log wealth is in the state "s" after \( T \) steps starting by \((i, j)\) state we have to sum the probabilities for the different variance states, i.e., 
\[ \pi_s(i, j, s, l) = \sum_{s=1}^n \pi_s(i, j, s, l) \]
where \( \pi_s(i, j, s, l) \) is the probability corresponding to the k-th raw of the matrix \( M^T \) to go in the state \((s, l)\) after \( T \) steps. Doing so we easily obtain the cumulative distribution of the forecasted final wealth for any described GARCH type model.
IV. PORTFOLIO VALUE DYNAMICS AS MARKOVIAN TREE PROCESS

In the portfolio selection problem we assume the initial wealth $W_0 = 1$ and all admissible wealth processes $W_t = \{ W_t(x) \}_{t=0}^T \text{ depending on an initial portfolio } x \in S$ are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

The portfolio selection problem when no short sales are allowed, can be represented as the maximization of a functional $f : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}$ applied to the random final wealth $W_T(x)$ obtained with the portfolio weights belonging to the $n$-dimensional simplex

$$S = \{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 1; x_i \geq 0 \},$$

i.e.,

$$\max_{x \in S} f(W_T(x)).$$

Typical examples are the performance type measure type functionals $f(X) = \frac{\rho_1(-X)}{\rho_2(X)}$ where $\rho_1(.)$, $\rho_2(.)$ are two positive increasing functions of coherent risk measures (see Rachev et al. (2008) and the reference therein). These functionals are isotone with the monotony order, but $\rho_1(.)$ and $\rho_2(.)$ are consistent with risk averse preferences. Thus, the functional $f(X) = \frac{\rho_1(-X)}{\rho_2(X)}$ is not isotone neither with risk lover nor with risk average preferences. We refer to Rachev et al. (2008) for further examples of the above measures. Here in the following we introduce the two performance type measure used in the choice problem: the Sharpe ratio and the Rachev ratio.

**Sharpe ratio (SR).** The Sharpe ratio (see Sharpe (1994)) serves to value the expected excess return for unity of risk (standard deviation), i.e.,

$$\text{SR}(X) = \frac{\mathbb{E}(X - r_b)}{\sigma_{X-r_b}},$$

where $r_b$ is a given benchmark and $\sigma_{X-r_b}$ is the standard deviation of the random variable $X - r_b$. When the benchmark $r_b$ is the riskfree rate and $X$ is the portfolio return, the Sharpe ratio is isotonic with non-satiable risk averse preferences.

**Rachev ratio** This performance functional is defined as

$$\text{OA-RR}_{(\omega, T)}(W_T(x)) = \frac{\text{ETL}_\omega(W_T(r_t - z_{\omega}) - 1)}{\text{ETL}_\omega(W_T(z_{\omega}) - r_t - 1)},$$

where

$$z_{\omega} = \exp(r_t \omega)$$

is the chosen portfolio gross return (i.e. $z_{\omega}^t = \exp(r_t \omega)$) and

$$W_T(z_{\omega}) - r_t - 1$$

is the final wealth at time $T$ we obtain investing in the excess return $z_{\omega}^t - r_t$.

**ETL** is the Expected Tail Loss or Average Value at Risk (AVaR) which is a coherent measure defined as

$$\text{ETL}_\omega(Y) = \frac{-1}{\alpha} \int_{\alpha F^{-1}(u)} \mathbb{P}(Y \leq t \exists u)$$

where $F^{-1}(u) = \inf \{ t \in \mathbb{R} | \mathbb{P}(Y \leq t) = u \}$ is the left inverse of the distribution function. Recall that the classic consistent estimator of expected tail loss is given by

$$\text{ETL}_\omega(Y) = \frac{-1}{\alpha T} \sum_{t=1}^{\infty} Y I_{\{Y \leq F^{-1}(\alpha)\}} + \{ 1 \text{ if } Y \leq F^{-1}(\alpha) \} \text{ otherwise}.$$

When the benchmark $r_t$ is the riskfree rate, $X$ is the portfolio return, and the numerator and the denominator are positive (negative), then the Rachev ratio is isotonic (consistent) with non-satiable preferences of investors who are neither risk averse nor risk lover (see Rachev et al. (2008)).

