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To my parents:

for their endless support,

motivation and encouragement.

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“One of the things I like about doing science, the thing that is the most fun, is coming up with something that seems ridiculous when you first hear it but finally seems obvious when you're finished.”

Fischer Black

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Abstract

Asset allocation theory and practice has been applied to many problems of institutional investors. In this dissertation, we consider the following two problems:

i) Optimal portfolio and spending rules for endowment funds.

ii) Capital adequacy management for banks in the Lévy market.

Part I: We investigate the role of different spending rules in a dynamic asset allocation model for an endowment fund. In particular, we derive the optimal portfolios under the consumption-wealth ratio rule (CW strategy) and the hybrid rule (hybrid strategy) and compare them with a theoretically optimal (Merton's) strategy for both spending and portfolio allocation. Furthermore, we show that the optimal portfolio is less risky with habit as compared with the optimal portfolio without habit. Similarly, the optimal portfolio under hybrid strategy is less risky than both CW and Merton's strategy for given set of constant parameters. Thus, endowments following hybrid spending rule use asset allocation to protect spending. Our calibrated numerical analysis on US data shows that the consumption under hybrid strategy is less volatile as compared to other strategies. However, hybrid strategy comparatively outperforms the conventional Merton's strategy and CW strategy when the market is highly volatile but under-performs them when there is a low volatility. Overall, the hybrid strategy is effective in terms of stability of spending and intergenerational equity because, even if it allows fluctuation in spending in the short run, it guarantees the convergence of spending towards its long term mean.

Part II: We investigate the capital adequacy management and asset allocation problems for a bank whose risk process follows a jump-diffusion process. Capital adequacy management problem is based on regulations in Basel III Capital Accord such as the capital adequacy ratio (CAR) which is calculated by the dividing the bank capital by total risk-weighted assets (TRWAs). Capital adequacy management requires a bank to reserve a certain amount for liquidity. We derive the optimal investment portfolio for a bank with constant absolute risk aversion (CARA) preferences and then the capital adequacy ratio process of the bank is derived, conditional on the optimal policy chosen.

Keywords: Endowment funds, spending rules, habit formation, martingale approach, capital adequacy management, HJB equation, dynamic programming, asset allocation.

Chapter 1

Introduction

Mathematical models in finance constitute deep and tremendous applications of differential equations and probability theory. Stochastic processes were introduced in finance in 1900 by Louis Bachelier in his PhD thesis "*Théorie de la Spéculation*" (Theory of Speculation), where he studied option pricing in a continuous time stochastic framework (Brownian motion). Bachelier's contribution remained unnoticed by prominent academics and the industry at that time. In 1905, Albert Einstein discovered the same equations in his mathematical theory of Brownian motion when he proposed a model of the motion of small particles suspended in a liquid. The theory of Brownian motion was further developed by some of the most eminent physicists and mathematicians of the Twentieth century in a series of papers. 1960's is considered as the beginning of modern mathematical finance when Paul Samuelson, in his two papers, explained the random fluctuation of stock prices and showed that an appropriate model for stock prices is geometric Brownian motion (GBM) which ensures that stock prices are always positive (Jarrow and Protter, 2004).

Differential equations and Brownian motion are crucial components on which historical developments in finance were established. The most significant contribution in mathematical finance is by Fischer Black, Myron Scholes, and Robert Merton who independently worked on the model of option pricing. For their outstanding work on the development of new method to determine the value of derivatives, the Royal Swedish Academy of Sciences announced the Nobel Prize in Economics in 1997. Black and Scholes [1973] derived the first quantitative model for the valuation of stock options by using the capital asset pricing model, satisfying a partial differential equation and they solved it using a combination of earlier pricing formulas and economic intuition. Merton [1973] extended the Black-Scholes theory by deducing a set of restriction and assumptions for option pricing formulas. Harrison and Kreps [1979] and Harrison and Pliska [1981] presented the solution in more abstract form as a mathematical model called martingale which provides more generality.

In modern finance, consumption and investment problems in continuous time for an investor to meet long-term financial goals and targets is viewed as a fundamental problem. Markowitz [1952] solved the problem of optimal investment in a static setting where a portfolio, consisting of riskless and risky assets, is selected with an objective to minimize variance of the portfolio. Markowitz identified that diversification of the portfolio is beneficial, as it reduces the risk by minimizing the variance of the portfolio for the same level of return. Merton [1969] and Merton [1971] studied the problem of a representative agent who aims at maximizing power utility function of both terminal wealth and intertemporal consumption in continuous time with risky assets following GBM. Since then, various investment problems of institutional and individual investors including banks, annuitants, pension funds, endowment funds, insurers, and insurance holders have been the subject of substantial research.

1.1 Objectives of the thesis

During recent years, there has been a growing interest among mathematicians and financial practitioners, in the institutional investment problems. The problem of maximizing the expected utility of wealth is the primary goal of an institutional investor. To achieve this goal, the investors are inclined to place their surplus funds in risky assets as they have a higher expected return. Nevertheless, if the investments are selected on the basis of returns only, then the risky investments may result into heavy losses. Therefore, investors have to strike a balance between the investments in risky and riskless assets. In this thesis, we consider the investment and risk management problem of endowment funds and banks. The problem of endowment funds is to allocate optimal amount for intertemporal spending and investment in risky assets whereas the problem of banks is to allocate their resources for investment in the financial market and loans, while maintaining regulatory requirements.

The study in this thesis is motivated by the subprime mortgage crisis 2007-09 which stemmed from expansion of mortgage credit. The crisis induced the most severe recession over the past decade and had a wide-ranging global effects on the financial institutions which resulted in the bankruptcy of many major financial firms including banks and insurance companies. It begun with the collapse of banks, then spread to the insurance companies and other institutional investors. In 2007 five major investment banks, Bear Stearns, Lehman Brothers, Goldman Sachs, Morgan Stanley and Merrill Lynch were operating with thin capital-leverage ratio. One of the largest and most successful insurance companies in the world at that time, American International Group (AIG), was on the brink of a collapse. These organizations failed to regulate their risks and were unable to take into account the consequences of derivatives trading on their capital structure. Furthermore, as the whole structure of many financial derivatives was based on the payments made by the debtors, when debtors started defaulting, these companies lost their worth. The crisis caused the major review of investment strategies applied by institutional investors and standards imposed by financial regulators. Regulators responded to the financial crisis by tightening existing capital ratios, introducing new capital ratios, and imposing global liquidity standards.

The main objective of the thesis is to study investment strategies applied by institutional investors subject to some rules or financial regulations. For this purpose, firstly, we study the optimal investment problem for endowment funds, under various spending rules, with an objective to achieve intergenerational equity, i.e. a reasonably smooth earnings and consequent smooth stream of spending for the current and future beneficiaries. Secondly, we study the problem of optimal investment for banks in risky assets while considering the regulatory requirements under Basel accord. In particular, we study the behavior of capital adequacy ratio conditional on the optimal portfolio chosen.

This research is based on stochastic optimal control, applied to measure and manage optimal investments. In this chapter, we introduce the problems and an outline of the thesis is provided at the end of the chapter.

1.2 Literature review

The fundamental problem of an institutional investor is to allocate assets to maximize its utility of the terminal wealth. The first major breakthrough was made in portfolio selection under uncertainty when Markowitz [1952] introduced modern portfolio theory, which proposes that the choice of a portfolio should be based on applicable future predictions established on mean-variance framework. This framework has variance as a measure of risk and expected return of the portfolio as a selection criterion. The most distinct and significant contribution of the framework was to identify that a security's contribution to the variance of the entire portfolio is more crucial than the risk of the security itself. Markowitz treated the problem of portfolio selection and risk management as a mathematical optimization problem. This approach is now prevalent for performance measurement and portfolio selection among institutional portfolio managers (Rubinstein [2002]). However, there are some limitations of mean-variance framework when it is applied in the dynamic settings as it is constructed under static settings, due to the fact that in the real world, the investors seldom make asset allocations based on single-period perspective and additionally the portfolios selected by this technique may not be diversified enough, as optimal asset allocations are highly sensitive to parameters especially the expected returns.

Another pioneering development was the introduction of optimal investment and consumption problem by Merton [1969] and Merton [1971], when the problem was solved in the continuous time using dynamic programming. The dynamic programming approach transforms Markovian stochastic optimal control problem into the problem of solving Hamilton-Jacobi-Bellman equation (HJB) which is a non-linear deterministic partial differential equation. For a discrete time framework, a similar study had been conducted before by Samuelson [1969]. The earlier paper by Merton examined the individual investor's continuous time consumption problem using stochastic optimal control where income is generated by capital gains on the investments in assets. An explicit solution for optimal consumption and investment was derived under the assumption that the asset prices were assumed to satisfy GBM and constant relative or constant absolute risk-aversion utility function. Merton [1971] extended the previous work to more general utility functions and included the income generated by non-capital gains sources and constructed a control by solving a non-linear partial differential equation. These notable works have been generalized and refined in many different ways by innumerable subsequent papers.

The main alternative method proposed for solving optimal investment problem is the martingale approach, a more direct and probabilistic method than dynamic programming and it does not need Markovian structure. Martingale approach for the complete market was implemented by Karatzas et al. [1987] to obtain the optimal consumption and wealth process explicitly. In their study, the problem is separated into the problems of maximizing utility of consumption only and maximizing utility of terminal wealth only and then these problems are properly compiled. The paper generalized the problem by allowing general utility function in finite-time horizon and arbitrary stock price fluctuations. Cox and Huang [1989] focused on explicit construction of optimal controls using a martingale technique and taken into account the non-negativity constraints on consumption and final wealth. The main advantage of this approach is that only a linear partial differential equation is required to be solved instead of nonlinear partial differential equation as in the case of dynamic programming. In dynamic programming, to establish the existence of a solution to the investment/consumption problems, two methods can be used: (i) applying the existence theorem from the stochastic control theory involving admissible controls (taking its values in a compact set), or (ii) constructing control by solving a nonlinear partial differential equation and verifying the solution by using verification theorem. The relatively more complicated case of incomplete market was explored by Karatzas et al. [1991], who determined the conditions required for the existence of optimal portfolio following non-Markov model and represented this optimal portfolio in terms of the solution to a dual optimization problem. Kramkov and Schachermayer [1999] uses the key idea of solving dual variational problem and then by convex duality finds the solution of original problem.

Optimal control theory was originally developed by engineers to study the properties of differential equations, since then it has been applied to many problems in finance. With the advancement in computer science and computational techniques, macroeconomic models based on optimal control are being developed to provide policy analysis and economic forecasts. In addition, innovation and evolution of important theoretical tools, such as stochastic dynamic programming, state feedback controllers, linear quadratic programming etc., has made the application of optimal control theory in the economic systems much more attractive. The references given above have been modified in many ways, the relevant literature about the main areas of application of optimal control in finance include investment optimization problems, pension funds, investment funds and insurance. We will briefly discuss below some of the aforementioned areas.

Firstly, we will review the literature of some notable modifications of Merton [1969, 1971]. Lehoczky et al. [1983] considered the investment consumption problem with various bankruptcy models and consumption constraint from below. Their work was later generalized by Karatzas et al. [1986] which also allowed general utility functions and general rates of return. In addition, the paper explicitly exhibits the value function and uses it to provide the conditions for the existence of optimal consumption/investment policies and finally the analysis were extended to consider more general risky investments. Sethi and Taksar [1992] extended the study further by modeling recovery from bankruptcy and showing one-to-one correspondence between the model with recovery and the model with terminal bankruptcy in Karatzas et al. [1986]. Cadenillas and Sethi [1997] further extended Karatzas et al. [1987] by considering the problem in a financial market with random coefficients and permitting general continuously differentiable concave utility functions. Karoui et al. [2005] looked into utility maximization of portfolio strategies of a fund manager with constraint applied on the terminal date (European guarantee) or on an every intermediary date (American guarantee). The investor chooses a tactical allocation using traded assets and then insures it by using a dynamic strategy called therein as strategic allocation and obtained the optimal strategic allocation satisfying the guarantee. Choulli et al. [2003] studied model of corporation facing constant liability with the objective to choose business policy and dividend distribution plan in order to maximize the expected present value of future dividends up to the time of bankruptcy. Zariphopoulou [1994] studied a problem akin to the previous one but without bankruptcy, to determine the value functions for examining their smoothness and characterization of the optimal policies. The primary tools used was theory of viscosity solutions for second-order partial differential equations and elliptic regularity from the theory of partial differential equations. Firstly, the value functions are shown to be unique constrained viscosity solutions of the associated HJB equations and then their viscosity solutions are proved to be smooth. Finally, explicit feedback form for the optimal policies is obtained.

There are several other studies on individual investment and consumption and macroeconomic problems. Yao and Zhang [2005] examined the optimal dynamic consumption, housing, and portfolio choices for an investor who acquires housing service from either renting or owning a house. Stein [2010] showed the effectiveness of application of stochastic optimal control (SOC)/dynamic risk management to determine the optimal degree of leverage, the optimum and excessive risk and the probability of a debt crisis. Stein [2011] used stochastic optimal control analysis to derive an optimal debt ratio and defined the difference between the actual and optimal debt ratio of household as an excess debt, as an indicator of Early Warning Signal (EWS) of debt crisis. Stein [2005] used stochastic optimal control to model optimum foreign debt and presented how vulnerable economies are by measuring the divergences of actual debt from an optimal one.

One of the interesting applications of stochastic optimal control for institutional investor is with the inclusion of risk control processes. For instance, Browne [1995] considered a firm with a uncontrollable stochastic cash flow or random risk process and modeled the risk by a continuous diffusion process. He focused on the criteria of minimizing the probability of ruin and maximizing the expected exponential utility of terminal wealth. Recently, insurance companies are finding it essential to apply control theory, previously it had been exclusive only to the other areas of finance. Typically, the objective of an insurance company is to maximize (or minimize) the objective function, where the insurance companies can invest in the stock market with the control variable of new investment, premium levels, reinsurance policy and dividend policy with a risk control process to model the insurance claims. Classical references in this area include, Buhlmann [1970], Dayananda [1970] and Martin-Lüf [1973]. The compound Poisson process is the most useful and popular process to describe the claims process since the classical Cramer-Lundberg model, introduced by Lundberg in 1903 and then republished by Cramer in 1930. This model can also be included in bank's asset allocation and capital adequacy management problem to model loan losses. The limiting of compound Poisson process is a diffusion process (see Taksar [2000]), hence, the subsequent research by Wang et al. [2007], Yang and Zhang [2005] and Zou and Cadenillas [2014] modeled the risk process as a jump diffusion process, providing a much better description of claims. Yang and Zhang [2005] studied optimal investment policies for an insurer with jump-diffusion risk process and obtained a closed form solution of optimal policy of exponential utility function under the assumptions that the risk process follows a compound Poisson process. They also studied general utility function and proved the verification theorem using martingale optimality principle. Wang et al. [2007] applied the martingale approach for optimal investment problem under utility maximization criterion and used jump diffusion model for the risk process. Sheng et al. [2014] investigated the optimal control strategy of excess-of-loss reinsurance (reinsurance in which reinsurer compensates the ceding company for losses that exceed a pre-specified limit) and investment problem for an insurer with compound Poisson jump diffusion risk process and analyzed the model with risky asset being priced by constant elasticity of variance (CEV) model. Moreover, they obtained a closed-form solution for HJB equation satisfying the verification theorem. Schmidli [2001] considered the insurer objective to minimize the probability of ruin and obtained optimal proportional reinsurance strategies in a classical risk model. Liu et al. [2013] also considered maximizing the expected exponential utility function for insurer with value-at-risk constraint on the portfolio and simplified the problem by using a decomposition approach both in complete and incomplete market. Zhu et al. [2015] analyzed the optimal proportional reinsurance and investment problem for default able market by decomposing the problem into the problems of a pre-default case and post-default case. The study extends the insurer's problem of optimal investment and reinsurance by considering a corporate bond and explicitly deriving optimal reinsurance and investment strategy that maximize the expected CARA utility of the terminal wealth.

Stochastic optimal control framework has also been applied to bank's optimal investment problems. Mukuddem-Petersen and Petersen [2006] considered the minimization of bank's market and capital adequacy risks and derived optimal portfolio and rate of bank's capital inflow. Pantelous [2008] examined the discrete stochastic framework for managing lending rate policy through a suitable investment strategy for loan portfolios and proposed the optimization model for banks described in a quadratic functional with control variables, stochastic inputs and a smoothness criterion. Mulaudzi et al. [2008] applied the stochastic optimization theory to generate optimal asset allocation between loans and treasury bills. They used a utility function with regret attribute alongside a risk component. Petersen et al. [2012] studied stochastic optimal problem of credit default insurance for subprime residential mortgage-backed securities and solved the credit default insurance problem with the cash outflow rate satisfying depositor obligations. Mukuddem-Petersen et al. [2010] studied an optimal securitization problem for banks that use the cash outflow rate for financing a portfolio of mortgage-backed securities where bank's investment is the control variable.

1.3 The research problems and organization of the thesis

In this thesis, we focus on the application of stochastic optimal control framework to solve optimal investment and consumption problems of institutional investors. We consider two main problems:

(i) We study the optimal asset allocation problem for endowment funds under various spending rules. We consider the spending rules practically applied by the endowments. In particular, the consumption-wealth ratio method, and the weighted average or hybrid method, which is very popular among the large endowment funds, like Yale and Stanford. We compare the optimal solutions under each rule with the classical Merton's optimal investment and consumption.

(ii) We take into account the asset allocation and capital adequacy management problem of banks with risk process and risky asset following jump-diffusion processes. We find optimal investment strategy to maximize the expected exponential utility of the bank's wealth for a finite time horizon. The stochastic optimal control problem is solved by using martingale approach.

The remainder of this thesis is organized in following way. Chapter 2 is devoted to the introduction of stochastic optimal control framework, portfolio management problems of institutional investor in the existing literature and some preliminaries. Chapter 3 is about optimal investment for endowment funds under various spending rules. Chapter 4 covers the capital adequacy management and optimal investment problem for banks.

Chapter 2

Stochastic optimal control and asset allocation problems

2.1 Introduction

Stochastic optimal control with investor preferences and assumptions on asset returns was introduced in finance by Merton in his pioneering work, since then it has become the natural formulation for asset allocation problems of institutional investor (such as, mutual funds, commercial banks, pension funds, insurance companies and manufacturing companies) and individual investors (such as households). From mathematical perspective, stochastic optimal control is suitable for optimal allocation problem with the control variables and constraints on the state variables. There are two main approaches to solve stochastic optimal control problems, a classical approach referred as dynamic programming approach and a modern approach called as martingale or duality approach. Dynamic programming was developed by R. Bellman in 1950's, based on the concept to embed the problem in a family of problems indexed by starting point in space and time and a relationships is established among these problems through Hamilton-Jacobi-Bellman equation (HJB), a second order partial differential equation. Through verification technique, we can obtain an optimal feedback control to minimize or maximize the Hamiltonian involved in the HJB equation. In this approach, the main mathematical difficulties are to show the equivalence of the optimal control problem and existence and uniqueness of the solution. The martingale method is a more direct approach, based on the martingale representation of wealth. The approach preserves the probabilistic nature of the problem. However, it is only feasible in complete market settings whereas the dynamic programming approach is also applicable in incomplete market but it requires Markovian structure. We briefly introduce the components of stochastic optimal control and general framework of asset allocation problems.

2.1.1 The components of optimal control

We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with an associated filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[t,T]}$. In general terms, a stochastic optimal control problem is formulated with the following components:

- **State variables:** The state variables provide the minimum information required to describe the problem. State variables follow a Markovian structure and can only be affected by the control variables. Typically in finance, the state process represents the wealth $R(t)$ by a stochastic differential equation. However, there can be more than one state processes, the other common state processes include interest rate, inflation etc.

- **Control processes:** The set of controls $\pi(t)$ are chosen by the optimizer to solve the optimization problem. The control variables must takes some values at each time instant t. The control variables which satisfy the constraints placed on them are called an admissible control. The set of all the feasible controls satisfying the requirements is represented by Π and it may depend on initial value of the state variables.

- **The objective function:** The objective is to either maximizes or minimizes the utility over all admissible controls.

2.1.2 General asset allocation problem

Asset allocation is the problem dealt by the investors who desire to allocate funds in an optimal manner across different assets or asset classes and current consumption. For our general settings, we define the investment/consumption problem. We consider the economic environment described by s state variables, whose values $z(t)$ solve the matrix stochastic differential equation

$$\underset{s\times 1}{dz(t)} = \underset{s\times 1}{\mu_z(t,z)}dt + \underset{s\times n}{\Omega(t,z)'}\underset{n\times 1}{dW(t)}, \tag{2.1}$$

where the prime denotes transposition and $dW(t)$ is the vector of n independent Brownian motions (normally distributed with zero mean and variance dt). We consider the following assets in the complete, arbitrage free and continuously open market:

- a riskless asset $G(t)$ which evolves according to

$$\frac{dG(t)}{G(t)} = r(t)dt, \tag{2.2}$$

where $G(t_0) = 1$, and which is the *numéraire* and $r(t)$ is the instantaneous nominal interest rate;

- n risky assets whose prices in vector $S(t)$ solve the matrix differential equation

$$\underset{n\times n}{I_S^{-1}}\underset{n\times 1}{dS(t)} = \underset{n\times 1}{\mu(t,z)}dt + \underset{n\times n}{\Sigma(t,z)'}\underset{n\times 1}{dW(t)}, \tag{2.3}$$

where I_S is a diagonal matrix containing the prices of the n assets, here we assume that $\exists\Sigma(t,z)^{-1}$ so that there exists only one vector of market prices of risk $\xi(t,z)$ which solves

$$\Sigma'(t,z)\,\xi(t,z) = \mu(t,z) - r(t,z)\,\mathbf{1},$$

where $\mathbf{1}$ is a vector of 1's. The uniqueness of $\xi(t,z)$ means that $\Sigma(t,z)$ is invert-able.

If θ_S is the vector containing the number of risky assets held in the portfolio. $\theta_G(t)$ contains the units of risk-less asset $G(t)$, then at any instant of time t the agents wealth $R(t)$ is given by

$$R(t) = \theta_G(t)G(t) + \underset{1\times n}{\theta_S(t)'}\underset{n\times 1}{S(t)}.$$

The differential of wealth can be written as

$$dR(t) = \big(R(t)r(t,z) + \theta_S(t)' I_S(\mu(t,z) - r(t,z)\mathbf{1}) - c(t)\big)dt + \theta_S(t)' I_S \Sigma(t,z)' dW(t). \tag{2.4}$$

We can also write the above equation in terms of amount of wealth invested in risky assets as follows

$$dR(t) = \big(R(t)r(t,z) + \pi_S(t)(\mu(t,z) - r(t,z)\mathbf{1}) - c(t)\big)dt + \pi_S(t)\Sigma(t,z)' dW(t), \tag{2.5}$$

where $\pi_S(t)$ is the vector of amounts invested in the risky assets.

If we consider an agent who wants to maximize his expected utility of intertemporal consumption $c(t)$ and terminal wealth $R(T)$ under subjective discount rate ρ for the time period $[t_0, T]$, and if the control variable is defined as $\pi(t) := \left\{c(t), \theta_S(t)'\right\}$, then it is reasonable to assume the following objective function

$$\max_{\pi(t)_{t\in[t_0,T]}} \mathbb{E}_{t_0}\left[\int_{t_0}^{T} U_c(c(t))\, e^{-\int_{t_0}^{t}\rho(s)ds}dt + U_R(R(T))\, e^{-\int_{t_0}^{T}\rho(s)ds}\right],$$

where U is Von Neumann-Morgenstern, twice differentiable, increasing and concave utility function. The objective is to find an optimal strategy $\pi(t)^*$ which provides expected utility as high as any other feasible strategy but may not be unique. The control variables can also be expressed in the form of the proportions of wealth by $\tilde{\pi}(t) := \{\tilde{c}(t), \tilde{\pi}_S(t)\}$, these variables may take any values in $\mathbb{R}$. The control variables must be $\mathcal{F}_t$-measurable, i.e., they can depend on the information available at time t. Some technical requirements include the consumption process $c(t)$ must be an $\mathcal{L}^1$-process, i.e. $\int_t^T \|c(s)\|\, ds < \infty$ with probability one. The portfolio strategy $\theta_S(t)$ is a progressively measurable process and satisfies $\pi_S(t)'\mu(t,z)$ is an $\mathcal{L}^1$-process and $\pi_S(t)'\Sigma(t,z)$ is $\mathcal{L}^2$-process, i.e. $\int_t^T \left\|\pi(s)'\Sigma(t,z)\right\|^2 ds < \infty$.

2.2 Dynamic programming approach

Dynamic programming requires the admissible strategies take values in the compact set and there must exist an optimal investment strategy. In this approach, we must apply verification theorem to verify that the optimal value function is optimal. The solution to the HJB equation, under some technical conditions give us the both the indirect utility function and the optimal control. The indirect utility function $J = J(t, R(t), z(t))$ at time t is

$$J(t, R(t), z(t)) \equiv \max_{\pi(s)_{s\in[t,T]}} \mathbb{E}_t \left[\int_t^T U_c(c(s)) e^{-\int_t^s \rho_u du} ds + U_R(R(T)) e^{-\int_t^T \rho_s ds} \right]. \tag{2.6}$$

Now, we can split the problem into two sub-problems for two sub-problems $[t, t+dt]$ and $[t+dt, T]$:

$$\begin{aligned} J(t, R(t), z(t)) = \max_{\pi(s)_{s\in[t,T]}} \mathbb{E}_t \Bigg[&\int_t^{t+dt} U_c(c(s)) e^{-\int_t^s \rho_u du} ds \\ &+ \int_{t+dt}^T U_c(c(s)) e^{-\int_t^{t+dt} \rho_u du} e^{-\int_{t+dt}^s \rho_u du} ds \\ &+ U_R(R(T)) e^{-\int_t^{t+dt} \rho_s ds} e^{-\int_{t+dt}^T \rho_s ds} \Bigg]. \end{aligned}$$

According to the Bellmans' principle, we assume that for the second period $[t+dt, T]$, the problem is already optimized, therefore we can write the optimization problem for the first sub-period plus the optimized value function for the second period.

$$\begin{aligned} J(t, R(t), z(t)) = \max_{\pi(s)_{s\in[t,t+dt]}} \mathbb{E}_t \Bigg[&\int_t^{t+dt} U_c(c(s)) e^{-\int_t^s \rho_u du} ds \\ &+ e^{-\int_t^{t+dt} \rho_u du} J(t+dt, R(t+dt), z(t+dt)) \Bigg]. \end{aligned}$$

If we subtract $J(t, R(t), z(t))$ from both sides and divide by dt, and take the limit $dt \to 0$, we get

$$0 = \max_{\pi(t)} \Big[U_c(c\,(t)) - \rho_t J(t, R\,(t)) + \frac{1}{dt}\mathbb{E}_t[dJ(t, R\,(t)\,, z\,(t))]\Big].$$

In order to simplify the notation we use $J = J(t, R\,(t)\,, z\,(t))$, we compute $dJ(t, R\,(t)\,, z\,(t))$ by using Ito's lemma and after taking the expected value the above equation becomes

$$0 = \max_{\pi(t)} \left\{ \begin{array}{c} U_c(c\,(t)) - \rho_t J + \frac{\partial J}{\partial t} + \frac{\partial J}{\partial R(t)}\big(R(t) r(t,z) + \pi_S\,(t)\,(\mu(t,z) - r(t,z)\mathbf{1}) - c(t)\big) \\ +\mu_z' \frac{\partial J}{\partial z(t)} + \frac{1}{2}\frac{\partial^2 J}{\partial R(t)} \pi_S\,(t)^2\, \Sigma' \Sigma + \frac{1}{2} \mathrm{tr}\left(\Omega' \Omega \frac{\partial^2 J}{\partial z(t)' \partial z(t)}\right) \\ +\pi_S\,(t)\, \Sigma' \Omega \frac{\partial^2 J}{\partial z(t) \partial R(t)} \end{array} \right\} \tag{2.7}$$

The above equation is called Hamilton Jacobi-Bellman equation (HJB). As we can see, it is a highly non-linear second order partial differential equation.

2.3 Optimal investment with habit formation

We take the dynamic asset allocation problem for an investor with habit formation in the preferences and time-varying investment opportunities in a complete financial market. The investor is allowed to invest in n risky assets and a risk less asset as given in the Sub-Section 2.1.2, thus the wealth process of the investor is given by

$$dR(t) = \big(R(t) r(t,z) + \theta_S(t)' I_S(\mu(t,z) - r(t,z)\mathbf{1}) - c(t)\big)dt + \theta_S(t)' I_S \Sigma(t,z)' dW(t).$$

We consider a price-taking investor with the fixed time horizon $[t_0, T]$. The optimization problem can be written as

$$\max_{\pi(t)} \mathbb{E}_{t_0} \left[\int_{t_0}^{T} e^{-\rho(s-t_0)} U_c\,(c\,(t)\,, h\,(t))\, ds + U_R\,(R\,(T))\, e^{-\rho(T-t_0)} \right],$$

where $\rho \geq 0$ is a subjective rate of time preference, and $h\,(t)$ is the habit level defined by

$$h(t) = h_0 e^{-\int_0^t \beta(u)du} + \int_0^t \alpha(s) c(s)\, e^{-\int_s^t \beta(u)du} ds,$$

where h_0 is the initial minimum amount of outflow, $\alpha(t)$ is the weighting function providing the relative importance to the past outflow in computing the threshold $h(t)$, while $\beta(t)$ is a discount rate. $h(t)$ can be written as

$$dh(t) = (\alpha(t)\, c(t) - \beta(t)\, h(t))\, dt.$$

We assume that the utility functions belong to Hyperbolic Absolute Risk Aversion (HARA) family and can be written as

$$U_c(c(t), h(t)) = \frac{(c(t) - h(t))^{1-\delta}}{1-\delta}, \quad U_R(R(T)) = \frac{(R(T) - R_m)^{1-\delta}}{1-\delta}, \tag{2.8}$$

where $c(t)$ is the instantaneous outflow or spending from the fund, the constant R_m can be interpreted as the minimum subsistence level of wealth. Thus the investor problem can be written as

$$\max_{c(t),\theta_S(t)} \mathbb{E}_{t_0} \left[\int_{t_0}^{T} \phi_C \frac{(c(s) - h(s))^{1-\delta}}{1-\delta} e^{-\rho(s-t_0)} ds + \phi_R \frac{(R(T) - R_m)^{1-\delta}}{1-\delta} e^{-\rho(T-t_0)} \right]. \tag{2.9}$$

Proposition 1. *Given the state variables wealth $R(t)$ and $z(t)$ described in (2.4) and (2.1) respectively, the optimal outflow and portfolio solving problem (2.9) are*

$$c^*(t) = h(t) + \phi_c^{\frac{1}{\delta}} \frac{(R(t) - h(t)B(t))(1 + B(t)\alpha(t))^{-\frac{1}{\delta}}}{A(t, z(t))}, \tag{2.10}$$

$$\begin{aligned} I_S\theta_S(t)^* = {} & \frac{R(t) - h(t)B(t, z(t))}{\delta} \Sigma(t, z)^{-1}\xi + h(t)\Sigma(t, z)^{-1}\Omega \frac{\partial B(t, z(t))}{\partial z} \\ & + \frac{R(t) - h(t)B(t, z(t))}{A(t, z(t))} \Sigma(t, z)^{-1}\Omega(t, z) \frac{\partial A(t, z(t))}{\partial z}, \end{aligned} \tag{2.11}$$

where

$$A(t,z(t)) = \mathbb{E}_t^{\mathbb{Q}_\delta}\left[\int_t^T \phi_c^{\frac{1}{\delta}}(1+B(s,z(s))\alpha(s))^{1-\frac{1}{\delta}} e^{-\frac{\delta-1}{\delta}\int_t^s\left(r(t,z)+\frac{\rho}{\delta-1}+\frac{1}{2\delta}\xi'\xi\right)du} ds \right.$$
$$\left. +\phi_R^{\frac{1}{\delta}} e^{-\frac{\delta-1}{\delta}\int_t^s\left(r(t,z)+\frac{\rho}{\delta-1}+\frac{1}{2\delta}\xi'\xi\right)du}\right], \tag{2.12}$$

$$B(t,z(t)) = \mathbb{E}_t^{\mathbb{Q}}\left[R_m e^{-\int_t^T(-\alpha(u)+\beta(u)+r(u,z))du} + \int_t^T e^{-\int_t^s(-\alpha(u)+\beta(u)+r(u,z))du} ds\right], \tag{2.13}$$

with z solving

$$dz(t) = \left(\mu_z(t,z) - \frac{\delta-1}{\delta}\Omega(t,z)'\xi\right)dt + \Omega(t,z)' dW(t)^{\mathbb{Q}_\delta},$$

and

$$dW(t)^{\mathbb{Q}_\delta} = \frac{\delta-1}{\delta}\xi dt + dW(t). \tag{2.14}$$

The Wiener process under $\mathbb{Q}_\delta$ can be represented as a weighted mean of the Wiener processes under the risk neutral and the historical probabilities (the weight is given by the inverse of δ):

$$dW^{\mathbb{Q}_\delta}(t) = \left(1-\frac{1}{\delta}\right) dW^{\mathbb{Q}}(t) + \frac{1}{\delta} dW(t).$$

Proof. See Appendix 2.A.1 □

The optimal portfolio (2.11) is a combination of three components:

- standard mean-variance portfolio, where the disposable wealth is invested in the risky assets proportionally to the risk aversion index δ and to the ratio between the diffusion matrix and market price of risk $(\Sigma(t,z)^{-1}\xi)$.
- a hedge portfolio which insures against the changes in the investment opportunities and the future cost of the consumption at the habit level,
- a component which ensures that an agent can consume minimum at the habit level.

2.3.1 Special cases

In this section, we present some of the special cases of habit formation considered in the literature.

2.3.1.1 Subsistence consumption

The main reference for this problem is Bajeux-Besnainou and Ogunc [2006]. Consider the problem of endowment funds with a self financing strategy, that can invest in the standard Black and Scholes market i.e. with two assets: a riskless assets $G(t)$ and a risky asset $S(t)$, then the wealth of the fund can be written as

$$\frac{dR(t)}{R(t)} = \big(r + \tilde{\pi}_S(t)(\mu - r) - c(t)\big)dt + \tilde{\pi}_S\,(t)\,\sigma dW(t), \tag{2.15}$$

where $\tilde{\pi}_S\,(t)$ is the proportion of wealth allocated for stock index and r is the constant risk less interest rate.

From Harrison and Kreps [1979] and Harrison and Pliska [1981], under the complete market the existence and uniqueness of a risk-neutral probability $\mathbb{Q}$ is characterized by Radon-Nikodym derivative, $Z(t) = \frac{d\mathbb{Q}}{d\mathbb{P}}$ satisfying:

$$\frac{dZ\,(t)}{Z\,(t)} = -\xi dW\,(t)\,,$$

where ξ is the instantaneous price of risk: $\xi = \frac{\mu - r}{\sigma}$ and where $Z\,(t_0) = 1$ for normalization.

The problem is to maximize the intertemporal utility of consumption/spending where the minimum amount for consumption needs to be made at every time period, called as minimum subsistence level consumption. The subsistence level is the minimum spending level given by:

$$\underline{c}(t) = \alpha R(0)e^{\lambda t},$$

where the constant inflation rate λ is always less than the risk-free rate $r(t)$. $0 < \alpha < 1$ represents the spending rate based on inflation-adjusted initial endowment's wealth $R(t_0)$.

The endowment fund preferences are represented by Hyperbolic Absolute Risk Aversion (HARA) utility function with a minimum subsistence level that increases with inflation rate. Similar to the habit formation structure, the utility function with subsistence level $\underline{c}(t)$ which is a deterministic function of time is given by

$$U_c(c(t)) = \frac{1}{1-\gamma}(c(t) - \underline{c}(t))^{1-\gamma},$$

where $\gamma > 0$ indicates the decreasing absolute risk aversion.

The self financing constraint 2.15 can be re-written under the assumption of complete markets. (see Cox and Huang [1989] and Karatzas et al. [1987] for details). The optimization problem can be written as

$$\max_{c(t)} \mathbb{E}_{t_0}\left[\int_{t_0}^{+\infty} e^{-\rho t} U_c(c(t))\, dt\right],$$

under the constraint

$$R(t_0) = \mathbb{E}_{t_0}\left[\int_{t_0}^{+\infty} e^{-\rho t} U_c(c(t))\, dt\right],$$

where $R(t_0)$ is the initial endowment wealth, $\rho > 0$ is a constant subjective discount rate and the spending stream $c(t)$ is the control variable.

2.3.1.2 Real subsistence consumption

Consider the problem of dynamic asset allocation with minimum subsistence level consumption in the presence of inflation-protected securities. These securities are used as a hedging tool against the inflation risk. It is an extension of minimum subsistence level consumption case considered above. In addition to the Standard Black and Scholes model, the nominal bond is added in the portfolio. The rate of inflation is stochastic. The price level $P(t)$ satisfies

$$\frac{dP(t)}{P(t)} = \lambda dt + \sigma_\lambda dW(t),$$

where $W(t)$ is a two-dimensional Brownian motion. λ is the locally expected inflation rate in the economy and σ_λ is the instantaneous inflation volatility vector.

There are two markets for trading bonds, one for nominal bonds $G_N(t)$ and other is for real bonds $G(t)$, which satisfy

$$\frac{dG_N(t)P(t)}{G_N(t)P(t)} = r_N dt \quad , \frac{dG(t)}{G(t)} = rdt,$$

The objective of the agent is to maximize the utility of consumption for infinite time horizon. The value function for the problem can be written as

$$J(t, R(t)) \equiv \max_{c(t),\pi_S(t),\pi_N(t)} \mathbb{E}\left[\int_{t_0}^{+\infty} e^{-\rho t} U_c(c(t))\, dt\right],$$

subject to the wealth constraint

$$\begin{aligned} dR(t) &= (\pi_S(t)(\mu - r) + \pi_N(t)(r_N - \lambda - r) + rR(t) - c(t))\, dt \\ &\quad + (\pi_S(t)\sigma - \pi_N(t)\sigma_\lambda)\, dW(t), \end{aligned}$$

and

$$c(t) \geq \underline{c}(t), \quad R(t_0) = R_0,$$

where $\pi_S(t)$ is the wealth invested in the stock, $\underline{c}(t)$ is the lower bound imposed on consumption and $\pi_N(t)$ is the wealth invested in the nominal bonds.

Since there is a lower bound imposed on consumption there exist a lower bound on wealth $\underline{R}(t)$ such that the agent's optimization is feasible and $J(t, \underline{R}(t)) = \frac{U_c(\underline{c}(t))}{\rho}$. The lower bound on wealth is given by

$$\underline{R}(t) = \frac{\underline{c}(t)}{r}.$$

The study shows that there exists an optimal consumption plan and trading strategy which converges to the optimal policy for the "benchmark" (Merton's standard problem). The main reference for this extension is Gong and Li [2006].

2.4 Optimal investment with jump-diffusion risk process

Consider the wealth of an investor can be invested in a security market described by the standard Black–Scholes model. i.e. with two assets: a riskless assets $G(t)$ and a risky asset $S(t)$ given by:

$$\frac{dG(t)}{G(t)} = r(t)dt,$$

and

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW_1(t).$$

The risk of institutional investor is captured by using Cramer-Lundberg model, a classical model used in the insurance industry to model claims. The risk process of an investor is modeled by a jump-diffusion process

$$dR\,(t) = adt + bd\bar{W}\,(t) + \gamma dN(t), \tag{2.16}$$

where a, b are positive constants and $\bar{W}(t)$ is a one-dimensional standard Brownian motion.

The cumulative amount of losses is given by $\sum_{i=1}^{N(t)} l_i$, where $\{l_i\}$ is a series of independent and identically distributed (i.i.d) random variables and $N\,(t)$ is a homogeneous Poisson process with intensity λ and independent of l_i. If the mean of l_i and intensity of $N\,(t)$ are finite, then this compound Poisson process is a Lévy process with finite Lévy measure.

As the capital gains in the financial market are negatively correlated with the liabilities, we denote ρ as the correlation coefficient between $W_1(t)$ and $\bar{W}(t)$, which implies

$$\bar{W}(t) = \rho W_1(t) + \sqrt{1-\rho^2} W_2(t), \tag{2.17}$$

where $W_2(t)$ is another standard Brownian motion independent of $W_1(t)$. There are three special cases for (4.6): (i) if $\bar{W}(t)$ is not correlated with $W_1(t)$ that is, $\rho = 0$ then $\bar{W}(t)$ is equal to $W_2(t)$, (ii) if $\rho^2 = 1$, then $\bar{W}(t)$ equals $W_1(t)$. In this case, the risky assets and the liabilities are driven by the same source of randomness, and (iii) if $\rho^2 \leq 1$ it means that the risk from liabilities cannot be eliminated by trading the financial assets. After substituting (4.6) into (4.5) we can write

$$dR\left(t\right) = adt + b\rho dW_1(t) + b\sqrt{1-\rho^2}dW_2(t) + \gamma dN(t), \tag{2.18}$$

In addition, $W_1(t)$, $W_2(t)$, and $N(t)$ are mutually independent and are all defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ where $\mathcal{F}_t$ is the usual augmentation of natural filtration with $\mathcal{F} = \mathcal{F}_T$. If we set $\pi_G \equiv \theta_G(t)G(t)$, $\pi_S \equiv \theta_S(t)S(t)$, and define the investor's strategy by $\pi(t) := \pi_S(t)$, a $\mathcal{F}_t$ − predictable process, which represents the dollar amounts invested in stock index fund $\pi_S(t)$, then we can write the investor's assets portfolio $X(t)$ as a controlled stochastic process depending on a strategy $\pi\left(t\right)$ as

$$\begin{aligned} dX^{\pi}(t) &= \left(X^{\pi}(t)r(t) + \pi_S(t)\left(\mu_S - r(t)\right) - a\right)dt + \left(\pi_S(t)\sigma_1 - b\rho\right)dW_1(t) \\ &\quad - b\sqrt{1-\rho^2}dW_2(t) - \gamma dN(t). \end{aligned}$$

Suppose that the investor's has a utility function $U(X(T))$ of the terminal wealth $X\left(T\right)$, then the aim of the investor is to

$$\max_{\pi(t)} \mathbb{E}\left[U\left(X\left(T\right)\right)\right],$$

where $\mathbb{E}$ is the conditional expectation under probability measure $\mathbb{P}$ and the utility function U is assumed to be strictly increasing and concave with respect to the wealth. Π denotes the set of all admissible controls with initial asset portfolio $X(0) = x_0$.

The main reference for this problem is Wang et al. [2007].

2.4.1 Bank management problem

The assets that a bank can invest in consist of a bond, stock index and loans. On the financial market two assets are listed:

- a riskless asset $G(t)$ which evolves according to

$$\frac{dG(t)}{G(t)} = r(t)dt, \tag{2.19}$$

  where $r\left(t\right)$is the short rate process which evolves according to the following stochastic differential equation:

$$dr\left(t\right) = \left(a - br\left(t\right)\right)dt - \sigma_r\sqrt{r\left(t\right)}dW_r\left(t\right),$$

  where the coefficients a,b, $r\left(0\right)$ and σ_r are strictly positive constants and $2a \geq \sigma_r^2$, so that $\mathbb{P}\left\{r\left(t\right) > 0 \forall t \in [0,T]\right\} = 1$.

- a security whose value is denoted by $S\left(t\right), t \geq 0$. The dynamics of $S\left(t\right)$is given by

$$\frac{dS\left(t\right)}{S\left(t\right)} = r\left(t\right)dt + \sigma_S\left(\xi_S + dW\left(t\right)\right)dt + \sigma_r\sqrt{r\left(t\right)}\left(\xi_r\sqrt{r\left(t\right)}dt + dW_r\left(t\right)\right),$$

  where $S\left(t_0\right) = 1$ and ξ_S, ξ_r, σ_S and σ_r are positive constants.

- a loan $L\left(t\right)$ amortized over a period $[0,T]$, whose dynamic are

$$\frac{dL\left(t\right)}{L\left(t\right)} = r\left(t\right)dt + \sigma_L\left(\xi_r\sqrt{r\left(t\right)}dt + dW_r\left(t\right)\right),$$

The optimization problem can be written as

$$\max_{\pi(t)} \mathbb{E}\left[U\left(R\left(t\right)\right)\right],$$

Subject to bank's asset portfolio

$$\frac{dR\left(t\right)}{R\left(t\right)} = r(t)dt + \pi_S\left(t\right)\sigma_S\left(\xi_S + dW\left(t\right)\right)dt + \pi_S\left(t\right)\sigma_r\sqrt{r\left(t\right)}\left(\xi_r\sqrt{r\left(t\right)}dt + dW_r\left(t\right)\right)$$
$$+ \pi_L\left(t\right)\sigma_L\left(\xi_r\sqrt{r\left(t\right)}dt + dW_r\left(t\right)\right),$$

with strictly positive initial asset portfolio value $R\left(t_0\right)$. The main reference for this problem is Witbooi et al. [2011].

2.5 Some preliminaries

2.5.1 Jump diffusion theorems

Theorem 1. *(Ito-Lévy Decomposition [JS]). Let η_t be a Lévy process. Then η_t has the decomposition*

$$\eta_t = \alpha t + \sigma W(t) + \int_{|z|<R} z\widetilde{N}(t, dz) + \int_{|z|\geq R} zN(t, dz), \tag{2.20}$$

for some constants $\alpha \in \mathbb{R}$, $\sigma \in \mathbb{R}$, $R \in [0, \infty]$. Here

$$\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt,$$

is the compensated Poisson random measure of $\eta(.)$, and $W(t)$ is a Brownian motion independent of $\widetilde{N}(dt, dz)$. For each $A \in B_0$ the process

$$M_t := \widetilde{N}(t, A),$$

is a martingale. If $\alpha = 0$ and $R = \infty$, we call η_t a Levy martingale Øksendal and Sulem [2005]. (Theorem 1.7).

Theorem 2. *Existence and Uniqueness of solutions of Lévy SDE's). Consider the following Lévy SDE in $\mathbb{R}^n$: $X(0) = x_0 \in \mathbb{R}^n$ and,*

$$dX(t) = \alpha(t, X(t))dt + \sigma(t, X(t))dW(t) + \int_{\mathbb{R}^n} \gamma(t, X(t^-), z)\widetilde{N}(dt, dz),$$

where $\alpha : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ *and* $\gamma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$ *satisfy the following conditions*

(At most linear growth) There exists a constant $C_1 < \infty$ *such that*

$$\|\sigma(t,x)\|^2 + |\alpha(t,x)|^2 + \int_{\mathbb{R}} \sum_{k=1}^{l} |\gamma_k(t,x,z)|^2 \nu_k(dz_k) \leq C_1 \left(1 + |x|^2\right),$$

for all $x, y \in \mathbb{R}^n$.

(Lipschitz continuity) There exists a constant $C_2 < \infty$ *such that*

$$\begin{aligned} &\|\sigma(t,x) - \sigma(t,y)\|^2 + |\alpha(t,x) - \alpha(t,y)|^2 \\ &+ \sum_{k=1}^{l} \int_{\mathbb{R}} \left|\gamma^{(k)}(t,x,z_k) - \gamma^{(k)}(t,y,z_k)\right|^2 \nu_k(dz_k) \leq C_2 |x-y|^2, \end{aligned}$$

for all $x, y \in \mathbb{R}^n$.

Then there exists a unique cadlag adapted solution $X(t)$ *such that*

$$\mathbb{E}\left[|X(t)|^2\right] < \infty \textit{for all } t.$$

Solutions of Lévy SDE's in the time homogeneous case, i.e. when $\alpha(t,x) = \alpha(x)$, $\sigma(t,x) = \sigma(x)$, *and* $\gamma(t,x,z) = \gamma(x,z)$ *are called jump diffusion (or Lévy diffusion). Øksendal and Sulem [2005] (Theorem 1.19).*

2.5.2 Burkholder-Davis-Gundy inequalities

If M is a continuous local martingale, we write $M(t)^* = sup_{s\leq t} |M_s|$.

Theorem 3. *For every* $p \in]0,\infty[$, *there exist two constants* c_p *and* C_p *such that, for all continuous local martingales* M *vanishing at zero,*

$$c_p \mathbb{E}\left[\langle M, M\rangle_\infty^{p/2}\right] \leq \mathbb{E}\left[(M_\infty^*)^p\right] \leq C_p \mathbb{E}\left[\langle M, M\rangle_\infty^{p/2}\right].$$

For proof, see Revuz and Yor [1991].

2.5.3 Doléans-Dade exponential

Definition 1. For any Wiener process $W(t)$ and any kernel process φ, the Doléans exponential process ε is defined by

$$\varepsilon\left(\varphi \star W\right)(t) = \exp\left\{\int_0^t \varphi^\star\left(s\right) dW\left(s\right) - \frac{1}{2}\int_0^t \left\|\varphi\right\|^2\left(s\right) ds\right\},$$

Source: Bjork [2009].

2.A Appendix A

2.A.1 Proof of proposition 1

The value function is given by

$$J(t, R(t), z(t)) \equiv \max_{c(t),\theta_S(t)} \mathbb{E}_t\left[\int_t^T \phi_c \frac{(c(s)-h(s))^{1-\delta}}{1-\delta} e^{-\rho(s-t)} ds + \phi_R \frac{(R(T)-R_m)^{1-\delta}}{1-\delta} e^{-\rho(T-t)}\right].$$

For this objective function, we can write the following HJB equation

$$0 = \max_{c(t),\theta_S(t)} \left\{ \begin{array}{c} \phi_c \frac{(c(s)-h(s))^{1-\delta}}{1-\delta} - \rho J + \frac{\partial J}{\partial t} + \frac{\partial J}{\partial h}(\alpha(t)c(t) - \beta(t)h(t)) \\ + \frac{\partial J}{\partial R}\big(R(t)r(t,z) + \theta_S(t)' I_S(\mu(t,z) - r(t,z)\mathbf{1}) - c(t)\big) + \left(\frac{\partial J}{\partial z}\right)' \mu_z(t,z) \\ + \frac{1}{2}\frac{\partial^2 J}{\partial R^2}\theta_S(t)' I_S \Sigma(t,z)' \Sigma(t,z) I_S \theta_S(t) \\ + \frac{1}{2} tr\left(\Omega(t,z)'\Omega(t,z)\frac{\partial^2 J}{\partial z' \partial z}\right) + \theta_S(t)' I_S \Sigma(t,z)' \Omega(t,z) \frac{\partial^2 J}{\partial z \partial R} \end{array} \right\}, \tag{2.21}$$

and the first order conditions (FOCs) of (2.21) w.r.t. $\theta_S(t)$ and $c(t)$ are

$$\phi_c (c(s) - h(s))^{-\delta} = \frac{\partial J}{\partial R} - \frac{\partial J}{\partial h}\alpha(t), \tag{2.22}$$

$$I_S\theta_S(t)^* = -\left(\Sigma(t,z)'\Sigma(t,z)\right)^{-1}\frac{\frac{\partial J}{\partial R}}{\frac{\partial^2 J}{\partial R^2}}\left(\mu(t,z)-r(t,z)\mathbf{1}\right)$$
$$-\left(\Sigma(t,z)'\Sigma(t,z)\right)^{-1}\frac{\frac{\partial^2 J}{\partial z\partial R}}{\frac{\partial^2 J}{\partial R^2}}\Sigma(t,z)'\Omega(t,z). \quad (2.23)$$

We assume the following guess function

$$J(t,R(t),z(t)) = A(t,z(t))^{\delta}\frac{(R(t)-h(t)B(t,z(t)))^{1-\delta}}{1-\delta}, \quad (2.24)$$

with the boundary conditions

$$A(T,z(T)) = \phi_R^{\frac{1}{\delta}},$$

$$B(T,z(T)) = R_m.$$

If we substitute the partial derivatives into both the optimal consumption (2.22) and the optimal portfolio (2.23), we obtain

$$c^*(t) = h(t) + \phi_c^{\frac{1}{\delta}}\frac{(R(t)-h(t)B(t,z(t)))(1+B(t,z(t))\alpha(t))^{-\frac{1}{\delta}}}{A(t,z(t))}, \quad (2.25)$$

$$I_S\theta_S(t)^* = \frac{R(t)-h(t)B(t,z(t))}{\delta}\Sigma(t,z)^{-1}\xi + \frac{R(t)-h(t)B(t,z(t))}{A(t,z(t))}\Sigma(t,z)^{-1}\Omega(t,z)A_z$$
$$+ h(t)\Sigma(t,z)^{-1}\Omega B_z. \quad (2.26)$$

Substituting the partial derivatives of the guess function and the optimal portfolios (2.25) and consumption (2.26) into the HJB equation (2.21), we have

$$
\begin{aligned}
0 =& \frac{\delta}{1-\delta}\phi_c^{\frac{1}{\delta}}(1+B(t,z(t))\alpha(t))^{1-\frac{1}{\delta}} - A(t,z(t))\frac{\rho}{1-\delta} + \frac{\delta}{1-\delta}A_t \\
&- A(t,z(t))\frac{h(t)B_t}{R(t)-h(t)B(t,z(t))} - A(t,z(t))\frac{B(t,z(t))\alpha(t)h(t)}{R(t)-h(t)B(t,z(t))} \\
&+ \frac{A(t,z(t))}{R(t)-h(t)B(t,z(t))}B(t,z(t))\beta(t)h(t) + \frac{\delta}{1-\delta}\mu_z(t,z)'A_z \\
&+ A(t,z(t))r(t,z) + \frac{A(t,z(t))h(t)B(t,z(t))}{R(t)-h(t)B(t,z(t))}r(t,z) - \frac{A(t,z(t))}{R(t)-h(t)B(t,z(t))}h(t) \\
&+ \frac{1}{2}\frac{\delta}{1-\delta}tr\left(\Omega(t,z)'\Omega(t,z)A_{zz}\right) + \frac{A(t,z(t))\xi'\xi}{2\delta} + \xi'\Omega(t,z)A_z \\
&- \frac{A(t,z(t))h(t)}{R(t)-h(t)B(t,z(t))}B_z\left(\mu_z(t,z)' - \xi'\Omega(t,z)\right) \\
&- \frac{1}{2}\frac{A(t,z(t))h(t)}{R(t)-h(t)B(t,z(t))}tr\left(\Omega(t,z)'\Omega(t,z)B_{zz}\right),
\end{aligned}
$$

which can be separated into two differential equations, one that contains $(R(t)-h(t)B(t,z(t)))^{-1}$ and one without it and after few simplifications, we have

$$
\begin{cases}
0 & = A_t - \frac{\delta-1}{\delta}A(t,z(t))\varphi(t,z) + \frac{1}{2}tr\left(\Omega(t,z)'\Omega(t,z)A_{zz}\right) + \psi(t,z)A_z \\
& +\phi_c^{\frac{1}{\delta}}(1+B(t,z(t))\alpha(t))^{1-\frac{1}{\delta}}, \\
0 & = B_t + B(t,z(t))\left(\alpha(t)-\beta(t)-r(t,z)\right) + B_z\left(\mu_z(t,z)' - \xi'\Omega(t,z)\right) \\
& +\frac{1}{2}tr\left(\Omega(t,z)'\Omega(t,z)B_{zz}\right) + 1,
\end{cases}
\tag{2.27}
$$

where

$$
\varphi(t,z) \equiv r(t,z) + \frac{\rho}{\delta-1} + \frac{1}{2\delta}\xi'\xi,
$$

$$
\psi(t,z) \equiv \mu_z(t,z)' - \frac{\delta-1}{\delta}\xi'\Omega(t,z).
$$

We can represent the solution of $A(t,z(t))$ through the Feynman-Kač formula which is based on a modified SDE for the state variable $z(t)$,

$$dz(t) = \Big(\mu_z(t,z) - \frac{\delta-1}{\delta}\Omega(t,z)'\xi\Big)dt + \Omega(t,z)'dW(t)^{\mathbb{Q}_\delta},$$

$$dz(t) = \mu_z(t,z)dt + \Omega(t,z)'\Big(dW(t)^{\mathbb{Q}_\delta} - \frac{\delta-1}{\delta}\xi dt\Big),$$

where, by using the following version of Girsanov's theorem, we define a new probability measure

$$dW(t)^{\mathbb{Q}_\delta} = \frac{\delta-1}{\delta}\xi dt + dW(t). \tag{2.28}$$

The differential equations (2.27) together with their corresponding boundary conditions, have the following solutions

$$\begin{aligned} A(t,z(t)) &= \mathbb{E}_t^{\mathbb{Q}_\delta}\Bigg[\int_t^T \phi_C^{\frac{1}{\delta}}(1+B(s,z(s))\alpha(s))^{1-\frac{1}{\delta}} e^{-\frac{\delta-1}{\delta}\int_t^s\left(r(t,z)+\frac{\rho}{\delta-1}+\frac{1}{2\delta}\xi'\xi\right)du}ds \\ &\quad + \phi_R^{\frac{1}{\delta}} e^{-\frac{\delta-1}{\delta}\int_t^s\left(r(t,z)+\frac{\rho}{\delta-1}+\frac{1}{2\delta}\xi'\xi\right)du}\Bigg], \\ B(t,z(t)) &= \mathbb{E}_t^{\mathbb{Q}}\Bigg[R_m e^{-\int_t^T(-\alpha(u)+\beta(u)+r(u,z))du} + \int_t^T e^{-\int_t^s(-\alpha(u)+\beta(u)+r(u,z))du}ds\Bigg], \end{aligned}$$

where $\mathbb{Q}_\delta$ is a new probability defined in (2.28).