A. Computational Complexity and an heuristic for global optimization

Some recent studies (see Stoyanov et al. (2007), and Rachev et al. (2008)) have classified the computational complexity of reward-risk portfolio selection problems. In particular, Stoyanov et al. (2007) have shown that we can distinguish four cases of reward/risk ratios $f(1)(X)$ and $f(2)(X)$ that admit an unique optimum in myopic strategies:

1. The ratio is a quasi-concave function when the risk functional $f(2)(X)$ is convex and the reward functional $f(1)(X)$ is concave.

2. The optimal ratio problem reduces to a convex programming problem when in addition to the conditions of point 1, both functions $f(1)(X)$ and $f(2)(X)$ are positively homogeneous.

3. The optimal portfolio problem reduces to a quadratic problem if in addition to the conditions of point 2, the reward function $f(1)(X)$ is linear (or linearizable), and the risk function $f(2)(X)$ is an increasing function of a quadratic form.

4. The optimal ratio problem reduces to a linear programming problem if the reward function $f(1)(X)$ is linear and the risk function $f(2)(X)$ is linearizable.

While the maximization of the Sharpe ratio can be solved as a quadratic type problem (it enters in the third category), the Rachev ratio (that is the ratio between two convex measures) is not included in this classification and it could present more local maxima. Moreover, when we approximate the bivariate process with a Markov chain, the transition matrix change with the portfolio weights $x$. Thus, the discretization process
we adopt when we build up the approximating Markov chain implies that none of the above cases apply and the computational complexity increases. For example, Angelelli and Ortobelli (2009) have shown that non-parametric Markov portfolio models generally admit many local maxima even if it has to give a unique maximum as a consequence of the monotony of the utility functional when we adopt our discretization process.

In order to illustrate the situation we consider 2000 historical observations of three components i, j, and k of the Dow Jones Industrial index and plot the values of different performance measures by varying the portfolio composition in the 3-dimensional simplex $S = \{(x_i, x_j, x_k) | x_i + x_j + x_k = 1, x_i, x_j, x_k \geq 0\}$. In particular, we consider Rachev ratio with temporal horizon $T = 20$ days. We consider a Markov chain with 28 states for the portfolio of returns and 6 states for its variance. Figure 1 reports the value of the Rachev ratio when we consider a GARCH(1,1) process. As we could expect the problem present more local maximum.

![Figure 1. The Rachev ratio performance when we use a GARCH(1,1) model and we vary the composition of a 3 components portfolio.](image)

In order to solve this global optimization problem we implemented a local search algorithm whose required input is an objective function $f$ and an initial feasible solution $\hat{x}$ representing a portfolio from which the search is started. A current solution (portfolio) $x$ is first defined as the initial solution at hand. Then the algorithm tries to iteratively update the current solution by a better one. Improving solutions, if any, are searched on a predefined grid of points fixed on the directions $x - e_i$ for $i = 1, 2, \ldots, n$ where $x$ is the current portfolio and $e_i$ is the portfolio where the share of asset $i$ is equal to 1 and all other assets have share equal to 0. If a better solution is found on a search direction the current solution is updated and the search is continued from the new one. If no direction provides an improved solution the search ends. Actually, the search can be performed with two opposite orientations. Indeed the share of an asset $i$ can be either increased or decreased. Accordingly, the algorithm, performs the search in three distinct steps. In the first step the algorithms tries to improve the current solution by increasing the share of assets $i$ ($i = 1, \ldots, n$); in the second step the algorithm tries to improve the current solution by decreasing the share of assets $i$ ($i = 1, \ldots, n$), in the last step the algorithm the first two steps changing iteratively the ordering of the control and the distance between each portfolio. More details are provided below. The general scheme of the algorithm is defined by the following MATLAB-like pseudo code.

```matlab
function x = Optimize(f,xi)
    [x,improved] := improveByIncreasingSingleAssets(f,xi);
    Do:
    [x,improved] := improveByDecreasingSingleAssets(f,x);
    if improved
        [x,improved] := improveByIncreasingSingleAssets(f,x);
    end if
    While improved
        return x;
    end
end
```