Chapter 3

Optimal portfolio and spending rules for endowment funds

3.1 Introduction

University endowments rank among the largest institutional investors. According to the 2016 estimates of the National Association of College and University Business Officers (NACUBO), its member organizations held $515 billion in endowment assets. Usually, these institutions setup an endowment fund to achieve their objective of a reasonably smooth earnings and consequent smooth stream of spending for the current and future beneficiaries (also called intergenerational equity), as they have to manage funds for a very long (possibly infinite) time horizon. These institutions may also have income other than the endowment fund, so therefore the principal goal of an endowment manager is to stabilize earnings rather than growing its value. Due to this reason, usually, endowment funds tend to follow a sub-optimal investment strategy and invest more in less risky assets which generally have lower returns and consequently, they have lower funds for spending. Common objective of endowment funds is to maximize the utility of both intertemporal spending and terminal wealth. To achieve this objective, many endowment funds have pre-defined spending rules. This chapter analyses the effect of different spending rules on the portfolio choices. Specifically, we determine the optimal investment policies under different spending rules and compare them with the classical Merton's optimal investment and consumption. Thus, essentially in our framework, the consumption (spending) is not optimally chosen, but it is instead treated like a state variable.

The existing literature about endowment fund can be classified into four main areas: (i) the organization and governance of endowment funds, (ii) the asset allocation, (iii) the university endowment performance, and (iv) the spending policies. Our research is related to the areas of asset allocation and spending policies.

The seminal work on asset allocation in continuous time for an investor dates back to Samuelson [1969] and Merton [1969], who presented an optimal strategy for a market with constant investment opportunities with additive time-separable utility function. This preliminary work was later extended by Merton [1971], to a more general utility function which included the income generated by non-capital gains sources. Merton [1993], applied continuous time framework to the endowment fund's problem and derived optimal expenditures and asset allocation that included non-endowed funds as a part of the total university's wealth. He concluded that endowment fund prefers a safer portfolio in the presence of non-financial income risk.

Spending rules are important feature of a university endowment funds. Hansmann [1990] addresses the reason why endowments need some mechanism for spending rather than spending all the gifts at once and concluded that endowments has several purposes: (i) ensure the support of the parent institution in its ongoing mission, (ii) protect its reputation and intellectual freedom, and (iii) hedge against financial shocks. Litvack et al. [1974] analyzes the definition of endowment income and points out that the main objective of the endowment funds is to provide a reasonably stable income over time, so that the fund can finance a sustainable spending stream and hence concludes that the endowment fund should maximize the total rate of return of its investments while preserving the corpus of the fund. Tobin [1974] recognizes that endowment trustees may want to stabilize overall university income, as a stable income entails both a sustainable consumption and intergenerational equity. Ennis and Williamson [1976] present different spending rules adopted by the endowment funds along with their historical spending patterns. Kaufman and Woglom [2005] analyze the spending rules based on the inflation method, banded inflation, and hybrid method, using Monte Carlo simulations in scenario of volatile and uncertain asset returns. Sedlacek and Jarvis [2010] also present an analysis of current practices and spending policies at endowments and their relative merit and demerits. Cejnek et al. [2014], using some of the above mentioned references, give a complete account of the research on endowment funds including various spending rule applied in practice.

Many studies are focused on the comparison of different spending policies and investment strategies, which may provide a perpetual level of expected real income without impairing the real value of an endowment.

Using the continuous time framework, Dybvig [1999] propose that spending by endowment fund can be sustainable, if risky investments are used in combination with TIPS (Treasury Inflation Protection Securities). Based on the historical data of spending, this strategy calls for lower spending rates than commonly applied to obtain a non-decreasing future spending rates.

Bajeux-Besnainou and Ogunc [2006] address the asset allocation problem of endowment fund by including minimum spending amount up-rated with inflation in the objective function and derive an explicit formula for optimal spending and portfolio allocation rules in the minimum subsistence level framework. Gong and Li [2006] consider the general optimal investment/consumption problem for an agent, where the consumption, due to habit formation or pre-commitment, must not go below a certain level called the subsistence level and it rises above that level only when the wealth exceeds a certain threshold. Such a framework is also applicable to endowment fund problem and other portfolios where a withdrawal pre-commitment exists.

Institutional preferences may exhibit intertemporal complementarity where past spending habits generate a desire to maintain the same level. Such a behavior can be explained by the habit formation or habit persistence, which is a more general case of the subsistence level consumption proposed in Bajeux-Besnainou and Ogunc [2006]. Constantinides [1990] apply the habit formation to the equity puzzle problem and show that the high equity premium with low risk aversion can be explained by the presence of habit formation. Ingersoll [1992] examines continuous time consumption in non-time additive utility framework and characterizes the optimal consumption in the deterministic investment opportunities under a general framework. Following a lucid discussion on optimal investment and consumption with habit formation, Munk [2008] studies the optimal strategies with general asset price dynamics under two special cases of time varying investment opportunities: stochastic interest rate, and mean-reverting stock returns. He shows that, in order to finance the habit, investing in bonds and cash is more effective than investing in stocks.

In this chapter, we examine investment strategies under two spending rules in particular: (i) the consumption-wealth ratio rule, a simplified form of moving average method, and (ii) the weighted average or hybrid rule which is more commonly used by large endowment funds like Yale and Stanford (as stated in Cejnek et al. [2014]). Under consumption-wealth ratio (CW) rule, the spending is a percentage of the market value of the fund while hybrid rule calculates the spending as a weighted average of the inflation method and the moving average method. To analyze the effect of the spending rules on risk taking, we derive the optimal portfolio under the above mentioned spending rules for hyperbolic absolute risk aversion utility function and compare them with the classical Merton's optimal investment and consumption.

In Merton's case, we consider a general form of utility function including both the cases of habit formation and subsistence level. Under the spending rules mechanism, endowments following hybrid spending rule protect spending by investing in a less risky portfolio than Merton's. Similarly, investment in the risky asset with habit is less than the investment without habit. Thus, their strategy is similar to proportional portfolio insurance where the fund invests in safe assets to maintain the value needed for having smooth payouts over time. We calibrate parameters over three different time horizons to investigate the effectiveness of the spending rules. The hybrid strategy comparatively outperforms the conventional Merton's and CW strategies when the market is highly volatile but under-performs it when there is a low volatility.

The rest of the chapter is structured in the following way. Section 3.2 describes the various spending strategies commonly applied by the endowment funds, while Section 3.3 sets up the general framework for market dynamics and preferences of endowment fund and specifies the three different strategies. Section 3.4 focuses on the results of optimal investment and spending under different strategies. Section 3.5 presents a numerical application of different results obtained in the preceding sections, and, finally, Section 3.6 concludes the chapter and some technical derivations are left to the appendices.

3.2 Spending rules

In practice, there are various spending policies actually followed by endowment funds. According to survey data Cejnek et al. [2014] based on the original work by Sedlacek and Jarvis [2010], the spending rules are divided into four categories: (i) simple rules, (ii) inflation-based rules, (iii) smoothing rules, and (iv) hybrid rules. These rules are summarized in Table 3.1.

We consider the following three rules for our optimal investment and spending strategies:

- Inflation rule;
- Moving average method;
- Weighted average or hybrid method.

Although, all the spending rules are designed to accomplish the goal of avoiding volatility in income, however some rules are better than others.

The three above mentioned rules are presented in details in the following subsections.

3.2.1 Inflation rule

The rules is devised to acknowledge the corrosive effects of inflation. The objective of endowment fund is not the mere preservation of the fund but to strive for a value addition. It can be achieved as long as the overall return from the portfolio exceeds the rate of inflation. Inflation rule increases the previous period's spending at the predetermined inflation rate. The spending rule based on inflation protects

Table 3.1: Endowments spending rules. Source: Sedlacek and Jarvis [2010]

Categories	Method	Description
1. Simple Rules	Income-based	Spend the whole current income.
	Consumption-wealth ratio	Spend either the pre-defined percentage of the market value of the fund or decide the percentage every year.
2. Inflation-Based Rules	Inflation-protected	Spending grows at the rate of inflation.
	Banded-Inflation	Same as inflation-protected but with the upper and lower bands.
3. Smoothing Rules	Moving Average	Pre-defined percentage of moving average of market values, generally based on three-years starting market values.
	Spending Reserve	5-10 percent of the market value is held in the reserve account and then invested in 90-day Treasury bills. The amount is withdrawn only when the fund's performance is below target.
	Stabilization Fund	The excess endowment returns are used to make a fund which is then used to control the long term growth of the total endowment.
4. Hybrid Rules	Weighted Average or Hybrid (Yale/Stanford) Method	Spending is calculated as the weighted average of spending adjusted for inflation and the policy spending rate.

the purchasing power of endowment fund. Spending in a year equals the spending in the previous year, increased at the inflation rate λ:

$$c(t) = c_0 e^{\lambda t},$$

where the initial value of consumption is a fixed ratio of wealth, i.e. $c_0 = yR_0$. The differential of $c\,(t)$ is

$$dc(t) = \lambda c(t)dt. \tag{3.1}$$

The inflation method is static and trivial, so we will not consider it here separately, however it is included in hybrid method along with moving average methods.

3.2.2 Moving average method

The most popular and commonly used spending rule is the moving average. As in Dimmock [2012], it is typically based on pre-specified percentage of moving average of 3-years quarterly market value. The main feature of this rule is that it saves some income and reinvests it. This method does smooth the volatility in spending. However, the method is flawed because it uses the market value of the endowment. Therefore, when the endowment value is rising, the institution may spend more than it is prudent and when endowment values are falling sharply, the formula will suggest a budget cut that may curtail the institution's mission.

In discrete-time, this spending rule can be algebraically written as follows

$$c\,(t) = \frac{y}{q}\left(R\,(t) + ... + R\,(t-(q-1))\right).$$

In continuous-time, instead, we can write

$$c\,(t) = \frac{y}{q}\int_{t-q}^{t} R\,(\tau)\,d\tau.$$

Here, for the sake of simplicity, we take the limit of the previous rule for q which tends towards zero:[1]

$$c(t) = \lim_{q \to 0} \frac{y}{q} \int_{t-q}^{t} R(\tau)\, d\tau = \lim_{q \to 0} yR(t-q) = yR(t). \tag{3.2}$$

The goal of moving average rule is to dampen the volatility of spending. Therefore, during the period of boom, this process results in an accelerating curve of upward spending and it causes a false sense of security of sustainable spending. The smoothing effect of this rule is limited and it may give a misplaced belief that the higher spending levels are sustainable. Furthermore, the resulting shrinkage in endowment values due to market decline results into the deep cuts in spending.

3.2.3 Weighted average or hybrid method

The weighted average method is generally followed by the large endowment funds and it is also known as the Yale/Stanford rule. It is a weighted average of the inflation method and the moving average method.

In discrete-time it is given by

$$c(t+1) = \omega c(t) e^{\lambda} + (1-\omega) \frac{y}{q} \left(R(t+1) + ... + R(t+1-(q-1)) \right), \tag{3.3}$$

where ω is the weight. We can simplify it by considering $q = 1$ to get

$$c(t+1) = \omega c(t) e^{\lambda} + (1-\omega) yR(t+1). \tag{3.4}$$

If we ignore inflation, i.e. $\lambda = 0$, we obtain

$$c(t+1) = \omega c(t) + (1-\omega) yR(t+1). \tag{3.5}$$

We make the above process stationary by assuming $|(1-\omega) y| < 1$.

Remark 1. If we take the limit $t+1 \to t$, we get the same consumption-wealth ratio as in (3.2)

[1] De l'Hôpital theorem is used, by recalling that $\frac{\partial}{\partial q} \int_{t-q}^{t} R(\tau)\, d\tau = R(t-q)$.

$$c(t) = yR(t).$$

Under the hybrid rule, during the boom, the spending will not increase as fast as compared with the moving average rule. Conversely, this rule will not call for spending cuts as deep as the moving average method. Evidence suggests that more and more institutions are changing their spending rules to inflation-based and hybrid method from moving average method Sedlacek and Jarvis [2010].

This section has provided an overview of the main spending rules applied by the endowments and now we will only consider two spending rules, the moving average method simplified as consumption-wealth ratio rule and hybrid spending rule given by (3.2) and (3.5) respectively.

3.3 General framework

Endowment funds usually invest in a variety of assets, however for the purpose of tractability we examine the aforementioned spending rules in the simplest framework. We consider two types of assets, a riskless asset and a risky asset in a complete and arbitrage free, continuously open financial market. On the financial market two assets are listed:

- a riskless asset $G(t)$ which evolves according to

$$\frac{dG(t)}{G(t)} = r(t)dt, \tag{3.6}$$

  where $G(t_0) = 1$, and which is the *numéraire* and $r(t)$ is the instantaneous nominal interest rate;

- a risky asset $S(t)$ having the price dynamics given by

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t). \tag{3.7}$$

The endowment fund holds $\theta_S(t)$ units of the risky asset $S(t)$, and $\theta_G(t)$ units of risk-less assets $G(t)$. Thus, at any instant in time t, the investor's wealth $R(t)$ is given by

$$R(t) = \theta_G(t)G(t) + \theta_S(t)S(t).$$

The differential of wealth can be written as

$$dR(t) = \big(R(t)r(t) + \theta_S(t)S(t)(\mu - r(t)) - c(t)\big)dt + \theta_S(t)S(t)\sigma dW(t), \tag{3.8}$$

where $c\,(t)$ is the consumption, following one of the above mentioned rules except when it is also a decision variable.

3.3.1 Endowment fund investment strategies

We consider the following three strategies for our analysis:

- **Merton's strategy:** Both investment and spending are decision variables.
- **Consumption-wealth ratio (CW) strategy:** Investment is the only decision variable and spending is given by the fixed consumption-wealth ratio rule.
- **Hybrid strategy:** Investment is the only decision variable and spending is given by weighted average spending rule.

3.3.2 General settings

According to Fraser and Jennings [2010], an endowment fund must define its investment policy statement identifying the investment beliefs, specific investment objectives, re-balancing policy and performance benchmark which are evaluated periodically. Since an endowment fund must report its performance for each accounting period, it is reasonable to consider the optimization problem for a finite time horizon $[t, T]$. If we assume the objective of an endowment fund is to maximize the sum of expected utility of spending $c(t)$ and the expected utility of the final wealth $R(T)$ then it can be stated as

$$\max_{\pi(t)} \mathbb{E}_t \left[\int_t^T U_c\left(c\left(t\right)\right) e^{-\rho(s-t)} ds + U_R\left(R\left(T\right)\right) e^{-\rho(T-t)} \right], \tag{3.9}$$

where endowment fund chooses the decision variables $\pi(t)$, which may include consumption and investment depending on the strategy considered and ρ is a constant subjective discount rate[2]. We assume the fund's preferences are defined by the following utility function

$$U\left(x\left(t\right)\right) = \frac{\left(x\left(t\right) - \alpha_X\right)^{1-\delta}}{1-\delta}, \tag{3.10}$$

where $\delta > 1$. The Arrow-Pratt Absolute Risk Aversion (ARA) index is

$$-\frac{\frac{\partial^2 U(x(t))}{\partial x(t)^2}}{\frac{\partial U(x(t))}{\partial x(t)}} = \frac{\delta}{x\left(t\right) - \alpha_X}.$$

If α_X is a positive constant, then this form of the utility function belongs to Hyperbolic Absolute Risk Aversion (HARA). If $\alpha_X = 0$, ARA index is given by

$$-\frac{\frac{\partial^2 U(x(t))}{\partial x(t)^2}}{\frac{\partial U(x(t))}{\partial x(t)}} = \frac{\delta}{x\left(t\right)},$$

which belongs to Constant Relative Risk Aversion (CRRA).

If $\alpha_X \neq 0$, the utility functions belong to HARA and can be written as

$$U_c\left(c\left(t\right), h\left(t\right)\right) = \frac{\left(c\left(t\right) - h\left(t\right)\right)^{1-\delta}}{1-\delta}, \quad U_R\left(R\left(T\right)\right) = \frac{\left(R\left(T\right) - R_m\right)^{1-\delta}}{1-\delta}, \tag{3.11}$$

where $c\left(t\right)$ is the instantaneous outflow or spending from the fund, the constant R_m can be interpreted as the minimum subsistence level of wealth, whereas $h(t)$ depends on the context, it is either a function representing the habit formation or a constant representing the subsistence level. Given the utility function (3.11), it is always

[2] While Tobin [1974] suggests that endowment trustees have a zero subjective discount factor, we consider the general case of positive ρ.

optimal to have outflows higher than the threshold $h(t)$, in fact when $c(t) = h(t)$, the marginal utility of the outflow tends towards infinity and, accordingly, it is sufficient to increase it by an infinitesimal amount in order to have an infinite increase in the utility level.

The corresponding ARA indices of (3.11) are given by

$$-\frac{\frac{\partial^2 U_c(c(t),h(t))}{\partial c(t)^2}}{\frac{\partial U_c(c(t),h(t))}{\partial c(t)}} = \frac{\delta}{c(t) - h(t)}, \quad -\frac{\frac{\partial^2 U_R(R(T))}{\partial R(t)^2}}{\frac{\partial U_R(R(T))}{\partial R(t)}} = \frac{\delta}{R(T) - R_m},$$

respectively, which implies that the higher δ the higher the risk aversion. Moreover, the higher $h(t)$ the higher is the risk aversion. This result shows that having a higher level of minimum outflows means that it is necessary to invest bigger amounts of wealth in the riskless asset in order to guarantee the outflows, which implies a higher risk aversion.

3.4 The optimal solutions

3.4.1 Merton's strategy

Investment and spending are both decision variables, i.e. $\pi(t) \equiv \{c(t), \theta_S(t)\}$. We can define the value function using (3.9) as

$$J(t, R(t)) \equiv \max_{c(t),\theta_S(t)} \mathbb{E}_t \left[\int_t^T \phi_c \frac{(c(s) - h(s))^{1-\delta}}{1-\delta} e^{-\rho(s-t)} ds + \phi_R \frac{(R(T) - R_m)^{1-\delta}}{1-\delta} e^{-\rho(T-t)} \right], \tag{3.12}$$

where ϕ_c and ϕ_R are constants and $h(t)$ is given by

$$h(t) = \begin{cases} h_0 e^{-\int_0^t \beta(u)du} + \int_0^t \alpha(s) c_s e^{-\int_s^t \beta(u)du} ds, & \text{habit formation}, \\ h, & \text{subsistence level}, \\ 0, & \text{classical problem}, \end{cases} \tag{3.13}$$

where h_0 is the initial minimum amount of outflow, $\alpha(t)$ is the weighting function providing the relative importance to the past outflow in computing the threshold $h(t)$, while $\beta(t)$ is a discount rate. In habit formation case, $h(t)$ can be rewritten in continuous time as

$$dh\left(t\right) = \left(\alpha\left(t\right) c\left(t\right) - \beta\left(t\right) h\left(t\right)\right) dt, \tag{3.14}$$

Proposition 2. *Given the state variable wealth $R(t)$ described in (3.8), the optimal consumption and portfolio solving problem (3.12) are*

- in the case with habit formation:

$$c(t)^{*} = h(t) + \phi_c^{\frac{1}{\delta}} \frac{\left(R\left(t\right) - h\left(t\right) B\left(t\right)\right)\left(1 + B\left(t\right)\alpha\left(t\right)\right)^{-\frac{1}{\delta}}}{A\left(t\right)}, \tag{3.15}$$

$$\theta_S(t)^{*} = \frac{\mu - r(t)}{S\left(t\right)\sigma^2} \frac{R\left(t\right) - h\left(t\right) B\left(t\right)}{\delta}, \tag{3.16}$$

where

$$\begin{aligned} A(t) &= \phi_R^{\frac{1}{\delta}} e^{-\int_t^T \left(\frac{\delta-1}{\delta} r(u) + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu - r(u))^2}{2\delta^2\sigma^2}\right) du} \\ &\quad + \int_t^T \phi_c^{\frac{1}{\delta}} \left(1 + B(s)\alpha(s)\right)^{\frac{\delta-1}{\delta}} e^{-\int_t^s \left(\frac{\delta-1}{\delta} r(u) + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu - r(u))^2}{2\delta^2\sigma^2}\right) du} ds, \\ B(t) &= R_m e^{-\int_t^T (-\alpha(u) + \beta(u) + r(u)) du} + \int_t^T e^{-\int_t^s (-\alpha(u) + \beta(u) + r(u)) du} ds, \end{aligned}$$

- in the case of a subsistence level:

$$c(t)^{*} = h + \phi_c^{\frac{1}{\delta}} \frac{R(t) - hB(t)}{A(t)}, \tag{3.17}$$

$$\theta_S(t)^{*} = \frac{\mu - r(t)}{S(t)\sigma^2} \frac{R(t) - hB(t)}{\delta}. \tag{3.18}$$

- in the classical case:

$$c(t)^{*} = \phi_c^{\frac{1}{\delta}} \frac{R(t)}{A(t)} \text{ and } \theta_S(t)^{*} = \frac{\mu - r(t)}{S(t)\sigma^2} \frac{R(t)}{\delta}, \tag{3.19}$$

where for both the subsistence level case and the classical case we have

$$A(t) = \phi_R^{\frac{1}{\delta}} e^{-\int_t^T \left(\frac{\delta-1}{\delta} r(u) + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu - r(u))^2}{2\delta^2\sigma^2}\right) du} + \phi_c^{\frac{1}{\delta}} \int_t^T e^{-\int_t^s \left(\frac{\delta-1}{\delta} r(u) + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu - r(u))^2}{2\delta^2\sigma^2}\right) du} ds,$$

$$B(t) = R_m e^{-\int_t^T r(u) du} + \int_t^T e^{-\int_t^s r(u) du} ds.$$

Proof. See Appendix 3.A.1. □

The function $A(t)$ in the optimal solutions is the weighted sum of two discount factors: (i) the discount factor for the final date T multiplied by $\phi_R^{\frac{1}{\delta}}$, and (ii) a kind of intertemporal discount factor for the intertemporal utility, multiplied by $\phi_c^{\frac{1}{\delta}}$. The function $B(t)$ in the optimal solutions, is the sum of two terms, the subsistence wealth R_m appropriately discounted from time T and a sum of discount factors. We can see that habit formation has an effect on the optimal portfolio of the risky asset, as it changes the allocation due to the reason that the riskless asset (treasury) is comparatively a better investment than the risky asset (stock) to ensure that the future consumption will not decline below the habit level.

We consider the optimal investment and consumption in the habit formation case defined in (3.16) and (3.15), respectively, in detail below:

Assumption 1. *We assume that all the parameters $\alpha, \beta, \mu, \sigma$, and r are constant over time and additionally $r - \alpha + \beta > 0$.*

Corollary 1. *Under Assumption 1, (i) The wealth process is itself a Markov process and the functions $A(t)$ and $B(t)$ can be written as*

$$\begin{aligned} A(t) &= \phi_R^{\frac{1}{\delta}} e^{-\left(\frac{\delta-1}{\delta} r + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu-r)^2}{2\delta^2\sigma^2}\right)(T-t)} \\ &+ \int_t^T \phi_c^{\frac{1}{\delta}} (1 + B(s)\alpha)^{\frac{\delta-1}{\delta}} e^{-\left(\frac{\delta-1}{\delta} r + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu-r)^2}{2\delta^2\sigma^2}\right)(s-t)} ds, \\ B(t) &= R_m e^{-(r-\alpha+\beta)(T-t)} + \int_t^T e^{-(r-\alpha+\beta)(s-t)} ds \\ &= R_m e^{-(r-\alpha+\beta)(T-t)} + \frac{1 - e^{-(r-\alpha+\beta)(T-t)}}{r - \alpha + \beta}, \end{aligned} \tag{3.20}$$

and finally, if we substitute the value of $B(t)$ into $A(t)$, we get

$$A(t) = \phi_R^{\frac{1}{\delta}} e^{-\left(\frac{\delta-1}{\delta} r + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu-r)^2}{2\delta^2\sigma^2}\right)(T-t)}$$
$$+ \int_t^T \phi_c^{\frac{1}{\delta}} (1 + R_m e^{-(r-\alpha+\beta)(T-s)} + \frac{1-e^{-(r-\alpha+\beta)(T-s)}}{r-\alpha+\beta} \alpha)^{\frac{\delta-1}{\delta}} e^{-\left(\frac{\delta-1}{\delta} r + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu-r)^2}{2\delta^2\sigma^2}\right)(s-t)} ds. \tag{3.21}$$

(ii) The optimal consumption and portfolio are

$$c(t)^* = h(t) + \phi_c^{\frac{1}{\delta}} \frac{(R(t) - h(t) B(t))(1 + B(t)\alpha)^{-\frac{1}{\delta}}}{A(t)}, \tag{3.22}$$

$$\theta_S(t)^* = \frac{\mu - r}{S(t)\sigma^2} \frac{R(t) - h(t) B(t)}{\delta}, \tag{3.23}$$

where $A(t)$ and $B(t)$ are given by (3.20) and (3.21).

The term

$$\frac{1-e^{-(r-\alpha+\beta)(T-s)}}{r-\alpha+\beta}$$

is positive and decreasing over time as $r - \alpha + \beta > 0$.

(iii) The optimal portfolio is less risky with habit as compared with the optimal portfolio without habit:

$$S(t)\theta_S(t)^* < S(t)\theta_S(t)^*|_{h(t)=0}.$$

Proof. The optimal portfolio $S(t)\theta_S(t)^*$ is given by

$$\frac{\mu - r}{S(t)\sigma^2} \frac{R(t) - h(t) B(t)}{\delta},$$

Since by construction $h(t) \geq 0$ and $B(t) \geq 0$, thus

$$\frac{\mu - r}{S(t)\sigma^2} \frac{R(t) - h(t) B(t)}{\delta} < \frac{\mu - r}{S(t)\sigma^2} \frac{R(t)}{\delta}.$$

□

Assumption 2. *We assume $T \to \infty$.*

Corollary 2. *Under Assumptions 1 and 2, the optimal consumption and portfolio are*

$$c(t)^* = h(t) + \phi_c^{\frac{1}{\delta}} \frac{(R(t) - h(t)\frac{1}{r-\alpha+\beta})(1 + \frac{\alpha}{r-\alpha+\beta})^{-\frac{1}{\delta}}}{A(t)},$$

$$\theta_S(t)^* = \left(R(t) - \frac{h(t)}{r-\alpha+\beta}\right) \frac{\mu - r}{S(t)\sigma^2\delta}.$$

In this case the dynamics of optimal wealth and habit are

$$\begin{aligned} dR(t) = &\left(R(t)r + \left(R(t) - \frac{h(t)}{r-\alpha+\beta}\right)\frac{(\mu-r)^2}{\sigma^2\delta} - h(t)\right. \\ &\left. + \phi_c^{\frac{1}{\delta}} \frac{\left(R(t) - h(t)\frac{1}{r-\alpha+\beta}\right)\left(1 + \frac{\alpha}{r-\alpha+\beta}\right)^{-\frac{1}{\delta}}}{A(t)}\right) dt \\ &+ \left(R(t) - \frac{h(t)}{r-\alpha+\beta}\right)\frac{\mu - r}{\sigma\delta} dW(t), \end{aligned}$$

$$dh(t) = (\beta - \alpha)\left(\frac{\alpha}{\beta - \alpha}\phi_c^{\frac{1}{\delta}} \frac{\left(R(t) - h(t)\frac{1}{r-\alpha+\beta}\right)\left(1 + \frac{\alpha}{r-\alpha+\beta}\right)^{-\frac{1}{\delta}}}{A(t)} - h\left(t\right)\right) dt,$$

where we can see that $h(t)$ *is a mean reverting process if* $\beta - \alpha > 0$ *and is instead exploding if* $\beta - \alpha < 0$.

3.4.2 CW strategy

In this case, the investment $\theta_S(t)$ is the only decision variable and the spending $c(t)$ is given by

$$c(t) = yR(t), \tag{3.24}$$

therefore we can put $\phi_c = 0$ and $\phi_R = 1$ in (3.12), then the value function can be defined as

$$J(t, R(t)) \equiv \max_{\theta_S(t)} \mathbb{E}_t \left[\frac{(R(T) - R_m)^{1-\delta}}{1-\delta} e^{-\rho(T-t)} \right]. \qquad (3.25)$$

Proposition 3. *Given the state variable $R(t)$ and $c(t)$ described in (3.8) and (3.24) respectively, the optimal portfolio solving problem (3.25) is*

$$\theta_S(t)^* = \frac{\mu - r(t)}{S(t)\sigma^2} \frac{R(t) - B(t)}{\delta}, \qquad (3.26)$$

where

$$B(t) = R_m e^{-\int_t^T (r(s)-y)ds},$$

and y is the constant defined in (3.24).

Proof. See Appendix 3.A.2. □

The function $B(t)$ in the optimal solution, is the subsistence wealth R_m appropriately discounted by a discount factor.

Corollary 3. *Under Assumptions 1,2 and $r > y$, the optimal portfolio (3.26) becomes*

$$\theta_S(t)^* = \frac{\mu - r}{S(t)\sigma^2} \frac{R(t)}{\delta}.$$

In this case the dynamics of optimal wealth is

$$\frac{dR(t)}{R(t)} = \left(r + \frac{1}{\delta} \frac{(\mu - r)^2}{\sigma^2} - y \right) dt + \frac{1}{\delta} \frac{\mu - r}{\sigma} dW(t).$$

3.4.3 Hybrid strategy

In this strategy, investment is the only decision variable $\pi \equiv \theta_S(t)$, while spending evolves according to the weighted average spending rule (3.5). We can write the wealth dynamics (3.8) in a discrete time as

$$R(t+1) = R(t)(1 + r(t)) + \theta_S(t)S(t)(\mu - r(t)) - c(t) + \theta_S(t)S(t)\sigma Z(t), \qquad (3.27)$$

and the consumption in a discrete time is given by

$$c(t+1) = \omega c(t) + (1-\omega)\, yR(t+1). \tag{3.28}$$

We substitute (3.27) into (3.28), and we get

$$c(t{+}1) = \omega c(t){+}(1-\omega)\, y\left(R(t)(1+r(t)) + \theta_S(t)S(t)(\mu - r(t)) - c(t) + \theta_S(t)S(t)\sigma Z(t)\right),$$

which can be rewritten in continuous time as

$$\begin{aligned} dc(t) &= (1-\omega)\,(1+y)\left(\frac{yR(t)(1+r(t)) + y\theta_S(t)S(t)(\mu - r(t))}{1+y} - c(t)\right) dt \\ &\quad + (1-\omega)\, y\theta_S(t)S(t)\sigma dW(t). \end{aligned} \tag{3.29}$$

We set $a \equiv (1-\omega)y$, and we can write

$$\begin{aligned} dc(t) &= (1-\omega)\,(1+y)\left(\frac{yR(t)(1+r(t)) + y\theta_S(t)S(t)(\mu - r(t))}{1+y} - c(t)\right) dt \\ &\quad + a\theta_S(t)S(t)\sigma dW(t), \end{aligned}$$

where we see that the consumption is a mean reverting process, whose strength of mean reversion is $(1-\omega)(1+y)$. The consumption reverts towards its long term mean:

$$\frac{yR(t)(1+r(t)) + y\theta_S(t)S(t)(\mu - r(t))}{1+y},$$

which depends on the portfolio choice. The higher the value of ω, the more slowly $c(t)$ converges towards its long term mean and vice versa.

As the consumption is given by (3.29), we include it as an additional state variable and put $\phi_c = 0$ and $\phi_R = 1$ in (3.12), hence the value function can be defined as

$$J\left(t,R\left(t\right),c\left(t\right)\right)\equiv\max_{\theta_S(t)}\mathbb{E}_t\left[\frac{\left(R\left(T\right)-R_m\right)^{1-\delta}}{1-\delta}e^{-\rho(T-t)}\right]. \tag{3.30}$$

Proposition 4. *Given the state variables $R(t)$ and $c(t)$ as in (3.8) and (3.29) and under the assumption that interest rate r is constant the optimal portfolio solving problem (3.30) is*

$$\theta_S\left(t\right)^*=\frac{\mu-r}{S\left(t\right)\sigma^2}\frac{R\left(t\right)-B\left(t,c\left(t\right)\right)}{\delta\left(1-\eta^*a\right)}, \tag{3.31}$$

where

$$B\left(t,c\left(t\right)\right)=\eta^*c(t)+R_me^{-(r-\eta^*a(1+r))(T-t)},$$

and η takes one of the following values

$$\eta^*=\frac{\left(1+r+a-\omega\right)\pm\sqrt{\left(1+r+a-\omega\right)^2-4a\left(1+r\right)}}{2a\left(1+r\right)},$$

such that $-\infty<\eta^*<\frac{1}{a}$ and $a=y\left(1-\omega\right)<1$.

Proof. See Appendix 3.A.3. □

Corollary 4. *Under Assumptions 1,2 and if $r>\eta^*a\left(1+r\right)$, the optimal portfolio (3.31) becomes*

$$\theta_S(t)^*=\frac{\mu-r}{S(t)\sigma^2}\frac{R(t)-\eta^*c(t)}{\delta\left(1-\eta^*a\right)}.$$

In this case the dynamics of optimal wealth and optimal consumption are

$$dR(t)=\left(R(t)r+\left(\frac{\left(\mu-r\right)^2}{\sigma^2}\frac{R(t)-\eta^*c(t)}{\delta\left(1-\eta^*a\right)}\right)-c(t)\right)dt+\left(\frac{\mu-r}{\sigma}\frac{R(t)-\eta^*c(t)}{\delta\left(1-\eta^*a\right)}\right)dW(t),$$

$$\begin{aligned}dc(t)=&\left(aR(t)(1+r)+a\left(\frac{\left(\mu-r\right)^2}{\sigma^2}\frac{R(t)-B(t,c(t))}{\delta\left(1-\eta^*a\right)}\right)-\left(1-\omega\right)\left(1+y\right)c(t)\right)dt\\&+a\left(\frac{\mu-r}{\sigma}\frac{R(t)-B(t,c(t))}{\delta\left(1-\eta^*a\right)}\right)dW(t).\end{aligned}$$

Corollary 5. *Under Assumption 1, (i) the optimal portfolio is less risky for the Merton's strategy as compared with the optimal portfolio for CW strategy if* $\alpha-\beta > y$ *and* $r > y$.[3]

$$\theta_{S,M}(t)^* < \theta_{S,C}(t)^*,$$

(ii) Under an additional assumption $\eta = 0$, *we can write*

$$\theta_{S,H}(t)^* < \theta_{S,M}(t)^* < \theta_{S,C}(t)^*.$$

Proof. We suppose the negation of the given statement is true

$$\theta_{S,M}(t)^* \geq \theta_{S,C}(t)^*.$$

As it is reasonable to assume $\mu - r > 0$, which implies $\frac{\mu-r}{S(t)\sigma^2\delta}>0$. Therefore

$$R\left(t\right) - h\left(t\right)B\left(t\right) \geq R(t) - B(t).$$

We substitute the values of the unknown functions to obtain

$$R\left(t\right) - h\left(t\right)\left(R_m e^{-(r-\alpha+\beta)(T-t)} + \frac{1-e^{-(r-\alpha+\beta)(T-t)}}{r-\alpha+\beta}\right) \geq R(t) - R_m e^{-(r-y)(T-t)},$$

since $h(t) > 0$ and $\frac{1-e^{-(r-\alpha+\beta)(T-t)}}{r-\alpha+\beta}$ is also positive and decreasing over time as we have $r-\alpha+\beta > 0$ under Assumption 1. Therefore, the above statement can only be correct only if

$$e^{-(r-\alpha+\beta)(T-t)} < e^{-(r-y)(T-t)},$$

as $r-\alpha+\beta > 0$, $r > y$ and $\alpha-\beta > y$. Therefore, we conclude that the following statement is true

$$\theta_{S,M}(t)^* < \theta_{S,C}(t)^*.$$

[3] In what follows we use subscripts M,C and H with $\theta_S(t)$ or $c(t)$ to indicate optimal portfolio or consumption under Merton's, CW and hybrid strategies, respectively.

For part (ii), we suppose

$$\theta_{S,H}(t)^* \geq \theta_{S,M}(t)^*.$$

We substitute the values of the unknown functions to obtain

$$R(t) - R_m e^{-r(T-t)} \geq R(t) - h(t) R_m e^{-(r-\alpha+\beta)(T-t)} - h(t) \frac{1 - e^{-(r-\alpha+\beta)(T-t)}}{r - \alpha + \beta}.$$

We can clearly see that the left-hand side is smaller than the right-hand side, therefore we conclude that the statement to be proved is true. □

Corollary 6. *Under Assumption 1, the optimal consumption under Merton's strategy as described in (3.15) can be equivalent to the consumption under CW strategy as described in (3.2) depending on the chosen values of constant y.*

Proof. As the optimal consumption for Merton's strategy is

$$c_M(t)^* = h(t) + \phi_c^{\frac{1}{\delta}} \frac{(R(t) - h(t)B(t))(1 + B(t)\alpha(t))^{-\frac{1}{\delta}}}{A(t)},$$

where

$$B(t) = R_m e^{-\int_t^T (-\alpha(u)+\beta(u)+r(u))du} + \int_t^T e^{-\int_t^s (-\alpha(u)+\beta(u)+r(u))du} ds,$$

and

$$\begin{aligned} A(t) = {} & \phi_R^{\frac{1}{\delta}} e^{-\int_t^T \left(\frac{\delta-1}{\delta} r(u) + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu - r(u))^2}{2\delta^2\sigma^2} \right) du} \\ & + \int_t^T \phi_c^{\frac{1}{\delta}} (1 + B(s)\alpha(s))^{\frac{\delta-1}{\delta}} e^{-\int_t^s \left(\frac{\delta-1}{\delta} r(u) + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu - r(u))^2}{2\delta^2\sigma^2} \right) du} ds. \end{aligned}$$

The consumption for CW strategy is given by

$$c_C(t)^* = yR(t).$$

Consumption under Merton's strategy and CW strategy can be equivalent if y

is chosen such that

$$y = \frac{h(t)}{R(t)} + \phi_c^{\frac{1}{\delta}} \frac{(1 + B(t)\alpha(t))^{-\frac{1}{\delta}}}{A(t)} - \phi_c^{\frac{1}{\delta}} \frac{h(t)B(t)(1 + B(t)\alpha(t))^{-\frac{1}{\delta}}}{R(t)A(t)}.$$

if $h = 0$, then

$$y = \frac{\phi_c^{\frac{1}{\delta}}}{A(t)},$$

where $\phi_c^{\frac{1}{\delta}}$ can be suitably chosen to match y. □

3.5 A numerical application

To illustrate the results of the preceding section, a simplified market structure is taken into account under Assumption 1 and condition that $h(t) = 0$. We have estimated the parameters related to the financial market and interest rate over three different time horizons: (i) January 2nd, 1997 and December 29th, 2006 (1997-2006), (ii) January 3rd, 2007 and December 30th, 2011 (2007-2011), and (iii) January 3rd, 2012 and December 30th, 2016 (2012-2016). The parameters of the risky asset $S(t)$ are obtained from the S&P 500 and the value of constant interest rate r is estimated as the average return of US 3-Month Treasury Bill (on secondary market – daily data). We assume the risk aversion parameter δ is equal to 2 similar to the most common choices of risk aversion parameter in the habit formation and life cycle literature (Munk [2008]; Gong and Li [2006] and Horneff et al. [2015]; Gourinchas and Parker [2002]). We set the subjective discount factor ρ equal to the riskless interest rate r. The estimated parameters along with some assumptions about wealth and preferences are gathered in Tab. 3.2.

All the graphs in this section show the mean of 1000 simulations. We recall the general objective function under the assumption $h(t) = 0$:

$$J(t, R(t)) \equiv \max_{\pi(t)} \mathbb{E}_t \left[\int_t^T \phi_c \frac{(c(s))^{1-\delta}}{1-\delta} e^{-\rho(s-t)} ds + \phi_R \frac{(R(T) - R_m)^{1-\delta}}{1-\delta} e^{-\rho(T-t)} \right],$$

Table 3.2: Parameters calibrated on the S&P 500 and US 3-Month Treasury Bill time series between (i) January 2nd, 1997 and December 29th, 2006 (1997-2006), (ii) January 3rd, 2007 and December 30th, 2011 (2007-2011), and (iii) January 3rd, 2012 and December 30th, 2016 (2012-2016). Other assumptions include $R_0 = 100$, $T = 10$, $R_m = 0$ or 90 when ($R_m \to R_0$) and $\delta = 2$.

Parameters	1997-2006	2007-2011	2012-2016
μ	0.0816	0.0117	0.1198
σ	0.1816	0.2659	0.1279
r, ρ	0.0356	0.0122	0.0011

The results derived in the preceding section are summarized in Table 3.3.

3.5.1 Wealth and consumption

The weights assigned to the terminal wealth and intertemporal consumption in the objective function do affect the optimal portfolio and consumption. Figure 3.1 shows how the variation in weights results in a changes in the wealth and consumption. The graphs illustrates that comparatively higher weight must be given to ϕ_R as compared with ϕ_C to maintain wealth above zero in the long run. The consumption-wealth ratio for this case in shown in Figure 3.2.

Figure 3.3 presents the impact of the variation of consumption-wealth ratio on the paths of wealth and consumption for CW strategy, we assume that subsistence wealth $R_m = 0$. As y increases the level of consumption rises but the terminal wealth declines.

When the subsistence wealth R_m approaches the initial wealth R_0, the mean of the wealth and consumption becomes less volatile for all the values of y, as shown in the Figure 3.4. We can also see that the rise or decline in the wealth and consumption in the long run depends on the value of y.

For hybrid strategy, we assume that $R_m = 0$, Figure 3.5 shows the wealth and consumption for different weights ω. We can see from the graph that the weight ω for the inflation method must be chosen prudently, the higher the value of ω, the more slowly $c(t)$ converges towards its long term mean and vice versa. Similar to CW strategy, if R_m approaches the initial wealth R_0, the paths of wealth and

Table 3.3: Optimal solutions for different strategies.

Strategies (Decision variables)	Optimal or defined consumption rule	Optimal portfolio
Merton's strategy $(c(t), \theta_S(t))$	$c(t)^* = \phi_c^{\frac{1}{\delta}} \frac{R(t)}{A(t)}$, where $A(t) = \phi_R^{\frac{1}{\delta}} e^{-\left(\frac{\delta-1}{\delta}r+\frac{\rho}{\delta}+\frac{(\delta-1)(\mu-r)^2}{2\delta^2\sigma^2}\right)(T-t)} + \phi_c^{\frac{1}{\delta}} \left(\frac{1-e^{-\left(\frac{\delta-1}{\delta}r+\frac{\rho}{\delta}+\frac{(\delta-1)(\mu-r)^2}{2\delta^2\sigma^2}\right)(T-t)}}{\left(\frac{\delta-1}{\delta}r+\frac{\rho}{\delta}+\frac{(\delta-1)(\mu-r)^2}{2\delta^2\sigma^2}\right)} \right)$.	$\theta_S(t)^* = \frac{\mu-r}{S(t)\sigma^2} \frac{R(t)}{\delta}$.
CW strategy$(\theta_S(t))$	$c(t) = yR(t)$.	$\theta_S(t)^* = \frac{\mu-r}{S(t)\sigma^2} \frac{R(t)-B(t)}{\delta}$, where $B(t) = R_m e^{(y-r)(T-t)}$.
Hybrid strategy$(\theta_S(t))$	$c(t+1) = \omega c(t) + (1-\omega) yR(t+1)$.	$\theta_S(t)^* = \frac{\mu-r}{S(t)\sigma^2} \frac{R(t)-B(t,c(t))}{\delta(1-\eta^* a)}$, where $B(t,c(t)) = \eta^* c(t) + R_m e^{-(r-\eta^* a(1+r))(T-t)}$, where $\eta = \frac{(1+r+a-\omega)\pm\sqrt{(1+r+a-\omega)^2-4a(1+r)}}{2a(1+r)}$.

Figure 3.1: Wealth and consumption under Merton's strategy with different values of the weights (ϕ_c and ϕ_R) in the objective function (3.12) and the dashed lines show the confidence interval (i.e. mean plus and minus two standard deviations). The values of all parameters are estimated for the period 1997-2006 as stated in Tab. 3.2.

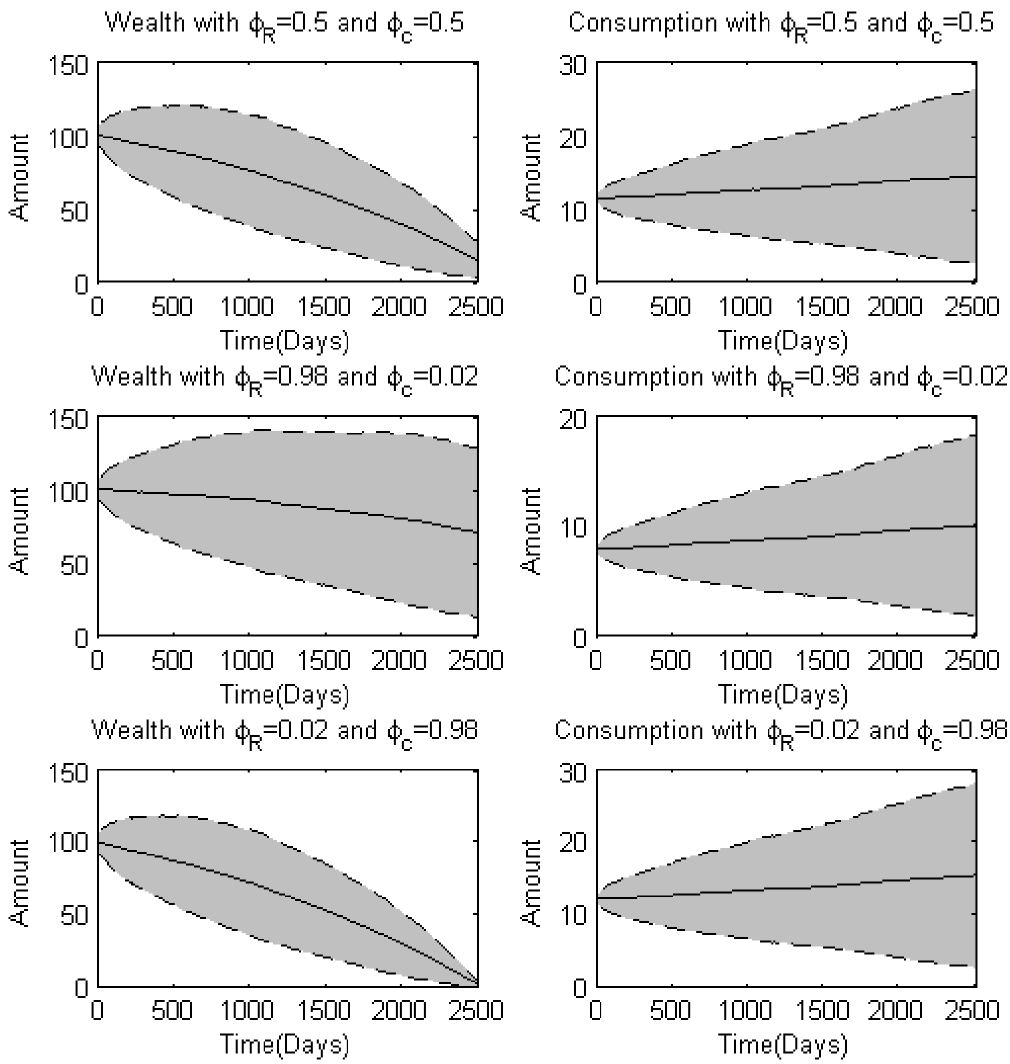


Figure 3.2: Consumption-wealth ratio for Merton's strategy with different values of the weights (ϕ_c and ϕ_R) in the objective function (3.12). The values of all parameters are estimated for the period 1997-2006 as stated in Tab. 3.2.

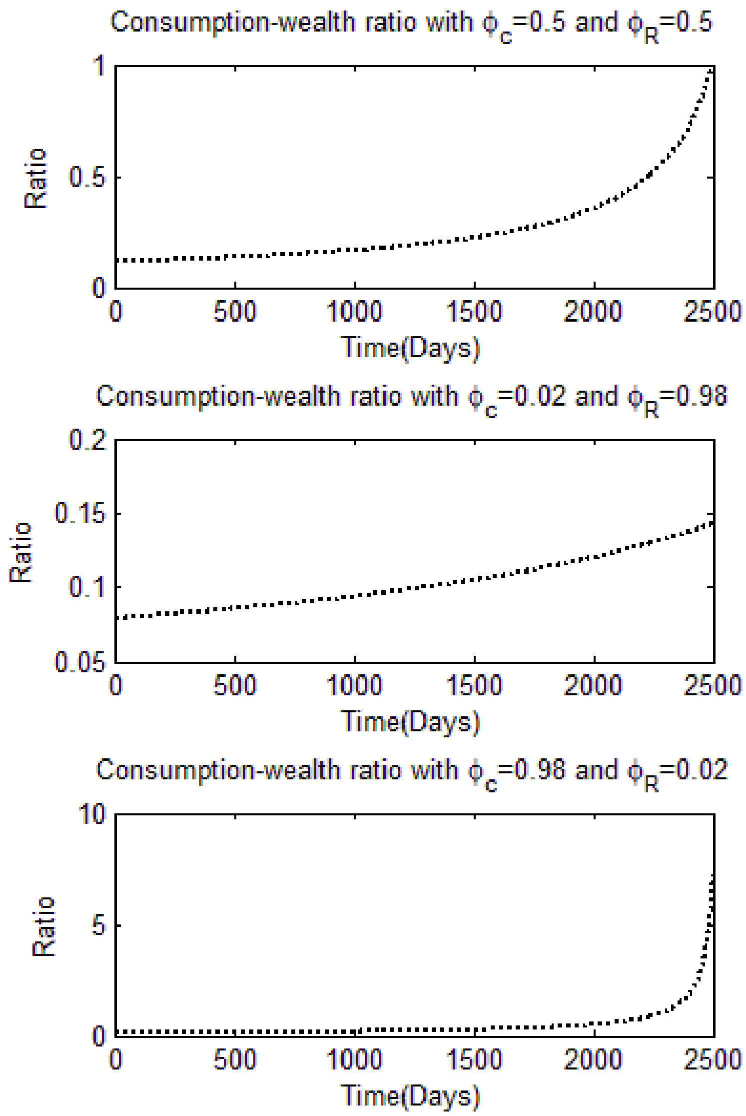


Figure 3.3: Wealth and consumption under CW strategy with different values of the consumption-wealth ratio y and the dashed lines show the confidence interval. The values of all parameters are estimated for the period 1997-2006 as stated in Tab. 3.2.