More in details, procedure `improveByIncreasingSingleAssets` tries to improve the current solution $x$ by iteratively increasing the share of single assets in the portfolio. The basic idea is to choose an asset $i$ such that $x_i > 0$, define a finite set of alternative portfolios

$$x(i) = (1 - \beta)x + \beta e_i,$$

where parameter $\beta$ is assigned values $\beta_h = \frac{m^p}{m}$ for $h = 1, \ldots, m$. with $p \geq 1$, and $m \in \mathbb{N}$. If there is a $h$ such that $f(x(\beta_h)) > f(x)$, then the current portfolio $x$ is updated by $x(\beta_h)$ where $h = \arg\max_h\{f(x(\beta_h))\}$. The integer $m$ defines the number of points in which the objective function will be evaluated, whereas the index $p \geq 1$ defines how the points are distributed on the simplex $(1 - \beta)x + \beta e_i$. In particular, for $p = 1$ the points are equidistributed, while if $p$ gets larger the points get more concentrated around $x$. The directions in the simplex are searched according to the increasing value of $i$. Without loss of generality we can assume that $f(e_i) > f(e_j)$ for any $i < j$ so that the attempt to increase the share of the asset with highest performance is made first. If a better solution is found on a search direction $x - e_i$ for some $k$, the current solution $x$ is updated by the new solution and the search is continued on the new directions.
consider historical iid returns.

We recalibrate daily the portfolio and for the dynamic strategies we use a temporal horizon T=20 working days.

Figure 2. Ex-post final wealth process when European strategies are applied with daily recalibration and temporal horizon T=20 days.

V. AN EX-POST EMPIRICAL COMPARISON AMONG GARCH TYPE MODELS

In this section, we compare portfolio selection strategies based on the GARCH models introduced in the previous sections. We use 32 assets quoted on the US markets (NYSE and NASDAQ) from 01/02/97 till 06/14/2010 for a total of 3384 daily observations. We compare the performance of:

1) Rachev ratio under the hypothesis the log wealth follows a GARCH(1,1), a GJR-GARCH or an E-GARCH.

2) Sharpe ratio (see Sharpe 1994) under the assumption we consider historical iid returns.

We forecast the future wealth using 28 states for the portfolio of returns and 6 states for its variance. As coefficients of AVaR in the Rachev ratio we use $\alpha = \beta = 0.05$. The comparison consists in the ex post evaluation of the wealth produced by the strategies. For each strategy, we consider an initial wealth $W_0 = 1$ at the date 04/30/2009, and at the $k$th recalibration ($k = 0, 1, 2, \ldots$), three main steps are performed to compute the ex-post final wealth:

Step 1 Determine the market portfolio $x_{M,k}$ that maximizes the performance ratio $(W(x))^U$, i.e. the solution of the following optimization problem:

$$
\max_{x \in U} W(x)
$$

s.t.

$x_i \geq 0; \quad i = 1, \ldots, n$. 

As shown by Angelelli and Ortobelli (2009) this type of problems could present more local optimum then we use the heuristic developed from them to approximate the global optimum.

Step 2 Determine the ex-post final wealth given by:

$$
W_{k+1} = W_k \left( \left( x_{M,k}^{(e)} \right) z^{(e,post)} \right),
$$

where $z^{(e,post)}$ is the vector of observed gross returns between $t_k$ and $t_{k+1}$.

Step 3 The optimal portfolio $x_{M,k}^{(e)}$ is the new starting point for the $(k+1)$-th optimization problem.

Steps 1, 2 and 3 are repeated until the observations are
available and for each performance ratio. The output of this analysis is represented in Figure 2. Figure 2 reports the ex-Post wealth process using different GARCH models. In particular these results emphasize the good performance of the classic GARCH(1,1) model that in the last year present earnings of about the 100%. Instead the other GARCH models are almost never comparable to the classic one. However the comparison with static classic strategy is amazing and it suggests us that we should never use the classic strategies in portfolio choices. Thus, the empirical results show that volatility GARCH models could be very important in portfolio theory.

VI. CONCLUSION

This paper examines the impact of GARCH type return evolution in portfolio selection problems. We describe how to approximate GARCH type processes with Markov chains and we deal the portfolio selection problem under these distributional assumptions. Thus we propose algorithms that permit to solve computationally complex problems in acceptable computational times. Finally, we propose an empirical comparison among the myopic portfolio selection models and those based on the GARCH approximation. The ex-post empirical comparison among classic approaches and those based on Markovian trees shows the greater predictable capacity of the latter.

REFERENCES