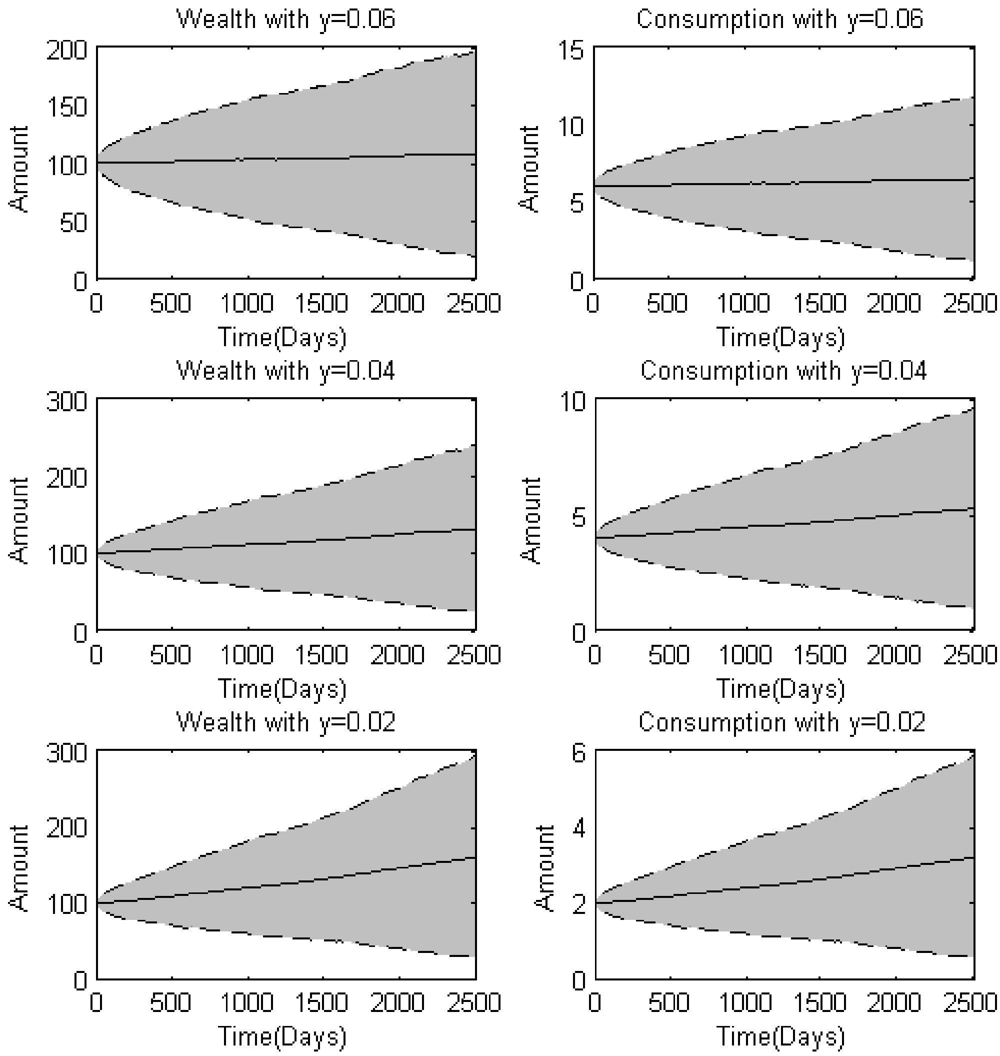
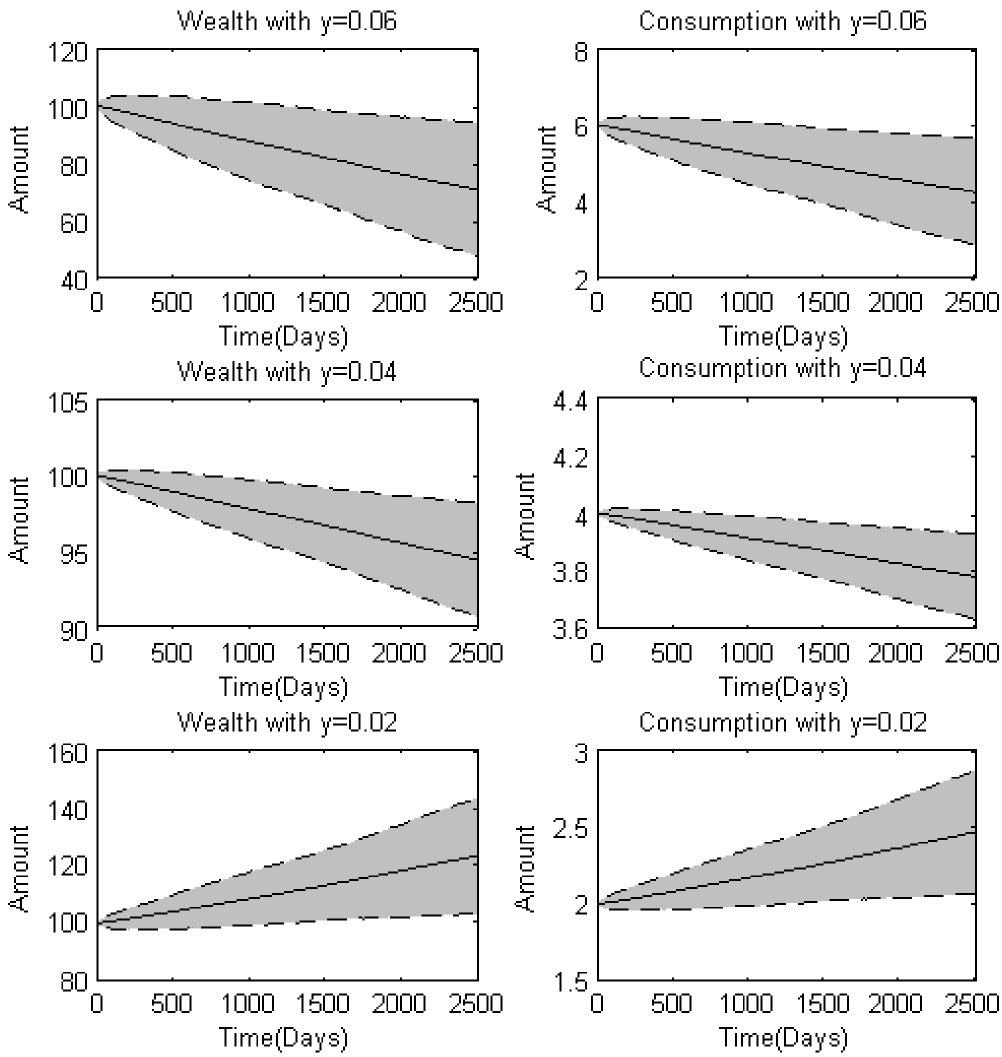


Figure 3.4: Wealth and consumption under CW strategy with different values of the consumption-wealth ratio y when $R_m \to R_0$ and the dashed lines show the confidence interval. The values of all parameters are estimated for the period 1997-2006 as stated in Tab. 3.2.



consumption becomes less volatile. For this strategy, the initial consumption must not be higher than a certain threshold, otherwise the wealth will decline to zero. Endowment fund managers face a dilemma. In fact, the board may want to have a spending above a certain level, but in order to achieve the long term objectives of the fund, the fund must have a lower spending than the threshold.

The consumption for hybrid strategy is also less volatile in the long run and it converges towards its long term mean. Hybrid spending rule is more effective than consumption-wealth ratio, as spending stream in this rule is less volatile. Actually, an excessive volatility in the spending is not desirable.

Figure 3.6 shows ratios between consumption and wealth with different values of the weight ω for hybrid strategy. It appears that for the higher weight ω of the inflation method, the consumption-wealth ratio is relatively linear.

Figure 3.7 shows the effect of different values of constant y. We see that while y increases, so does the volatility in wealth and consumption. This figure illustrates that y must be chosen cautiously, to have a growth in wealth. The path of consumption also depends on the value of y. If it is higher than the optimal value, then the wealth will not grow in the long run and eventually, the consumption will also decline to a much lower level. If y is lower than the optimal level, then the wealth will grow in the long run but the consumption will remain lower even in the long run, provided that the values of other parameters remain unchanged.

3.5.2 Wealth invested in the risky asset

Figure 3.8 shows the amount and percentage of wealth invested in the risky asset under Merton's strategy with different values of the weights (ϕ_c and ϕ_R). Figure 3.9 illustrates the amount and percentage of wealth invested in the risky asset under CW strategy with different values of consumption-wealth ratio y with subsistence level $R_m = 0$. Both the figures 3.8 and 3.9 shows a constant percentage of wealth invested in the risky asset over time as all the parameters are constant. Figure 3.10 and 3.11 shows the amount and percentage of wealth invested in the risky asset under hybrid strategy with different values of the weight ω and with different values of y respectively.

Figure 3.5: Wealth and consumption under hybrid strategy with different values of the weight ω and the dashed lines show the confidence interval. The values of all parameters are estimated for the period 1997-2006 as stated in Tab. 3.2.

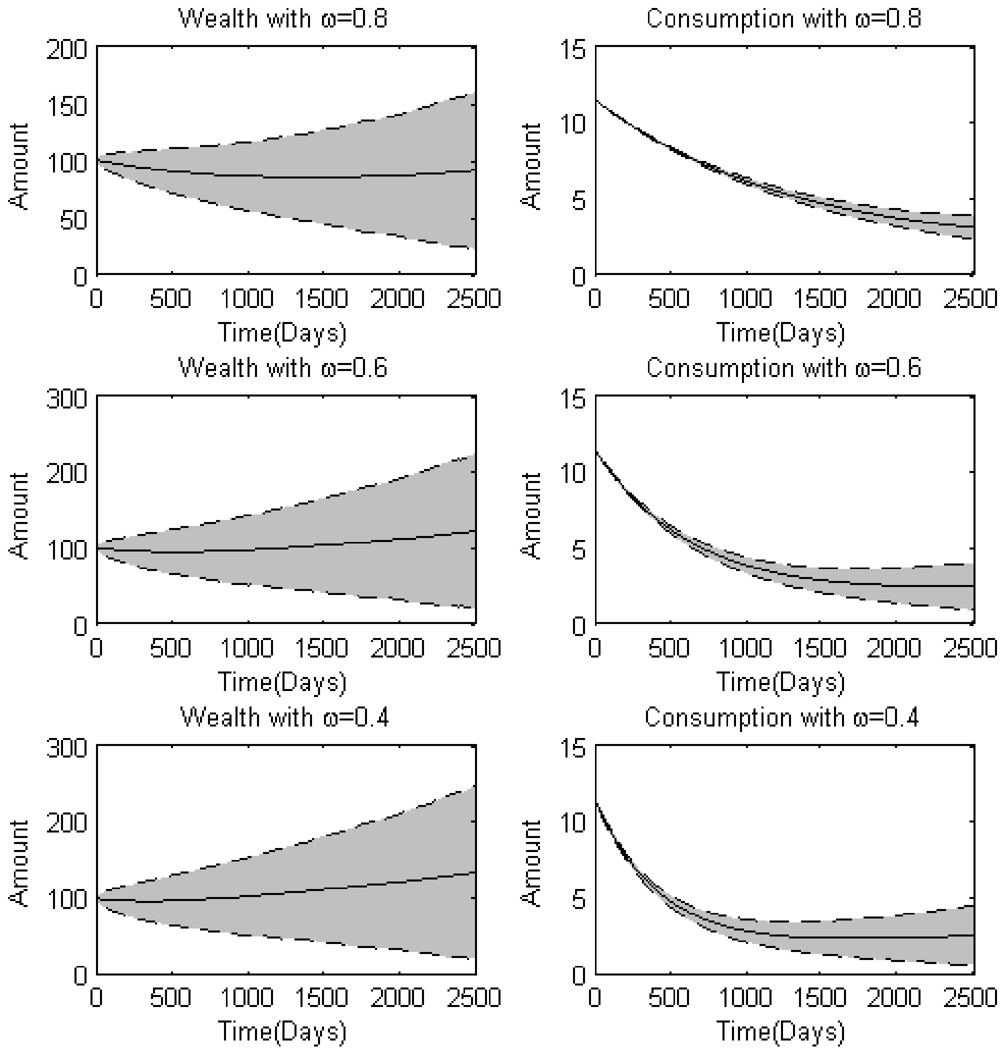


Figure 3.6: Hybrid strategy ratios between consumption and wealth with different values of the weight ω. The values of all parameters are estimated for the period 1997-2006 as stated in Tab. 3.2.

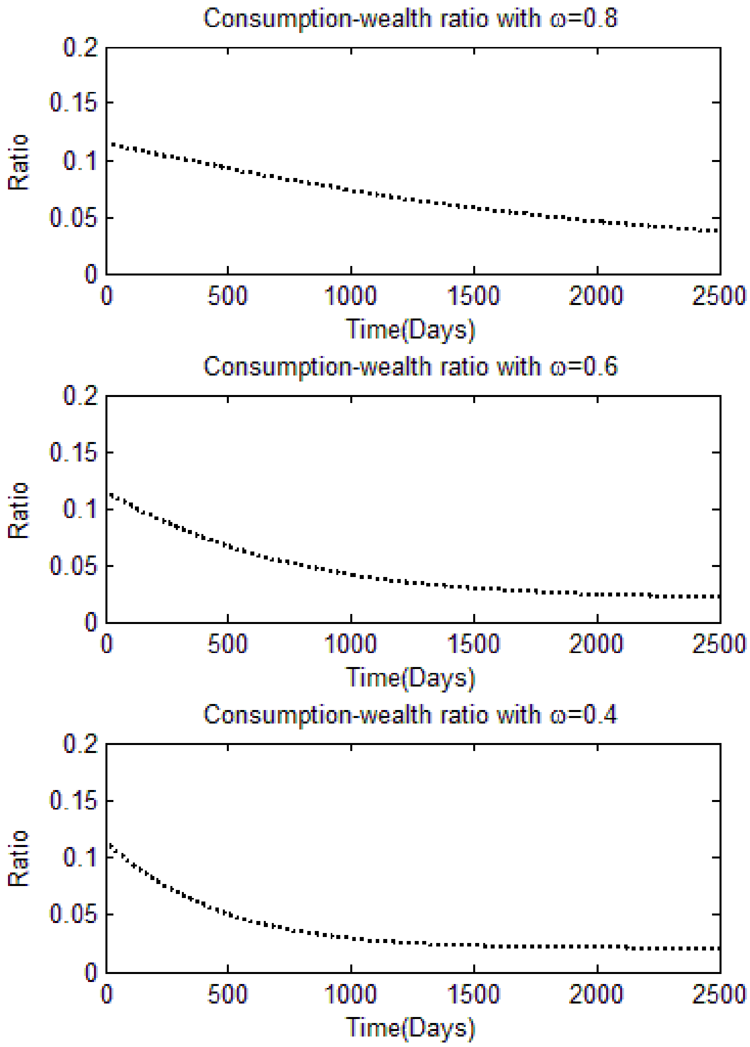


Figure 3.7: Wealth and consumption under hybrid strategy with different values of y and the dashed lines show the confidence interval. The values of all parameters are estimated for the period 1997-2006 as stated in Tab. 3.2.

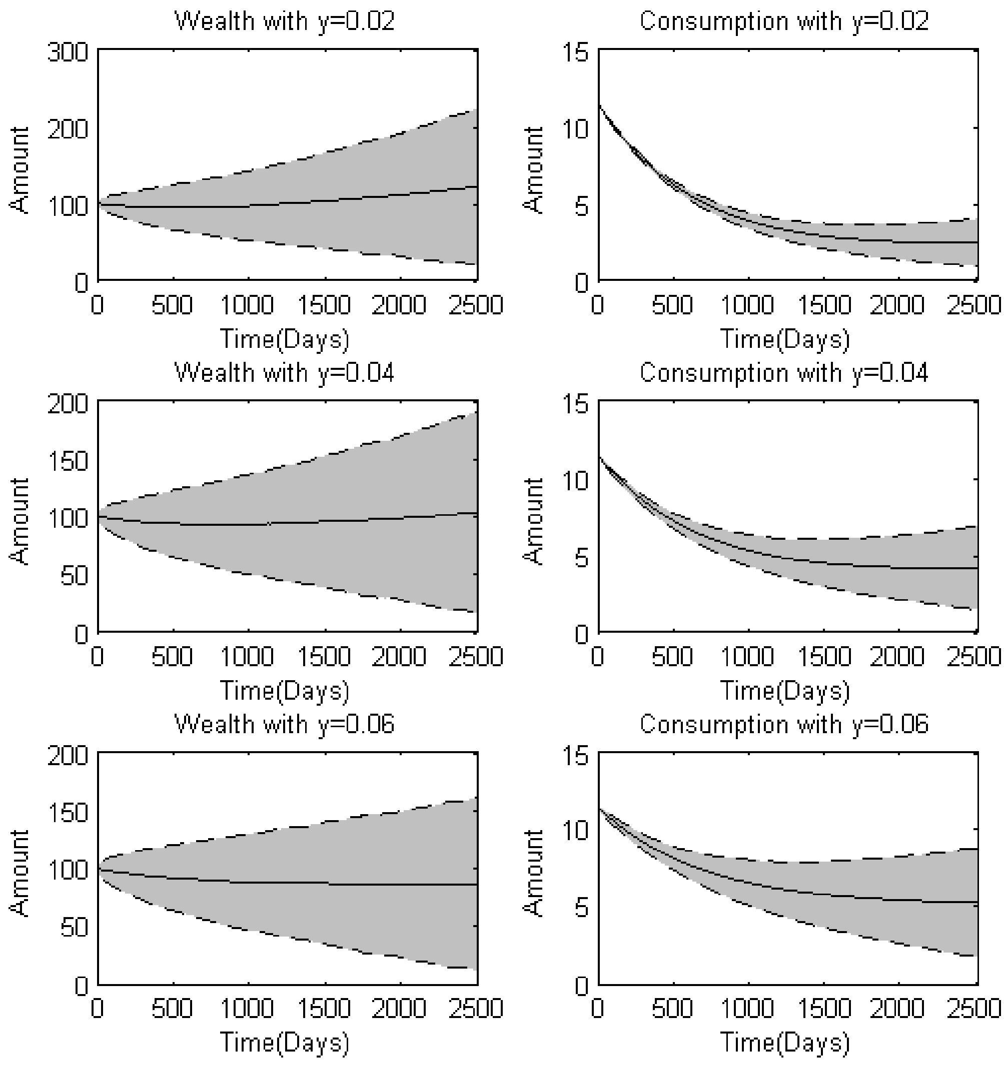


Figure 3.8: Wealth invested in the risky asset (left column graphs) and percentage of wealth invested in the risky asset (right column graphs) under Merton's strategy with different values of the weights (ϕ_c and ϕ_R). Dashed lines show the confidence interval. The values of all parameters are estimated for the period 1997-2006 as stated in Tab. 3.2.

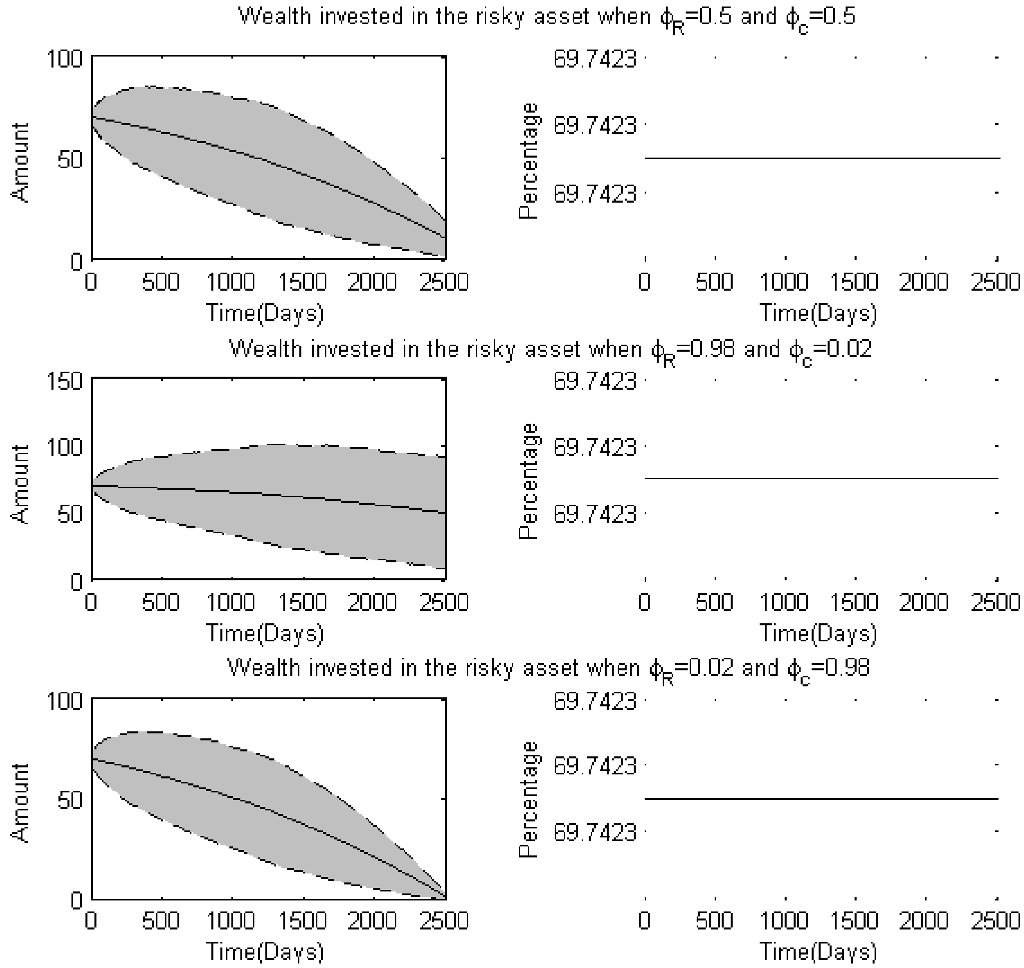


Figure 3.9: Wealth invested in the risky asset (left column graphs) and percentage of wealth invested in the risky asset (right column graphs) under CW strategy with different values of consumption-wealth ratio y with subsistence level $R_m = 0$. The values of all parameters are estimated for the period 1997-2006 as stated in Tab. 3.2.

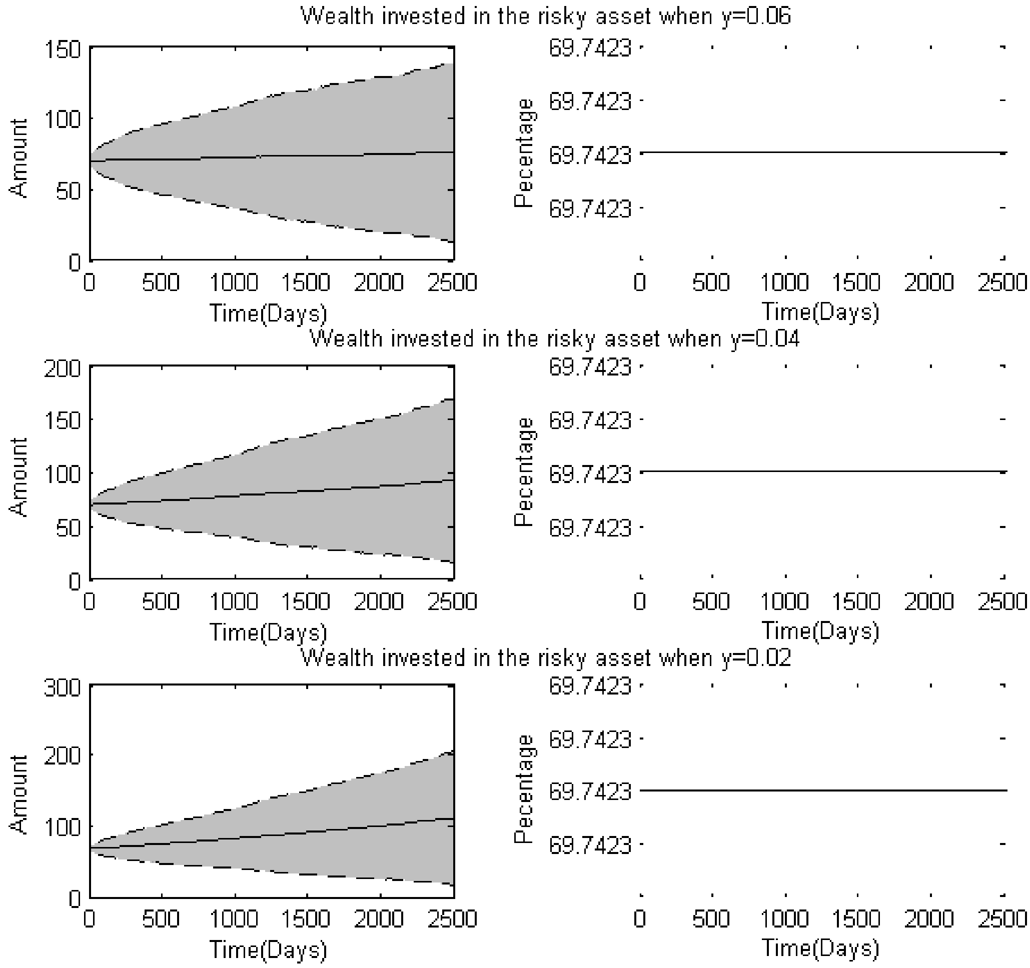


Figure 3.10: Wealth invested in the risky asset (left column graphs) and percentage of wealth invested in the risky asset (right column graphs) under hybrid strategy with different values of the weight ω when $y = 0.02$. The values of all parameters are estimated for the period 1997-2006 as stated in Tab. 3.2.

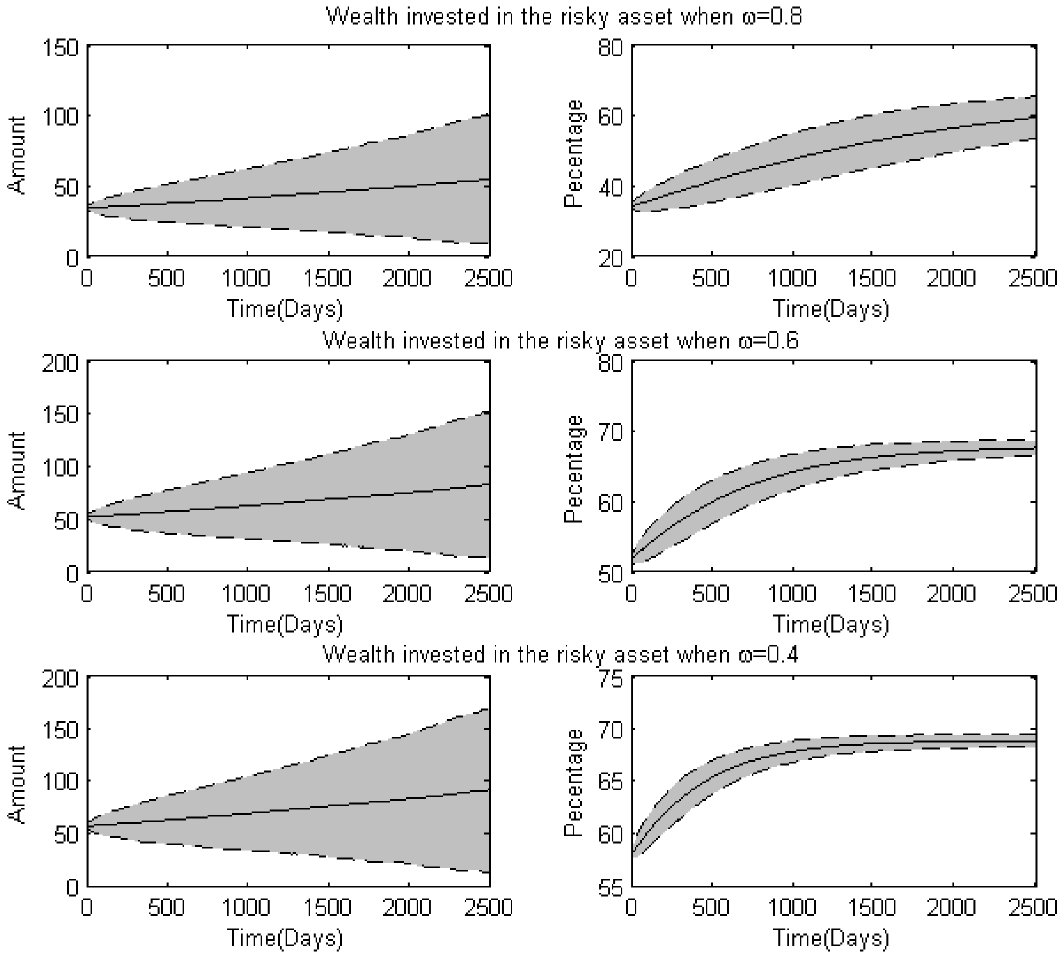


Figure 3.11: Wealth invested in the risky asset (left column graphs) and percentage of wealth invested in the risky asset (right column graphs) under hybrid strategy with different values of y when $\omega = 0.6$. The values of all parameters are estimated for the period 1997-2006 as stated in Tab. 3.2.

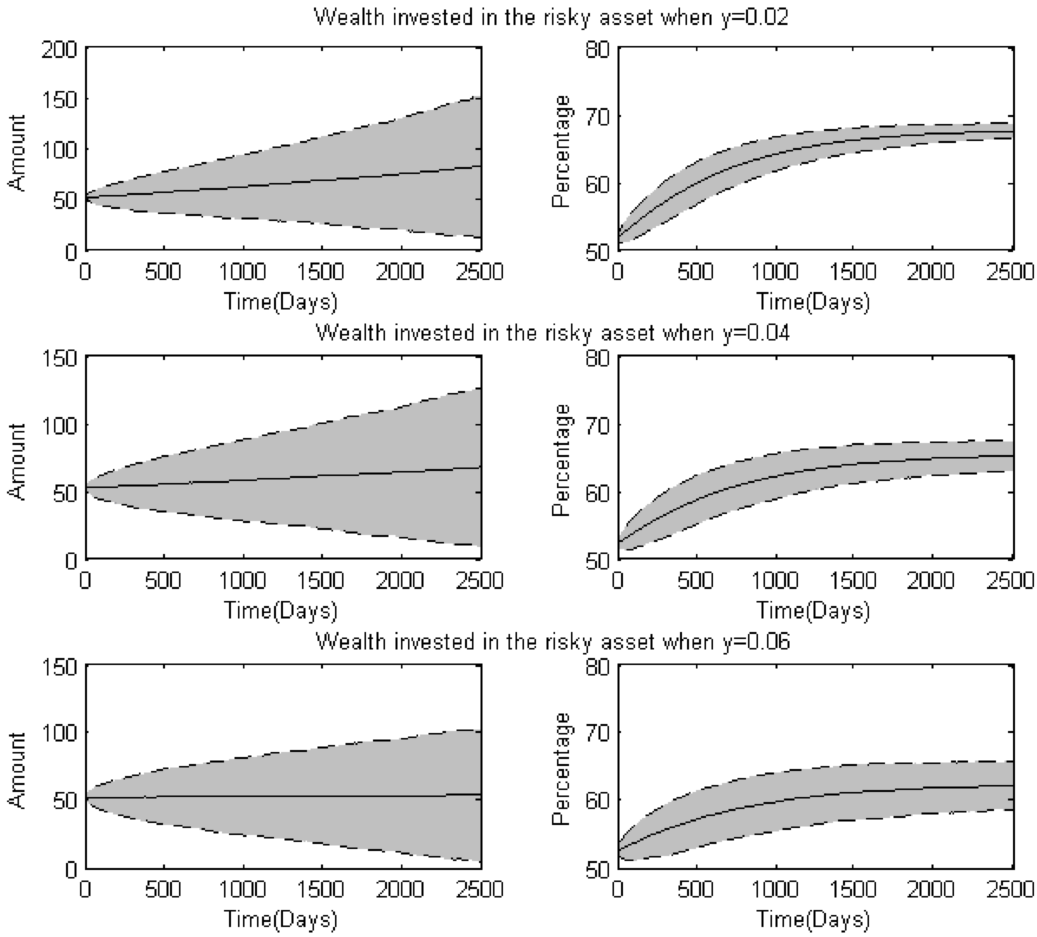
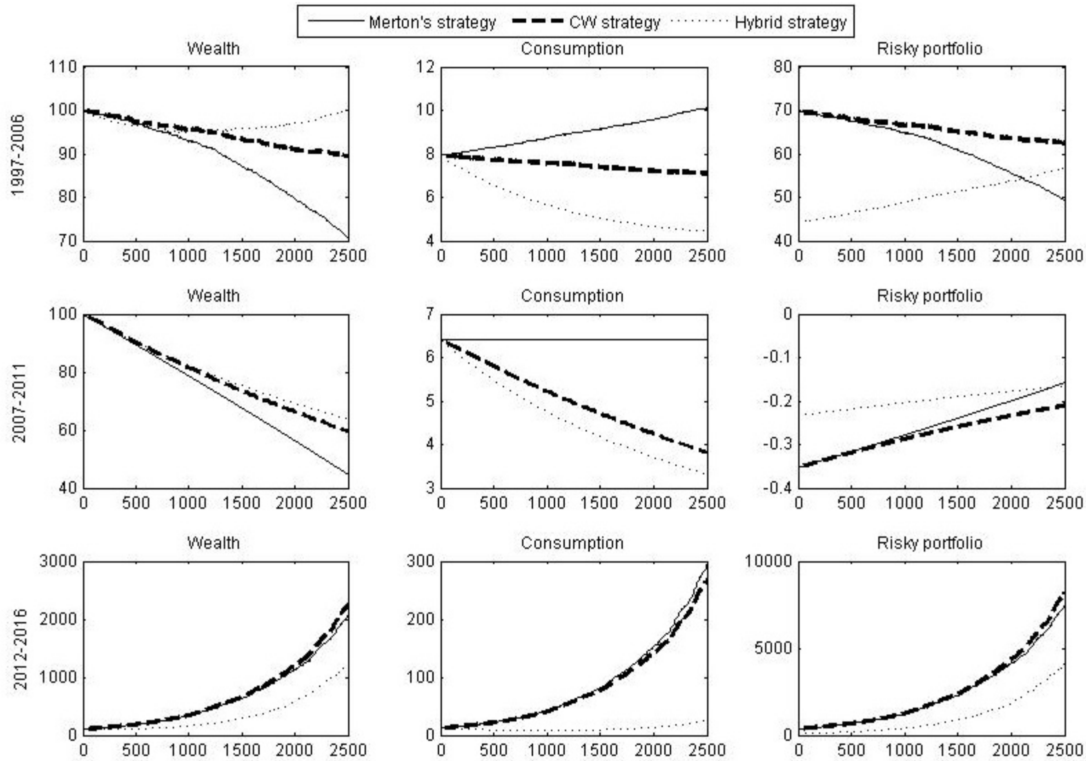


Figure 3.12: Comparison of Merton's strategy with $\phi_R = 0.98$ and $\phi_c = 0.02$ (continuous line), CW strategy with $y = 0.04$ (dashed line) and hybrid strategy with $\omega = 0.8$ and $y = 0.04$ (dotted line). The values of all parameters are estimated for the periods 1997-2006, 2007-2011, and 2012-2016 as stated in Tab. 3.2.



3.5.3 Comparison of the strategies

Fig. 3.12 shows the comparison of wealth, consumption and risky portfolio (all in monetary units) for Merton's, CW and hybrid strategies using the parameters estimated for three different periods: (i) 1997-2006, (ii) 2007-2011, and (iii) 2012-2016. The graphs of the first column show the dynamic behavior of wealth. We can see that:

- during the first period (1997-2006) under hybrid strategy, the wealth initially declines and then recovers while under Merton's strategy, it declines sharply.
- during the second period (2007-2011), wealth declines for all strategies but the magnitude of decline is far less for hybrid strategy compared to the other two strategies.
- during the third period (2012-2016), wealth rises for all three strategies but it

rises less strongly for hybrid strategy.

The graphs of the second column show the dynamic behavior of consumption. For the comparison of consumption we chose, the consumption-wealth ratio for CW strategy and initial consumption for hybrid strategy to match the initial optimal consumption under Merton's strategy. We can see that:

- during the first period, consumption rises under Merton's strategy while it declines under hybrid strategy in a short run and converges towards its long term mean.
- during the second period, consumption remains constant under Merton's strategy, however it declines under the other two strategies.
- during the third period, consumption increases greatly for CW and Merton's strategy while it increases with less intensity for hybrid strategy.

The third column graphs shows the risky portfolio. We can see that during the second period it is optimal to short sell the risky asset whereas in the third period it is optimal to short sell the riskless asset and invest in the risky asset.

3.6 Conclusion

This chapter has provided a brief overview of different spending rules applied by endowment funds. The endowment fund managers adopt these rules to effectively preserve the corpus of the fund and have a stable spending stream. We have obtained the optimal investment strategy under consumption-wealth ratio and hybrid spending rules. Furthermore, we have compared these optimal portfolios and defined spending rules with the classical Merton's optimal portfolio and consumption. We have found that the optimal Merton's portfolio is less riskier than that under consumption-wealth ratio rule, while the Merton's optimal consumption can be replicated using consumption-wealth rule by a suitable selection of the consumption-wealth ratio. The hybrid strategy, for some values of constant parameters, is less risky than both Merton's and consumption-wealth ratio, and consumption under this strategy is less volatile compared to other strategies. The unique characteristic of hybrid rule is that it allow fluctuation in spending during the short run. Also, it converges towards its long term mean regardless of the initial allocation for spending. However, hybrid strategy comparatively outperforms the conventional Merton's strategy and CW strategy when the market is highly volatile but under-performs them when there is a low volatility. Thus, an endowment fund must evaluate, review and modify its spending rule and investment policy periodically, depending on the conditions of the financial market.

3.A Appendix A

For simplification of the notation, we will use the following definitions throughout this appendix:

$$\frac{\partial A(t)}{\partial t} \equiv A_t, \quad \frac{\partial B(t,c(t))}{\partial t} \equiv B_t,$$

$$\frac{\partial B(t,c(t))}{\partial c} \equiv B_c, \quad \frac{\partial^2 B(t,c(t))}{\partial^2 c} \equiv B_{cc}.$$

3.A.1 Proof of proposition 2

The value function or indirect utility function is given by

$$J(t,R(t)) \equiv \max_{c(t),\theta_S(t)} \mathbb{E}_t \left[\int_t^T \phi_c \frac{(c(s)-h(s))^{1-\delta}}{1-\delta} e^{-\rho(s-t)} ds + \phi_R \frac{(R(T)-R_m)^{1-\delta}}{1-\delta} e^{-\rho(T-t)} \right].$$

We know that the above objective function must solve the following differential equation (so-called Hamilton-Jacobi-Bellman HJB equation):

$$0 = \max_{c(t),\theta_S(t)} \left\{ \begin{array}{c} \phi_c \frac{(c(s)-h(s))^{1-\delta}}{1-\delta} - \rho J + \frac{\partial J}{\partial t} + \frac{\partial J}{\partial h}(\alpha(t)c(t) - \beta(t)h(t)) \\ +\frac{\partial J}{\partial R}\big(R(t)r(t) + \theta_S(t)S(t)(\mu - r(t)) - c(t)\big) + \frac{1}{2}\frac{\partial^2 J}{\partial R^2}\theta_S(t)^2 S(t)^2\sigma^2 \end{array} \right\}. \tag{3.32}$$

The HJB equation (3.32) in $J(t, R(t))$, needs a boundary condition so that the value function coincide with the final utility function at the time T:

$$J(T, R(T)) = U(R(T)).$$

We assume the following guess function

$$J(t, R(t)) = A(t)^{\delta} \frac{(R(t) - h(t)B(t))^{1-\delta}}{1-\delta}, \tag{3.33}$$

where $A(t)$ and $B(t)$ are the functions that must be determined to solve equation (3.32). Both functions must satisfy boundary conditions as follows:

$$A(T)^{\delta} = \phi_R \Rightarrow A(T) = \phi_R^{\frac{1}{\delta}},$$

$$B(T) = R_m,$$

and the first order conditions (FOCs) of (3.32) w.r.t. $\theta_S(t)$ and $c(t)$ are:

$$\theta_S(t)^* = -\frac{\mu - r(t)}{S(t)\sigma^2} \frac{\frac{\partial J}{\partial R}}{\frac{\partial^2 J}{\partial R^2}}, \tag{3.34}$$

$$\phi_c \left(c(s) - h(s)\right)^{-\delta} = \frac{\partial J}{\partial R} - \frac{\partial J}{\partial h}\alpha(t). \tag{3.35}$$

By substituting the derivatives of the guess function into both the optimal consumption (3.35) and the optimal portfolio (3.34), we obtain

$$c(t)^* = h(t) + \phi_c^{\frac{1}{\delta}} \frac{\left(R(t) - h(t)B(t)\right)\left(1 + B(t)\alpha(t)\right)^{-\frac{1}{\delta}}}{A(t)}, \tag{3.36}$$

$$\theta_S(t)^* = \frac{\mu - r(t)}{S(t)\sigma^2} \frac{R(t) - h(t)B(t)}{\delta}. \tag{3.37}$$

Inserting the optimal portfolio (3.37), the optimal consumption (3.36), and the partial derivatives of the guess function into equation (3.32), we have

$$
\begin{aligned}
0 =& \frac{\delta}{1-\delta}\phi_c^{\frac{1}{\delta}}\left(1+B(t)\alpha(t)\right)^{1-\frac{1}{\delta}} - \frac{\rho A(t)}{1-\delta} \\
&+ \frac{\delta}{1-\delta}A_t - \frac{A(t)h(t)B_t}{R(t)-h(t)B(t)} \\
&- \frac{A(t)B(t)\alpha(t)h(t)}{R(t)-h(t)B(t)} + \frac{A(t)B(t)\beta(t)h(t)}{R(t)-h(t)B(t)} \\
&+ r(t)A(t) + \frac{A(t)h(t)B(t)r(t)}{R(t)-h(t)B(t)} + A(t)\frac{(\mu-r(t))^2}{2\sigma^2\delta} - \frac{A(t)h(t)}{R(t)-h(t)B(t)},
\end{aligned}
$$

which can be separated into two differential equations, one that consists of the terms containing $(R(t)-h(t)B(t))^{-1}$ and one without them and after few simplifications we have

$$
\begin{cases}
0 & = \phi_c^{\frac{1}{\delta}}\left(1+B(t)\alpha(t)\right)^{\frac{\delta-1}{\delta}} + A_t + A(t)\left(\frac{1-\delta}{\delta}r(t) - \frac{\rho}{\delta} + \frac{(1-\delta)(\mu-r(t))^2}{2\delta^2\sigma^2}\right) \\
0 & = B_t + B(t)\left(\alpha(t)-\beta(t)-r(t)\right) + 1
\end{cases}
\tag{3.38}
$$

The above ordinary differential equations, together with their corresponding boundary conditions, have the following unique solutions:

$$
\begin{aligned}
A(t) =& \phi_R^{\frac{1}{\delta}} e^{-\int_t^T\left(\frac{\delta-1}{\delta}r(u)+\frac{\rho}{\delta}+\frac{(\delta-1)(\mu-r(u))^2}{2\delta^2\sigma^2}\right)du} \\
&+ \int_t^T \phi_c^{\frac{1}{\delta}}\left(1+B(s)\alpha(s)\right)^{\frac{\delta-1}{\delta}} e^{-\int_t^s\left(\frac{\delta-1}{\delta}r(u)+\frac{\rho}{\delta}+\frac{(\delta-1)(\mu-r(u))^2}{2\delta^2\sigma^2}\right)du} ds, \\
B(t) =& R_m e^{-\int_t^T(-\alpha(u)+\beta(u)+r(u))du} + \int_t^T e^{-\int_t^s(-\alpha(u)+\beta(u)+r(u))du} ds.
\end{aligned}
$$

- when $h(t)$ is constant, i.e. $h(t)=h$, the optimal consumption and portfolio is given by

$$
c(t)^* = h + \phi_c^{\frac{1}{\delta}}\frac{R(t)-hB(t)}{A(t)},
$$

$$
\theta_S(t)^* = \frac{\mu-r(t)}{S(t)\sigma^2}\frac{R(t)-hB(t)}{\delta},
$$

where the unknown functions are given by

$$A(t) = \phi_R^{\frac{1}{\delta}} e^{-\int_t^T \left(\frac{\delta-1}{\delta} r(u) + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu-r(u))^2}{2\delta^2\sigma^2}\right) du} + \int_t^T \phi_C^{\frac{1}{\delta}} e^{-\int_t^s \left(\frac{\delta-1}{\delta} r(u) + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu-r(u))^2}{2\delta^2\sigma^2}\right) du} ds,$$

$$B(t) = R_m e^{-\int_t^T r(u) du} + \int_t^T e^{-\int_t^s r(u) du} ds.$$

- when $h(t) = 0$, the optimal consumption and portfolio is given by

$$c(t)^* = \phi_C^{\frac{1}{\delta}} \frac{R(t)}{A(t)},$$

$$\theta_S(t)^* = \frac{\mu - r(t)}{S(t)\sigma^2} \frac{R(t)}{\delta},$$

where the unknown functions are given by

$$A(t) = \phi_R^{\frac{1}{\delta}} e^{-\int_t^T \left(\frac{\delta-1}{\delta} r(u) + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu-r(u))^2}{2\delta^2\sigma^2}\right) du} + \int_t^T \phi_C^{\frac{1}{\delta}} e^{-\int_t^s \left(\frac{\delta-1}{\delta} r(u) + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu-r(u))^2}{2\delta^2\sigma^2}\right) du} ds,$$

$$B(t) = R_m e^{-\int_t^T r(u) du} + \int_t^T e^{-\int_t^s r(u) du} ds.$$

3.A.2 Proof of proposition 3

The value function is given by

$$J(t, R(t)) \equiv \max_{\theta_S(t)} \mathbb{E}_t \left[\frac{(R(T) - R_m)^{1-\delta}}{1-\delta} e^{-\rho(T-t)} \right].$$

For this objective function, we can write the following HJB equation

$$0 = \max_{\theta_S(t)} \left\{ \begin{array}{c} -\rho J + \frac{\partial J}{\partial t} + \frac{\partial J}{\partial R} \left(R(t) r(t) + \theta_S(t) S(t) (\mu - r(t)) - c(t) \right) \\ + \frac{1}{2} \frac{\partial^2 J}{\partial R^2} \theta_S(t)^2 S(t)^2 \sigma^2 \end{array} \right\}, \quad (3.39)$$

and the first order condition (FOC) of (3.39) w.r.t. $\theta_S(t)$ is

$$\theta_S(t)^* = -\frac{\mu - r(t)}{S(t)\sigma^2} \frac{\frac{\partial J}{\partial R}}{\frac{\partial^2 J}{\partial R^2}}. \quad (3.40)$$

We assume the following guess function

$$J(t,R(t)) = A(t)^{\delta}\frac{\left(R(t)-B(t)\right)^{1-\delta}}{1-\delta}, \tag{3.41}$$

where $A(t)$ is the function that must solve equation (3.39), with the boundary condition

$$A(T) = 1,$$

while $B(t)$ must satisfy the boundary condition

$$B(T) = R_m,$$

and the optimal portfolio process $\theta_S(t)^*$ in (3.40), for our guess function can be written as

$$\theta_S(t)^* = \frac{\mu - r(t)}{S(t)\sigma^2}\frac{R(t)-B(t)}{\delta}. \tag{3.42}$$

Substituting the optimal portfolio (3.42) and partial derivatives of the guess function into (3.39), we get

$$\begin{aligned} 0 = {} & \frac{\delta}{1-\delta}A_t + A(t)\left(r(t) - y - \frac{\rho}{1-\delta} + \frac{1}{2}\frac{\left(\mu - r(t)\right)^2}{\sigma^2\delta}\right) \\ & + \frac{A(t)}{R(t)-B(t)}\left(B(t)\big(r(t)-y\big) - B_t\right), \end{aligned}$$

which can be separated into two differential equations, one that consists of the terms containing $(R(t) - B(t))^{-1}$ and one without them and after few simplifications, we have

$$\begin{cases} 0 = & A_t + A(t)\left(\frac{1-\delta}{\delta}\big(r(t)-y\big) + \frac{1-\delta}{2\delta^2}\frac{(\mu-r(t))^2}{\sigma^2} - \frac{\rho}{\delta}\right), \\ 0 = & B(t)\big(r(t)-y\big) - B_t. \end{cases}$$

The above ordinary differential equations with their corresponding boundary conditions have the following solutions:

$$A(t) = e^{-\int_t^T \left(\frac{\delta-1}{\delta} r(s) - \frac{\delta-1}{\delta} y + \frac{\delta-1}{2\delta^2} \frac{(\mu - r(s))^2}{\sigma^2} - \frac{\rho}{\delta}\right) ds},$$

$$B(t) = R_m e^{-\int_t^T (r(s)-y) ds}.$$

3.A.3 Proof of proposition 4

The value function is given by

$$J(t, R(t), c(t)) \equiv \max_{\theta_S(t)} \mathbb{E}_t \left[\frac{(R(T) - R_m)^{1-\delta}}{1-\delta} e^{-\rho(T-t)} \right].$$

For this objective function, we can write the following HJB equation

$$0 = \max_{\theta_S(t)} \left\{ \begin{array}{c} -\rho J + \frac{\partial J}{\partial t} + \frac{\partial J}{\partial R}\big(R(t)r(t) + \theta_S(t)S(t)(\mu - r(t)) - c(t)\big) \\ +\frac{\partial J}{\partial c}\left(1-\omega\right)\left(1+y\right)\left(\frac{yR(t)(1+r(t)) + y\theta_S(t)S(t)(\mu - r(t))}{1+y} - c(t)\right) + \frac{1}{2}\frac{\partial^2 J}{\partial R^2}\theta_S(t)^2 S(t)^2 \sigma^2 \\ +\frac{1}{2}\frac{\partial^2 J}{\partial c^2} a^2 \theta_S(t)^2 S(t)^2 \sigma^2 + \frac{\partial^2 J}{\partial c \partial R} a \theta_S(t)^2 S(t)^2 \sigma^2 \end{array} \right\}, \tag{3.43}$$

and the first order condition (FOC) of (3.43) w.r.t. $\theta_S(t)$ is

$$\theta_S(t)^* = -\frac{\mu - r(t)}{S(t)\sigma^2} \frac{\frac{\partial J}{\partial R} + \frac{\partial J}{\partial c} a}{\frac{\partial^2 J}{\partial R^2} + \frac{\partial^2 J}{\partial c^2} a^2 + 2\frac{\partial^2 J}{\partial c \partial R} a}. \tag{3.44}$$

We assume the following guess function

$$J(t, R(t), c(t)) = A(t)^{\delta} \frac{(R(t) - B(t, c(t)))^{1-\delta}}{1-\delta}. \tag{3.45}$$

Thus, the optimal portfolio process $\theta_S(t)^*$ in (3.44), for our guess function can be written as

$$\theta_S(t)^* = -\frac{\mu - r(t)}{S(t)\sigma^2} A(t)^{\delta} \left(R(t) - B(t, c(t))\right)^{-\delta} \frac{1 - B_c a}{F(t, c(t))},$$

where we define

$$
\begin{aligned}
F(t,c(t)) = & -\delta A(t)^{\delta}(R(t)-B(t,c(t)))^{-\delta-1} - \delta A(t)^{\delta}(R(t)-B(t,c(t)))^{-\delta-1}B_c^2 a^2 \\
& - A(t)^{\delta}(R(t)-B(t,c(t)))^{-\delta}B_{cc}a^2 + 2\delta A(t)^{\delta}(R(t)-B(t,c(t))^{-\delta-1}B_c a.
\end{aligned}
$$

The HJB equation (3.43), under the hypotheses $B_c = \eta$ and $B_{cc} = 0$, becomes

$$
\begin{aligned}
0 = & -A(t)\frac{\rho}{1-\delta} + \frac{\delta}{1-\delta}A_t - A(t)\frac{B_t}{R(t)-B(t,c(t)} + A(t)r \\
& + A(t)\frac{rB(t,c(t))}{R(t)-B(t,c(t))} - A(t)\frac{c(t)}{R(t)-B(t,c(t))} - A(t)\eta a(1+r) \\
& - A(t)\frac{\eta a(1+r)B(t,c(t))}{R(t)-B(t,c(t))} + A(t)\frac{\eta\,(1-\omega)\,c(t)}{R(t)-B(t,c(t))} + A(t)\frac{\eta a c(t)}{R(t)-B(t,c(t))} \\
& + \frac{A(t)(\mu-r)^2}{2\sigma^2\delta},
\end{aligned}
$$

which can be separated into two differential equations, one that consists of the terms containing $(R(t)-B(t,c(t)))^{-1}$ and one without them and after few simplifications, we have

$$
\begin{cases}
0 & = A_t + A(t)\left(\frac{1-\delta}{\delta}r - \frac{\rho}{\delta} - \frac{1-\delta}{\delta}\eta a(1+r) + \frac{1-\delta}{\delta}\frac{(\mu-r)^2}{2\sigma^2\delta}\right), \\
0 & = B_t - B(t,c(t))\,(r-\eta a(1+r)) + c(t)\,(1-\eta\,(1-\omega)-\eta a)\,,
\end{cases}
\tag{3.46}
$$

and the optimal portfolio in this case can be written as using (3.44)

$$
\theta_S(t)^* = \frac{\mu-r}{\delta S(t)\sigma^2}\frac{R(t)-B(t,c(t))}{1-\eta a}.
$$

These equations are both ordinary linear differential equations and their boundary conditions can be obtained from the boundary condition of the HJB equation:

$$
A(T,c(T)) = 1,
$$

$$
B(T,c(T)) = R_m.
$$

The solution of the above ODE (3.46), together with their boundary conditions is given by

$$A(t) = e^{-\left(\frac{\delta-1}{\delta}r+\frac{\rho}{\delta}-\frac{\delta-1}{\delta}\eta a(1+r)+\frac{\delta-1}{\delta}\frac{(\mu-r)^2}{2\sigma^2\delta}\right)(T-t)}.$$

Since the second differential equation has been obtained under the hypothesis that $B_c = \eta$ and $B_{cc} = 0$, then the only consistent functional form for $B(t, c(t))$ is

$$B(t, c(t)) = \eta\left(t\right) c(t) + h\left(t\right),$$

where $\eta\left(t\right)$ and $h\left(t\right)$ may be functions of time. Thus the second ODE can be rewritten as follows

$$\left(\frac{\partial \eta\left(t\right)}{\partial t}c(t) + \frac{\partial h\left(t\right)}{\partial t}\right) - \left(r - \eta\left(t\right) a\left(1+r\right)\right)\left(\eta\left(t\right) c(t) + h\left(t\right)\right) - \left(\eta\left(t\right)\left(1-\omega+a\right)-1\right)c(t) = 0,$$

which can be separated into two ODE's, one which contains $c(t)$ and one which contains all the terms without $c(t)$

$$0 = \frac{\partial \eta\left(t\right)}{\partial t} - \left(r - \eta\left(t\right) a\left(1+r\right)\right)\eta\left(t\right) - \left(\eta\left(t\right)\left(1-\omega+a\right)-1\right), \tag{3.47}$$

$$0 = \frac{\partial h\left(t\right)}{\partial t} - \left(r - \eta\left(t\right) a\left(1+r\right)\right) h\left(t\right), \tag{3.48}$$

with the boundary condition

$$\eta\left(T\right) c(T) + h\left(T\right) = R_m,$$

as

$$B(T, c(T)) = R_m,$$

and consequently,

$$\eta\left(T\right) = 0,$$
$$h\left(T\right) = R_m,$$

which means that η must be a constant and the ODE (3.47) (with $\frac{\partial \eta(t)}{\partial t} = 0$) gives

$$\eta^2 a\left(1+r\right) - \eta\left(1+r+a-\omega\right) + 1 = 0,$$

which has two solutions. One of the two must be suitably chosen (η^*):

$$\eta = \frac{\left(1+r+a-\omega\right) \pm \sqrt{\left(1+r+a-\omega\right)^2 - 4a\left(1+r\right)}}{2a\left(1+r\right)}.$$

The ODE (3.48), together with the boundary condition, has a unique solution

$$h\left(t\right) = R_m e^{-(r-\eta^* a(1+r))(T-t)}.$$

Therefore,

$$B(t, c(t)) = \eta^* c(t) + R_m e^{-(r-\eta^* a(1+r))(T-t)}, \tag{3.49}$$

such that $-\infty < \eta^* < \frac{1}{a}$.

Chapter 4

Capital adequacy management for banks in the Lévy market

4.1 Introduction

Bank management mainly consists of four closely linked operations: (i) asset management, (ii) capital adequacy management, (iii) liability management, and (iv) liquidity management. Capital adequacy management, from the regulatory perspective, is one of the most important component in the banking operations. It defines, implements and monitors the banking operations by imposing certain limitations on risk-taking. The primary risks associated with capital adequacy management are credit risk, market risk, and operational risk. Capital adequacy management involves the determination of the amount of capital that a bank is required to hold compared to the amount of assets, to comply with the minimum capital requirements established by the regulator. Higher level of bank capital benefits the bank owners because it reduces the chances of bank failure while it is costly because it lowers the return on equity. The Basel Committee on Banking Supervision (BCBS), supervises and regulates the international banking industry. The committee formulates the international minimum standards on bank capital adequacy to ensure that banks can absorb unexpected losses during a period of crisis and promotes effectiveness and stability of banks. The capital accord Basel I was developed to assess the bank's capital compared to the bank's credit risk. It was revised to Basel II to make the minimal capital requirements more risk sensitive. Basel III, reinforces the capital requirement under previous accords by decreasing bank leverage, increasing bank liquidity and requiring reserves for different forms of deposits and borrowings.

Capital adequacy ratio acts as an index of capital adequacy of banks. In general, it is calculated by dividing a measure of bank capital by an indicator of the level of bank risk. In this study, we consider the ratio of total bank capital to the total risk-weighted assets (TRWAs) as capital adequacy ratio (CAR). CAR mandates the international banks to hold capital in proportion to their perceived risks, as portfolio position is important for risk management strategy. If we denote the total bank capital by C and the total risk-weighted assets by A then CAR Γ is given by

$$\Gamma = \frac{C}{A},$$

where TRWAs are constituted by the capital charges for credit, market and operational risks.

In this chapter, we consider the problem of a bank which can invest in three assets: a treasury, a stock index, and loans. We determine the optimal investment portfolio under constant absolute risk aversion (CARA) preferences and then this portfolio is used to model the dynamics of CAR. Our study has two roots in the literature: (i) capital adequacy management, and (ii) optimal investment in the Lévy market.

Several studies have applied stochastic optimization methods to asset management and capital adequacy management in banking. Mukuddem-Petersen and Petersen [2006] built a stochastic model for banks and minimized the market and capital adequacy risks by selection of security allocation and capital requirements, respectively. They suggest an optimal portfolio choice and rate of bank capital inflow that will keep the loan level as close as possible to an actuarially determined reference process. Fouche et al. [2006] constructed continuous-time stochastic models for the dynamics of non-risk-based and risk-based CAR's. Mukuddem-Petersen et al. [2007] solved the maximization problem involving the expected utility of discounted depository consumption, a consumption of the bank's profits by the taking and holding of deposits, over a random time interval and profit at terminal time.

They analyzed different aspects of their banking model against the regulations of the Basel II capital accord. Mukuddem-Petersen and Petersen [2008] optimized CAR by optimal allocation of bank's equity using the rate at which additional equity and debt are raised. Mulaudzi et al. [2008] investigated the bank's investment in loans and treasuries with the objective of generating an optimal final fund level in the presence of behavioral aspects such as risk and regret. Witbooi et al. [2011] incorporated the stochastic interest rate which follows CIR model and derived an optimal equity allocation strategy for banks. They monitored the performance of the Basel II CAR under the allocation strategy. Perera [2015] solved the optimal portfolio choice problem to maximize expected utility of wealth of bank's shareholders at a given investment horizon in an environment subject to stochastic interest rate and inflation uncertainty following correlated Ornstein–Uhlenbeck processes.

The second root of our research is optimal investment problems, which are based on two approaches: (i) the dynamic programming approach which dates back to the seminal work of Merton [1969, 1971], and (ii) the martingale approach developed by Cox and Huang [1989]. Merton's work has been modified and extended in many subsequent papers, such as Lehoczky et al. [1983], Karoui et al. [2005], and Choulli et al. [2003]. Further extensions have incorporated the Lévy market in Merton's framework. The pioneering contributions on portfolio optimization with jump-diffusion processes includes Aase [1982], Aase [1984], Aase [1986], Aase [1988], and Benth et al. [2001]. Recently, a number of authors have focused on the addition of Lévy processes in the insurers risk process in optimal investment-reinsurance problems. Yang and Zhang [2005] studied the optimal investment problem for an insurer with controlled jump-diffusion risk process and derived the closed form solution for general objective function. The objective function included both the terminal wealth and the survival probability of the insurer. Zhao et al. [2013] found the optimal investment–reinsurance strategy for an insurer, whose surplus process is governed by compound Poisson risk process perturbed by diffusion process in a financial market with one risk-free asset and one risky asset whose price follows the Heston model. Liu et al. [2013] studied the maximization of expected exponential utility function for an insurer who can purchase reinsurance with value-at-risk constraint on the portfolio. They solved the problem by using decomposition approach for both complete and incomplete markets. Sheng et al. [2014] investigated the optimal control strategy of excess-of-loss reinsurance and investment problem for an insurer. The model takes into account a compound Poisson jump diffusion risk process with the risk asset price modeled by a constant elasticity of variance (CEV) model. Zhu et al. [2015] analyzed the optimal proportional reinsurance-investment problem for a default able market by decomposing the problem into two sub-problems: a pre-default case, and a post-default case. The study extended the insurer's problem of reinsurance-investment by the addition of a corporate bond and the optimal strategy that maximizes the expected CARA utility of the terminal wealth was explicitly derived.

In this chapter, we use the martingale approach to solve our problem. It is based on equivalent martingale measures and martingale representation theorems, see Kramkov and Schachermayer [1999] for the detailed description. It was first developed and applied in the continuous time by Harrison and Kreps [1979]. Thereafter, it was applied to many problems including the optimal investment and consumption problem by Karatzas and Shreve [1998], and optimal investment problem for insurers with the risk process modeled by a Lévy process in Wang et al. [2007]. Zhou [2009] extended the problem considered by Wang et al. [2007] with the addition of a Lévy process for the risky asset. Lòpez and Serrano [2015] studied the optimal investment-consumption problem in a pure-jump asset pricing model with regime switching framework. Zou and Cadenillas [2014] modeled the insurer's risk process by a jump diffusion process which is negatively correlated with the capital gains in the financial market and obtained explicit solutions for various utility functions.

This chapter mainly addresses optimal portfolio selection problem incorporating the bank's risk process using the martingale approach. We first solve the asset allocation problem and then the capital adequacy ratio process of the bank is derived, conditional on the optimal policy chosen. In comparison with the Merton's approach, we add a jump-diffusion process in the stock that is simultaneous to the jump in the bank's risk process modeled by Cramer-Lundberg model. Furthermore, we have used a jump process to model the expected losses that covered the loan loss provision.

In our framework, the bank's investment manager dynamically chooses the amount for investment in the risky assets (i.e. stocks and loans). We derive the optimal investment strategy of the bank's investments in the stocks and loans with CARA utility function.

The outline of this chapter is as follows. Section 4.2 presents the stochastic model for a bank and formulates the optimization problem for the bank's asset portfolio. In Section 4.3, we derive and verify the optimal solution under exponential utility function. Section 4.4 shows the derivation of the dynamics of CAR and Section 4.5 concludes this chapter.

4.2 General framework

Stochastic framework can reasonably represent the dynamic nature of different items on the bank's balance sheet and the uncertainty associated with them. The model presented here is highly simplified for tractability purposes and omits many important aspects of bank operations.

4.2.1 Stochastic model of a bank

Banks sell liabilities and uses the proceeds from the sales to buy assets with different properties, this process is called asset transformation. Bank's balance sheet records the utilization of funds (assets) and sources of funds (liabilities). The items on the balance sheet exhibit uncertainty due to unpredictable nature of risky investments, loan demands and repayments, deposits, borrowings and regulatory capital. The bank's balance sheet has the following well-known relationship

$$\text{Bank Capital=Total Assets} - \text{Total Liabilities.}$$

As defined by Mukuddem-Petersen and Petersen [2006] and Witbooi et al. [2011], a typical commercial bank's balance sheet at time t can be represented as

$$\underbrace{C(t)}_{\text{Capital}} = \underbrace{G(t) + S(t) + L(t)}_{\text{Assets}} - \underbrace{(D(t) + B(t))}_{\text{Liabilities}}, \tag{4.1}$$

where G,S,L,D,B,C: $\Omega \times \mathbb{R}_{+} \to \mathbb{R}$ are treasury, security, loans, deposits, borrowings, and bank capital respectively.

We consider a continuously open and frictionless financial market over the fixed time interval $[0,T]$, where trading of assets in fractional units is permitted. We define a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$, where Ω is the information structure and $\mathbb{P}$ denotes the real world probability measure on Ω.

4.2.2 Assets

In the financial market there are three assets available for investment: (i) a treasury $G(t)$, (ii) a security $S(t)$, and (iii) loans $L(t)$. The bank's objective is to invest in assets with low default probability and follow a sufficiently diversified investment strategy. In this section, we model the risks associated with these assets by using stochastic processes.

The treasury is a bond issued by the national Treasury. We consider the treasury $G(t)$, as a riskless asset which evolves according to

$$\frac{dG(t)}{G(t)} = r(t)dt, \tag{4.2}$$

where $G(0) = 1$, and it is the *numéraire* and $r(t)$ is the instantaneous nominal riskless interest rate.

A marketable security is a stock index fund $S(t)$ whose price dynamics are governed by the following jump-diffusion process

$$\frac{dS(t)}{S(t)} = \mu_S dt + \sigma_1 dW_1(t) + \gamma_1 dN_1(t), \tag{4.3}$$

where μ_S is the appreciation rate and σ_1 is the positive volatility parameters, $W_1(t)$ is a one-dimensional standard Brownian motion and $N_1(t)$ is a one-dimensional Poisson component with the intensity λ and the size of the jump is γ_1 which satisfies $0 > \gamma_1 \geq -1$.

The evolution of the price dynamics of the total loans $L(t)$ is governed by the following diffusion process

$$\frac{dL(t)}{L(t)} = \mu_L dt + \sigma_2 dW_2(t), \tag{4.4}$$

where μ_L is the appreciation rate, $\sigma_2 \in \mathbb{R}_+$ is the volatility and $dW_2(t)$ is another standard Brownian motion.

We capture the bank's risk by using Cramer-Lundberg model, a classical model used in the insurance industry to model claims. The cumulative amount of losses is given by $\sum_{i=1}^{N(t)} l_i$, where $\{l_i\}$ is a series of independent and identically distributed (i.i.d) random variables and $N(t)$ is a homogeneous Poisson process with intensity λ and independent of l_i. If the mean of l_i and intensity of $N(t)$ are finite, then this compound Poisson process is a Lévy process with finite Lévy measure. We define $M_i(t) := N_i(t) - \lambda t$ as the compensated Poisson process of $N_i(t)$. Bank's risk process is given by

$$dR(t) = adt + bd\bar{W}(t) + \gamma_2 dN_2(t) + \gamma_{12} dN_1(t), \tag{4.5}$$

where a, b are positive constants and $\bar{W}(t)$ is a one-dimensional standard Brownian motion. As the capital gains in the financial market are negatively correlated with the liabilities, we denote ρ as the correlation coefficient between $W_1(t)$ and $\bar{W}(t)$, which implies

$$\bar{W}(t) = \rho W_1(t) + \sqrt{1-\rho^2} W_3(t), \tag{4.6}$$

where $W_3(t)$ is another standard Brownian motion independent of $W_1(t)$ and $W_2(t)$. There are three special cases for (4.6): (i) if $\bar{W}(t)$ is not correlated with $W_1(t)$ that is, $\rho = 0$ then $\bar{W}(t)$ is equal to $W_3(t)$, (ii) if $\rho^2 = 1$, then $\bar{W}(t)$ equals $W_1(t)$. In this case, the risky assets and the liabilities are driven by the same source of randomness, and (iii) if $\rho^2 \leq 1$ it means that the risk from liabilities cannot be eliminated by trading the financial assets. After substituting (4.6) into (4.5) we can write

$$dR(t) = adt + b\rho dW_1(t) + b\sqrt{1-\rho^2}dW_3(t) + \gamma_{12}dN_1(t) + \gamma_2 dN_2(t). \tag{4.7}$$

Bank's usually have provisions for some expected losses accumulated in the loan loss reserve. If the loss $l(t)$ is such a loss diminished by loss covered by provision of loan losses $P(l(s))$ then $\gamma_2 = [l(s) - P(l(s))]$ and γ_{12} is the size of jump causing unexpected losses. We can rewrite the risk process (4.7) as

$$dR(t) = adt + b\rho dW_1(t) + b\sqrt{1-\rho^2}dW_3(t) + \gamma_{12}dN_1(t) + [l(s) - P(l(s))]\, dN_2(t). \tag{4.8}$$

In our settings $N_1(t)$ produces simultaneous jump in bank's risk process and stock index fund. As under Basel III, banks must hold sufficient amount of capital to cover for unexpected losses where expected losses are provided by provisions. In addition, $W_1(t)$, $W_2(t)$, $W_3(t)$, $N_1(t)$ and $N_2(t)$ are mutually independent and are all defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ where $\mathcal{F}_t$ is the usual augmentation of natural filtration with $\mathcal{F} = \mathcal{F}_T$.

4.2.3 Loan loss reserve

The banks are required to hold provisions for the expected losses or bad debts. According to Gideon et al. [2007], we can define the provision for loan losses made by the banks which takes the form of a continuous contribution

$$k(t) = [1 + \psi(t)]\, \phi(t)\, \mathbb{E}[P(l(t))], \tag{4.9}$$

where $\psi(t)$ is the loading term which depends on the level of credit risk, $\phi(t)$ is the deterministic frequency parameter and $\mathbb{E}[P(l(t))]$ is the expected loan losses.

4.2.4 Bank's asset portfolio

The bank's assets portfolio is a linear combination of three positions namely, the units $\theta_G(t)$ of treasury $G(t)$ whose dynamics are given by (4.2), the units $\theta_S(t)$ of stock index fund $S(t)$ whose dynamics are given in (4.3), the units $\theta_L(t)$ of loan $L(t)$ whose dynamic are given in (4.4). Thus, at any instant in time t, the bank's assets portfolio is given by

$$X(t) = \theta_G(t)G(t) + \theta_S(t)S(t) + \theta_L(t)L(t), \tag{4.10}$$

whose differential is given by

$$\begin{aligned} dX(t) &= \underbrace{\theta_G(t)dG(t) + \theta_S(t)dS(t) + \theta_L(t)dL(t)}_{dX_1(t)} \\ &+ \underbrace{(S(t) + dS(t))\, d\theta_S(t) + (L(t) + dL(t))\, d\theta_L(t) + G(t)d\theta_G(t)}_{dX_2(t)}, \end{aligned} \tag{4.11}$$

where the term $dX_2(t)$ must be equal to the inflow-outflow to the bank's assets portfolio which is given by

$$dX_2(t) = -\left[k(t)\, dt + dR(t)\right],$$

where $k(t)$ equates the continuous contribution to the provision for loan losses given by (4.9) and $dR(t)$ is the bank's risk process modeled by (4.8).

Substituting the dynamics of $G(t)$, $S(t)$, $L(t)$ and $dX_2(t)$ into (4.11), we obtain

$$\begin{aligned} dX(t) &= \Big(X(t)r(t) + \theta_S(t)S(t)\left(\mu_S - r(t)\right) + \theta_L(t)L(t)\left(\mu_L - r(t)\right) - a \\ &- \left[1 + \psi(t)\right]\phi(t)\,\mathbb{E}\left[P\left(l(t)\right)\right]\Big)dt + \theta_S(t)S(t)\left(\sigma_1 dW_1(t) + \gamma_1 dN_1(t)\right) \\ &+ \theta_L(t)L(t)\left(\sigma_2 dW_2(t)\right) - b\rho dW_1(t) - b\sqrt{1-\rho^2}dW_3(t) - \gamma_{12}dN_1(t) \\ &- \left[l(t) - P\left(l(t)\right)\right]dN_2(t). \end{aligned}$$

If we set $\pi_G \equiv \theta_G(t)G(t)$, $\pi_S \equiv \theta_S(t)S(t)$, and $\pi_L \equiv \theta_L(t)L(t)$ and define the bank's strategy by $\pi(t) := (\pi_S(t), \pi_L(t), P(l(t)))$, a $\mathcal{F}_t -$ predictable process, which represents the dollar amounts invested in stock index fund $\pi_S(t)$, loans $\pi_L(t)$ and amount of provision for loan losses $P(t)$, then we can write bank's assets portfolio $X(t)$ as a controlled stochastic process depending on a strategy $\pi(t)$ as

$$\begin{aligned} dX^{\pi}(t) = \Big(& X^{\pi}(t)r(t) + \pi_S(t)\left(\mu_S - r(t)\right) + \pi_L(t)\left(\mu_L - r(t)\right) - a \\ & - \left[1 + \psi(t)\right]\phi(s)\,\mathbb{E}\left[P\left(l(t)\right)\right]\Big)dt + \left(\pi_S(t)\sigma_1 - b\rho\right)dW_1(t) \\ & + \pi_L(t)\sigma_2 dW_2(t) - b\sqrt{1-\rho^2}dW_3(t) + \left(\pi_S(t)\gamma_1 - \gamma_{12}\right)dN_1(t) \\ & - \left[l(t) - P\left(l(t)\right)\right]dN_2(t), \end{aligned} \tag{4.12}$$

where the initial asset portfolio is $X(0)^{\pi} = x_0 > 0$ and it can be written as

$$\begin{aligned} X^{\pi}(t) = e^{rt}x_0 + \int_0^t & e^{r(t-s)}\Big(\pi_S(s)\left(\mu_S - r(s)\right) + \pi_L(s)\left(\mu_L - r(s)\right) - a \\ & - \left[1 + \psi(s)\right]\phi(s)\,\mathbb{E}\left[P\left(l(s)\right)\right]\Big)ds + \int_0^t e^{r(t-s)}\left(\pi_S(s)\sigma_1 - b\rho\right)dW_1(s) \\ & + \int_0^t e^{r(t-s)}\pi_L(s)\sigma_2 dW_2(s) - \int_0^t e^{r(t-s)}b\sqrt{1-\rho^2}dW_3(s) \\ & + \int_0^t e^{r(t-s)}\left(\pi_S(t)\gamma_1 - \gamma_{12}\right)dN_1(s) \\ & - \int_0^t e^{r(t-s)}\left[l(s) - P\left(l(s)\right)\right]dN_2(s). \end{aligned} \tag{4.13}$$

Remark 2. The bank's assets portfolio process (4.12) without loan loss provision can be written as

$$\begin{aligned} dX^{\pi}(t) &= \left(X^{\pi}(t)r(t) + \pi_S(t)\left(\mu_S - r(t)\right) + \pi_L(t)\left(\mu_L - r(t)\right) - a\right)dt \\ &+ \left(\pi_S(t)\sigma_1 - b\rho\right)dW_1(t) + \pi_L(t)\sigma_2 dW_2(t) \\ &- b\sqrt{1-\rho^2}dW_3(t) + \left(\pi_S(t)\gamma_1 - \gamma_{12}\right)dN_1(t) \\ &- l\left(t\right)dN_2(t), \end{aligned} \tag{4.14}$$

where $\gamma_{12} = 0$, in the case of uncorrelated jumps.

4.2.5 Bank's optimization problem

The objective function defines the objective of the optimizer under the set of available actions and constraints imposed on those actions. However, in the case of a bank, the objective may vary, from a shareholder's perspective, it is imperative to maximize the utility of the asset portfolio so that their equity yields higher returns, in the form of either dividends or stock price appreciation. On the other hand, from the regulator's point of view, banks should make limited investment in the risky assets and set aside substantial buffer capital for a period with lower-than-expected earnings, whereas the bank management is usually concerned about its own remunerations and prestige attached to their positions because of which they strive for the higher value of firm assets.

We assume that the bank maximizes the utility of the terminal value of bank's assets portfolio, thus the bank optimizes

$$\max_{\pi\in\Pi} \mathbb{E}\left[U(X(T))\right], \tag{4.15}$$

where $\mathbb{E}$ is the conditional expectation under probability measure $\mathbb{P}$ and the utility function U is assumed to be strictly increasing and concave with respect to the wealth. Π denotes the set of all admissible controls with initial asset portfolio $X(0) = x_0$. The admissible control $\{\pi\}_{0\le t\le T}$ is progressively measurable with respect to filtration $\{\mathcal{F}\}_{0\le t\le T}$ and satisfies the conditions $\mathbb{E}\left[\int_0^T \left(\pi_S(s)\right)^2 dt\right] < \infty$, $\mathbb{E}\left[\int_0^T \left(\pi_L(s)\right)^2 dt\right] < \infty$and $\mathbb{E}\left[\int_0^T \left(P\left(l\left(s\right)\right)\right)^2 dt\right] < \infty$ and $P\left(l\left(s\right)\right) \ge 0, \forall t \in [0,T]$.

4.3 Optimal asset portfolio

We apply the martingale approach to solve our problem (4.15) which is based on two important Lemmas introduced below.

Lemma 1. *If there exists a strategy $\pi^* \in \Pi$, such that*[1]

$$\mathbb{E}\left[U'(X^{\pi^*}(T)) \times X^{\pi}(T)\right], \tag{4.16}$$

is constant over all admissible controls, i.e. $\pi \in \Pi$, then $\pi^*$is the optimal trading strategy.

Proof. For proof, see Wang et al. [2007] and Zhou [2009]. □

Lemma 2. *(Martingale Representation) For any local martingale $Z(t)$, there exists some $\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \in \Theta$, such that,*

$$Z(t) = Z(0) + \sum_{i=1}^{3} \int_0^t \theta_i(s) dW_i(s) + \sum_{i=4}^{5} \int_0^t \theta_i(s) dM_{i-3}(s),$$

for all $t \in [0, T]$. For some notation and measureability and integrability conditions for θ_i, see Tankov and Cont [2003] and Wang et al. [2007].

We obtain the optimal control to problem (4.15) through the following steps. We conjecture the form of π_S^*, $\pi_L^*$and P^* that satisfies the condition (4.16). We define

$$Z(T)^* := \frac{U'(X^{\pi^*}(T))}{\mathbb{E}\left[U'(X^{\pi^*}(T))\right]}, \qquad \text{and } Z(\tau)^* = \mathbb{E}\left[Z(T)^* \mid \mathcal{F}_\tau\right], \tag{4.17}$$

almost surely for any stopping time $\tau \leq T$ a.s. We also recall that the process Z is strictly positive (square-integrable) martingale under $\mathbb{P}$ with $\mathbb{E}\left[Z(t)\right] = 1$. Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$ such that $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z(T)^*$.

Using the expression of $X^{\pi}(t)$ in (4.13) and Lemma 1, for any stopping time $\tau \leq T$ a.s., let $\pi_S(t) = I_{[t \leq \tau]}$ and $\pi_L(t) = 0$, which is apparently an admissible control, we have

$$\mathbb{E}\left[Z(T)^* \int_0^{\tau} e^{-rs} \left((\mu_S - r(s))\, ds + \sigma_1 dW_1(s) + \gamma_1 dN_1(s)\right)\right],$$

[1] where $U'(X^{\pi^*}(T)) = \frac{dU(X^{\pi^*}(T))}{dX^{\pi^*}(T)}$

which can be written as

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau} e^{-rs}\left((\mu_S - r(s))ds + \sigma_1 dW_1(s) + \gamma_1 dN_1(s)\right)\right],$$

is constant over all $\tau \leq T$, which implies

$$\int_0^t e^{-rs}(\mu_S - r(s))ds + e^{-rs}\sigma_1 dW_1(s) + e^{-rs}\gamma_1 dN_1(s), \tag{4.18}$$

is a $\mathbb{Q}$ martingale. We define

$$K(t) := \int_0^t \frac{1}{Z(s-)^*} dZ(s)^*,$$

where $t \in [0,T]$ and it is a local martingale as Z is a local martingale. By Lemma 2, there exist some predictable processes $\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \in \Theta$ such that

$$dK(t) = \sum_{i=1}^{3} \theta_i(t) dW_i(t) + \sum_{i=4}^{5} \theta_i(t)\, dM_{i-3}(t),$$

i.e.,

$$dZ(t)^* = Z(t-)^* \left[\sum_{i=1}^{3} \theta_i(t) dW_i(t) + \sum_{i=4}^{5} \theta_i(t)\, dM_{i-3}(t)\right].$$

Moreover, from Doleans-Dade exponential formula, we have

$$\begin{aligned} Z(t)^* = Z(0) \exp \Bigg\{ & \int_0^t \left(\theta_1(s) dW_1(s) + \theta_2(s) dW_2(s) + \theta_3(s) dW_3(s)\right) \\ & - \frac{1}{2}\int_0^t \left(\theta_1^2(s) + \theta_2^2(s) + \theta_3^2(s) + 2\lambda\theta_4(s) + 2\lambda\theta_5(s)\right) ds \\ & + \int_0^t \ln\left(1 + \theta_4(s)\right) dN_1(s) + \int_0^t \ln\left(1 + \theta_5(s)\right) dN_2(s) \Bigg\}. \end{aligned} \tag{4.19}$$

By Girsanov's Theorem, we know that $dW_i(t) - \theta_i(t)dt$ for $i = 1-3$ and $dN_{i-3}(t) - \lambda\left(1 + \theta_i(t)\right) dt$ for $i = 4-5$ are martingale under $\mathbb{Q}$, which together with (4.18) imply that θ_1 must satisfy the equation

$$\sigma_1 \theta_1(t) + \gamma_1 \lambda \left(1 + \theta_4(t)\right) = -\left(\mu_S - r(t)\right). \tag{4.20}$$

By using the expression of $X^{\pi}(t)$ in (4.13) and Lemma 1, for any stopping time $\tau \leq T$ a.s., let $\pi_S(t) = 0$ and $\pi_L(t) = I_{[t\leq\tau]}$ and following the steps above we get

$$\theta_2(t) = -\frac{\mu_L - r(t)}{\sigma_2}. \tag{4.21}$$

Remark 3. The conditions (4.20) and (4.21) satisfy even for our two special cases, i.e. (i) uncorrelated jump processes, and (ii) in the absence of loan loss provision.

4.3.1 CARA utility function

Proposition 5. *When the utility function is $U(X(t)) = -\frac{1}{\delta}e^{-\delta X(t)}$ (CARA utility function) where $\delta > 0$ and $U'(X(t)) = e^{-\delta X(t)}$, then the optimal policies (π_S^*,π_L^*,P^*) are given by*

$$\begin{aligned}
\pi_S(t)^* &= \frac{b\rho}{\sigma_1} - \frac{\theta_1(t)e^{-r(T-t)}}{\delta\sigma_1}, \\
\pi_L(t)^* &= -\frac{\theta_2(t)e^{-r(T-t)}}{\delta\sigma_2}, \\
P\left(l\left(s\right)\right)^* &= l\left(s\right) - \frac{\ln\left(1+\theta_5(t)\right)}{\delta e^{r(T-t)}}, \\
\theta_1\left(t\right) + \frac{\gamma_1\lambda}{\sigma_1}\left(1+\theta_4\left(t\right)\right) &= -\frac{\mu_S - r(t)}{\sigma_1}, \\
\theta_2(t) &= -\frac{\mu_L - r(t)}{\sigma_2}, \\
\theta_3(t) &= \delta e^{r(T-t)} b\sqrt{1-\rho^2}, \\
\theta_4(t) &= \exp\left(-\delta e^{r(T-t)}\left(\pi_S\left(t\right)^*\gamma_1 - \gamma_{12}\right)\right) - 1, \\
\theta_5(t) &= \exp\left(\delta e^{r(T-t)}\left(l\left(s\right) - P\left(l\left(s\right)\right)\right)^*\right) - 1
\end{aligned}$$

Proof. The condition (4.16) for our utility function can be rewritten as

$$\mathbb{E}\left[e^{-\delta X^{\pi^*}(T)} X^{\pi}(T)\right], \tag{4.22}$$

Replacing $X^{\pi}(T)$ by solution given in (4.13), we obtain the expression

$$\mathbb{E}\Bigg[e^{-\delta X^{\pi^*}(T)}\int_0^T e^{-rs}\Big(\pi_S(s)\left(\mu_S - r(s)\right) + \pi_L(s)\left(\mu_L - r(s)\right)$$
$$- \left[1+\psi\left(s\right)\right]\phi\left(s\right)\mathbb{E}\left[P\left(l\left(s\right)\right)\right]\Big)ds$$
$$+ \int_0^T e^{-rs}\pi_S(s)\sigma_1 dW_1(s) + \int_0^T e^{-rs}\pi_L(s)\sigma_2 dW_2(s)$$
$$+ \int_0^t e^{r(t-s)}\pi_S\left(t\right)\gamma_1 dN_1(s)\Bigg], \tag{4.23}$$

which is constant over $({\pi_S}^*, {\pi_L}^*, P^*) \in \Pi$.

We apply the following three steps to prove the above theorem.

Step 1: We define the Radon-Nikodym process using the conjecture (4.17) which satisfies the condition (4.22), we have

$$Z(T)^* := \frac{e^{-\delta X^{\pi^*}(T)}}{\mathbb{E}\left[e^{-\delta X^{\pi^*}(T)}\right]}, \qquad \text{and } Z(\tau)^* = \mathbb{E}\left[Z(T)^* \mid \mathcal{F}_\tau\right], \tag{4.24}$$

a.s. for any stopping time $\tau \leq T$ a.s. Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$ such that $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z\left(T\right)^*$.

From stochastic differential equation (4.13), we can calculate

$$\exp\left\{-\delta X^{\pi^*}(T)\right\} = \exp\Bigg\{-\delta e^{rT}x_0 - \delta\int_0^T e^{r(T-s)}\Big(\pi_S(s)^*\left(\mu_S - r(s)\right) + \pi_L(s)^*\left(\mu_L - r(s)\right) - a$$
$$- \left[1+\psi\left(s\right)\right]\phi\left(s\right)\mathbb{E}\left[P\left(l\left(s\right)\right)\right]\Big)ds - \delta\int_0^T e^{r(T-s)}\left(\pi_S(s)^*\sigma_1 - b\rho\right)dW_1(s)$$
$$- \delta\int_0^T e^{r(T-s)}\pi_L(s)^*\sigma_2 dW_2(s) + \delta\int_0^T e^{r(T-s)}b\sqrt{1-\rho^2}dW_3(s)$$
$$- \delta\int_0^T e^{r(T-s)}\left(\pi_S\left(t\right)^*\gamma_1 - \gamma_{12}\right)dN_1(s)$$
$$+ \delta\int_0^T e^{r(T-s)}\left[l\left(s\right) - P\left(l\left(s\right)\right)^*\right]dN_2(s)\Bigg\}, \tag{4.25}$$

We compare the terms $dW_1(t)$,$dW_2(t)$,$dW_3(t)$,$dN_1(t)$ and $dN_2(t)$- terms in (4.19) with those in (4.25) to obtain the term required for (4.24), we conjecture that

$$\begin{cases} \theta_1(t) & = -\delta e^{r(T-t)} \left(\pi_S(t)^* \sigma_1 - b\rho\right) \\ \theta_2(t) & = -\delta e^{r(T-t)} \pi_L(t)^* \sigma_2 \\ \theta_3(t) & = \delta e^{r(T-t)} b\sqrt{1-\rho^2} \\ \ln\left(1+\theta_4(t)\right) & = -\delta e^{r(T-t)} \left(\pi_S\left(t\right)\gamma_1 - \gamma_{12}\right) \\ \ln\left(1+\theta_5(t)\right) & = \delta e^{r(T-t)} \left(l\left(s\right) - P\left(l\left(s\right)\right)^*\right) \\ \sigma_1\theta_1\left(t\right) & = -\left(\mu_S - r(t)\right) - \gamma_1\lambda\left(1+\theta_4\left(t\right)\right) \\ \theta_2(t) = & -\frac{\mu_L - r(t)}{\sigma_2} \end{cases},$$

i.e.,

$$\begin{cases} \pi_S(t)^* & = \frac{b\rho}{\sigma_1} - \frac{\theta_1(t)e^{-r(T-t)}}{\delta\sigma_1} \\ \pi_L(t)^* & = -\frac{\theta_2(t)e^{-r(T-t)}}{\delta\sigma_2} \\ \theta_3(t) & = \delta e^{r(T-t)} b\sqrt{1-\rho^2} \\ \theta_4(t) & = \exp\left(-\delta e^{r(T-t)}\left(\pi_S\left(t\right)^*\gamma_1 - \gamma_{12}\right)\right) - 1. \\ P\left(l\left(s\right)\right)^* & = l\left(s\right) - \frac{\ln(1+\theta_5(t))}{\delta e^{r(T-t)}} \\ \theta_1\left(t\right) & = -\frac{(\mu_S - r(t)) + \gamma_1\lambda(1+\theta_4(t))}{\sigma_1} \\ \theta_2(t) & = -\frac{\mu_L - r(t)}{\sigma_2} \end{cases} \tag{4.26}$$

Step 2: We verify that $Z(T)^*$ defined in (4.19) is consistent with our conjecture in (4.24). We rewrite (4.25) as

$$\exp\left\{-\delta X^{\pi^*}(T)\right\} = I(T)H(T), \tag{4.27}$$

where

$$
\begin{aligned}
I(T) = \exp\bigg\{ & -\delta e^{rT} x_0 - \delta \int_0^T e^{r(T-s)} \Big(\pi_S(s)^* \left(\mu_S - r(s)\right) + \pi_L(s)^* \left(\mu_L - r(s)\right) - a \\
& - \left[1 + \psi\left(s\right)\right] \phi\left(s\right) \mathbb{E}\left[P\left(l\left(s\right)\right)^*\right] \Big) ds \bigg\}, \\
H(T) = \exp\bigg\{ & -\delta \int_0^T e^{r(T-s)} \left(\pi_S(s)^* \sigma_1 - b\rho\right) dW_1(s) - \delta \int_0^T e^{r(T-s)} \pi_L(s)^* \sigma_2 dW_2(s) \\
& + \delta \int_0^T e^{r(T-s)} b\sqrt{1-\rho^2} dW_3(s) - \delta \int_0^T e^{r(T-s)} \left(\pi_S\left(t\right)^* \gamma_1 - \gamma_{12}\right) dN_1(s) \\
& + \delta \int_0^T e^{r(T-s)} \left[l\left(s\right) - P\left(l\left(s\right)\right)^*\right] dN_2\left(s\right) \bigg\}.
\end{aligned}
$$

By substituting (4.26) back into (4.19), we obtain

$$
Z(T)^* = J(T)H(T), \tag{4.28}
$$

i.e.,

$$
\begin{aligned}
J(T)^* = \exp\bigg\{ & \int_0^T \delta^2 e^{2r(T-s)} \Big(-\frac{1}{2}\pi_S^2(s)^* \sigma_1^2 + \pi_S(s)^* \sigma_1 - \frac{1}{2}\pi_L^2(s)^* \sigma_2^2 - \frac{1}{2} b^2 \Big) ds \\
& - \int_0^T \lambda \Big(\theta_4\left(s\right) \exp\left(-\delta e^{r(T-s)} \left(\pi_S\left(s\right)^* \gamma_1 - \gamma_{12}\right)\right) - 1 \Big) ds \\
& - \int_0^T \lambda \Big(\exp\Big(\delta e^{r(T-s)} \left[l\left(s\right) - P\left(l\left(s\right)\right)^*\right] \Big) - 1 \Big) ds \bigg\}
\end{aligned}
$$

is constant. Since Z^* is a martingale, we have $\mathbb{E}\left[Z(T)^*\right] = 1$ and hence

$$
\mathbb{E}\left[H(T)^*\right] = J(T)^{-1}. \tag{4.29}
$$

Finally, from (4.27),(4.28) and (4.29), we obtain

$$\begin{aligned}\frac{\exp\left\{-\delta X^{\pi^*}(T)\right\}}{\mathbb{E}\left[\exp\left\{-\delta X^{\pi^*}(T)\right\}\right]} &= \frac{I(T)H(T)}{I(T)\mathbb{E}\left[H(T)\right]}\\ &= J(T)H(T)\\ &= Z(T)^*,\end{aligned}$$

which is just what we desired.

Step 3: We verify that our optimal policies $\pi^* = (\pi_S^*, \pi_L^*, P^*)$, given in (4.26) are indeed the optimal strategies that satisfy the condition (4.22). For any admissible strategy π, we define a new process $M^\pi(t)$ as:

$$\begin{aligned}M^\pi(t) := \int_0^t e^{-rs}\Big(&\pi_S(s)\left(\mu_S - r(s)\right) + \pi_L(s)\left(\mu_L - r(s)\right)\\ &- \left[1+\psi(s)\right]\phi(s)\,\mathbb{E}\left[P\left(l\left(s\right)\right)^*\right]\Big)ds + \int_0^t e^{-rs}\pi_S(s)\sigma_1 dW_1(s)\\ &+ \int_0^t e^{-rs}\pi_L(s)\sigma_2 dW_2(s) + \int_0^t e^{-rs}\pi_S\left(t\right)\gamma_1 dN_1(s),\end{aligned}$$

is a local martingale under $\mathbb{Q}$ by Girsanov's Theorem $\mathbb{E}\left[Z(T)^*\right] < \infty$ for any admissible strategy (π_S, π_L, P) and by Burkholder-Davis-Gundy Inequality Revuz and Yor [1991], we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\sup_{0\le t\le T}|M^\pi(t)|\right] &= \mathbb{E}\left[Z(T)^*\sup_{0\le t\le T}|M^\pi(t)|\right]\\ &\le \sqrt{\mathbb{E}\left[\left(Z(T)^*\right)^2\right]}\sqrt{\mathbb{E}\left[\sup_{0\le t\le T}|M^\pi(t)|^2\right]} < \infty.\end{aligned}$$

This implies, for a family of stopping times $\{M^\pi(\tau) : \tau \text{ is a stopping time and } \tau \le T\}$ is uniformly integrable under $\mathbb{Q}$ and $M^\pi(t)$ is a $\mathbb{Q}$-martingale and hence $\mathbb{E}_{\mathbb{Q}}\left[M^\pi(t)\right] = 0$. □

Corollary 7. *If we assume the problem (4.15) in the absence of loan loss reserve, then the optimal policies* (π_S^*,π_L^*) *are given by*

$$
\begin{aligned}
\pi_S(t)^* &= \frac{b\rho}{\sigma_1} - \frac{\theta_1(t)e^{-r(T-t)}}{\delta\sigma_1} \\
\pi_L(t)^* &= -\frac{\theta_2(t)e^{-r(T-t)}}{\delta\sigma_2} \\
\theta_1(t) + \frac{\gamma_1\lambda}{\sigma_1}(1+\theta_4(t)) &= -\frac{\mu_S - r(t)}{\sigma_1}, \\
\theta_2(t) &= -\frac{\mu_L - r(t)}{\sigma_2} \\
\theta_3(t) &= \delta e^{r(T-t)} b\sqrt{1-\rho^2} \\
\theta_4(t) &= \exp\left(-\delta e^{r(T-t)}\left(\pi_S(t)\gamma_1 - \gamma_{12}\right)\right) - 1 \\
\theta_5(t) &= \exp\left(\delta e^{r(T-t)}\gamma_2\right) - 1
\end{aligned}
$$

where $\gamma_{12} = 0$ *if the jump in the risk process is not correlated with jump in the financial market.*

4.4 Capital dynamics

Bank's capital include retained earnings, capital raised by selling new equity, and debt acquired. The total bank's capital $C(t)$ can be divided into two tiers according to Basel III capital accord:

$$C(t) = C_1(t) + C_2(t),$$

where $C_1(t)$ is the sum of book value of stocks $E(t)$ and retained earnings $E_r(t)$. $C_2(t)$ (also known as supplementary capital) is a sum of subordinate loans $S_D(t)$ and loan-loss reserves or provisions for bad debts $R_L(t)$. As a result, the total capital can be written as

$$C(t) = E(t) + E_r(t) + S_D(t) + R_L(t), \tag{4.30}$$

Here, for the sake of simplicity, we assume the market value of S_D is given by

$$S_D(t) = S_D(0)\exp\left(\int_0^t r(u)\,du\right).$$

We follow the dynamics of capital as reported in the existing literature (see for instance, Perera [2015] and Witbooi et al. [2011]) and assume that bank holds its capital in $n+1$ categories, one category is related to the subordinated debt and n categories are bank's equity. The return on the ith bank equity can be written as

$$de_i(t) = e_i(t)\left[\left(r(t) + \sum_{j=1}^{n}\sigma_{ij}\upsilon_j\right)dt + \sum_{j=1}^{n}\sigma_{ij}d\hat{W}_j(t)\right],$$

where the market price of risk and co-variance matrix are constants and are given by $\varUpsilon = (\upsilon_1, ...\upsilon_n)^{'}$ and $\varPhi = (\sigma_{ij})_{i,j=1}^{n}$ respectively. As retained earnings $E_r(t)$ and loan-loss reserves $R_L(t)$ are non-dynamic in nature, we consider them as inactive components of bank's capital which implies $dE_r = 0$ and $dR_L = 0$, similar to the previous literature (see for instance Mukuddem-Petersen and Petersen [2008]; Witbooi et al. [2011] and Perera [2015]). We assume the bank's capital is being converted into loans and securities is $\beta X(t)$, where β is a constant and $X(t) = G(t) + S(t) + L(t)$ which represents the total asset portfolio of the bank at time t. Thus, the capital dynamics can be represented as

$$\begin{aligned} dC(t) &= C(t)\left[\sum_{i=1}^{n}\tilde{\pi}_i(t)\frac{de_i(t)}{e_i(t)} + \left(1 - \sum_{i=1}^{n}\tilde{\pi}_i(t)\right)\frac{dS_D(t)}{S_D(t)}\right] - \beta X(t)\,dt, \\ &= C(t)\left[\left(r(t) + \tilde{\pi}(t)^{'}\varPhi\varUpsilon\right)dt + \tilde{\pi}(t)^{'}\varPhi d\hat{W}(t)\right] - \beta X(t)\,dt, \end{aligned} \tag{4.31}$$

where $\tilde{\pi}(t)^{'}$ is a transposed vector containing the proportions invested in securities and loans and $\tilde{\pi}(t)^{'}\varPhi d\hat{W}(t)$ forms the correlation between total risk-weighted assets and bank's capital.

4.4.1 Capital Ratio

Capital adequacy requirements in Basel III instructs the bank's to maintain capital adequacy ratio Γ such that, $\Gamma \geq 0.08$, for an adequately capitalized bank. At a time

t it is given by

$$\Gamma(t) = \frac{C(t)}{A\,(t)}, \tag{4.32}$$

where the total TRWAs $A\,(t)$, are the risk weighted bank's assets, comprised of treasuries $G\,(t)$, securities $S\,(t)$ and loans $L\,(t)$ assigned the weights ω_G, ω_S and ω_L respectively. The weights are assigned to the assets according to perceived risks, which means that a more risky the asset has a higher risk weight.

The CAR may increase while the actual levels of capital $C\,(t)$ may decrease as it also depends on TRWAs which is sensitive to risk changes. Thus in this case the bank should hold more regulatory capital means that a given CAR can only be sustained if banks hold more regulatory capital.

Remark 4. TRWAs is a weighted sum of the different assets of the bank whose dynamics may be expressed as:

$$dA(t) = \omega_G \tilde{\pi}_G\,(t)\,\frac{dG\,(t)}{G\,(t)} + \omega_S \tilde{\pi}_S\,(t)\,\frac{dS\,(t)}{S\,(t)} + \omega_L \tilde{\pi}_L\,(t)\,\frac{dL\,(t)}{L\,(t)},$$

where $\tilde{\pi}_G\,(t) \equiv \pi_G\,(t)\,X(t)$, $\tilde{\pi}_S\,(t) \equiv \pi_S\,(t)\,X(t)$, and $\tilde{\pi}_L\,(t) \equiv \pi_L\,(t)\,X(t)$, representing the proportion of asset portfolio invested in the treasury, stock index fund and loans. We can rearrange the above equation as

$$\begin{aligned} \frac{dA(t)}{A(t)} &= \left(\omega_G \tilde{\pi}_G\,(t)\,r(t)dt + \omega_S \tilde{\pi}_S\,(t)\,\mu_S dt + \omega_L \tilde{\pi}_L\,(t)\,\mu_L\right) dt \\ &\quad + \omega_S \tilde{\pi}_S\,(t)\,\sigma_1 dW_1(t) + \omega_L \tilde{\pi}_L\,(t)\,\sigma_2 dW_2(t) + \omega_S \tilde{\pi}_S\,(t)\,\gamma_1 dN_1(t), \end{aligned} \tag{4.33}$$

Proposition 6. *Under the dynamics of the total bank capital described in (4.31) and dynamics of TRWAs given by (4.33), the dynamics of the Capital Ratio (CAR) can be expressed as*

$$
\begin{aligned}
d\Gamma(t) = \Gamma(t)\Big[&(\alpha_0 - \alpha_1 + \alpha_2)\,dt - \alpha_3 dW_1(t) - \alpha_4 dW_2(t) - \alpha_5 dN_2 + \alpha_6 d\hat{W}(t)\Big] \\
&- \beta\frac{X(t)}{A(t)}dt,
\end{aligned}
$$

where,

$$
\alpha_0 = r(t) + \tilde{\pi}(t)' \varPhi\varUpsilon, \quad \alpha_1 = \omega_G\tilde{\pi}_G(t)\,r(t)dt + \omega_S\tilde{\pi}_S(t)\,\mu_S dt + \omega_L\tilde{\pi}_L(t)\,\mu_L,
$$

$$
\alpha_2 = \omega_S^2\tilde{\pi}_S^2(t)\,\sigma_1^2 + \omega_L^2\tilde{\pi}_L^2(t)\,\sigma_2^2, \quad \alpha_3 = \omega_S\tilde{\pi}_S(t)\,\sigma_1, \quad \alpha_4 = \omega_L\tilde{\pi}_L(t)\,\sigma_2,
$$

$$
\alpha_5 = \frac{\omega_S\tilde{\pi}_S(t)\,\gamma_1}{A(t) + \omega_S\tilde{\pi}_S(t)\,\gamma_1}, \quad \alpha_6 = \tilde{\pi}(t)'\,\varPhi.
$$

Proof. Let $f(A(t)) = (A(t))^{-1}$, by the application of Ito's lemma for jump diffusion processes we obtain,

$$
\begin{aligned}
df(t, A(t)) = \frac{1}{A(t)}\Bigg(&- (\omega_G\tilde{\pi}_G(t)\,r(t)dt + \omega_S\tilde{\pi}_S(t)\,\mu_S dt + \omega_L\tilde{\pi}_L(t)\,\mu_L) + \omega_S^2\tilde{\pi}_S^2(t)\,\sigma_1^2 \\
&+ \omega_L^2\tilde{\pi}_L^2(t)\,\sigma_2^2\Bigg)dt - \frac{1}{A(t)}\left(\omega_S\tilde{\pi}_S(t)\,\sigma_1 dW_1(t) + \omega_L\tilde{\pi}_L(t)\,\sigma_2 dW_2(t)\right) \\
&+ \frac{1}{A(t)}\left(\frac{-\omega_S\tilde{\pi}_S(t)\,\gamma_1}{A(t) + \omega_S\tilde{\pi}_S(t)\,\gamma_1}\right)dN_2. \qquad (4.34)
\end{aligned}
$$

The capital ratio (4.32) is given by

$$
\Gamma(t) = C(t)\,f(A(t))\,.
$$

We apply Ito's product rule to $\Gamma(t)$. As $W_1(t)$, $W_2(t)$, $\hat{W}(t)$, and $N_1(t)$ are mutually independent, as a result we have

Table 4.1: Risk weights of different asset categories.

Asset Types	Risk weights
Cash, reserves, treasuries	0
Securities	$\frac{1}{5}$
Loans	$\frac{1}{2}$

$$d\Gamma(t) = C(t)\, d(f(A(t))) + f(A(t))\, dC(t) + df(A(t))\, dC(t),$$

$$\begin{aligned} d\Gamma(t) = \Gamma(t) \Bigg[& -\left(\omega_G \tilde{\pi}_G(t)\, r(t) dt + \omega_S \tilde{\pi}_S(t)\, \mu_S dt + \omega_L \tilde{\pi}_L(t)\, \mu_L\right) dt \\ & + \left(\omega_S^2 \tilde{\pi}_S^2(t)\, \sigma_1^2 + \omega_L^2 \tilde{\pi}_L^2(t)\, \sigma_2^2\right) dt - \left(\omega_S \tilde{\pi}_S(t)\, \sigma_1 dW_1(t) + \omega_L \tilde{\pi}_L(t)\, \sigma_2 dW_2(t)\right) \\ & - \frac{\omega_S \tilde{\pi}_S(t)\, \gamma_1}{A(t) + \omega_S \tilde{\pi}_S(t)\, \gamma_1} dN_2 \Bigg] \\ & + \Gamma(t) \left[\left(r(t) + \tilde{\pi}(t)' \Phi \Upsilon\right) dt + \tilde{\pi}(t)' \Phi d\hat{W}(t)\right] - \beta \frac{X(t)}{A(t)} dt \end{aligned} \tag{4.35}$$

$$\begin{aligned} d\Gamma(t) = \Gamma(t) & \left[(\alpha_0 - \alpha_1 + \alpha_2)\, dt - \alpha_3 dW_1(t) - \alpha_4 dW_2(t) - \alpha_5 dN_2 + \alpha_6 d\hat{W}(t)\right] \\ & - \beta \frac{X(t)}{A(t)} dt. \end{aligned}$$

□

Example 1. For illustration purposes, we consider an example of risk weights for different asset categories given in Table 4.1.

We can substitute the risk weights into the dynamics of CAR given in (4.35) to get

$$
\begin{aligned}
d\Gamma(t) = \Gamma(t)\Bigg[&-\left(\frac{1}{5}\tilde{\pi}_S\left(t\right)\mu_S dt+\frac{1}{2}\tilde{\pi}_L\left(t\right)\mu_L\right)dt \\
&+\left(\left(\frac{1}{5}\right)^2\tilde{\pi}_S^2\left(t\right)\sigma_1^2+\left(\frac{1}{2}\right)^2\tilde{\pi}_L^2\left(t\right)\sigma_2^2\right)dt-\left(\frac{1}{5}\tilde{\pi}_S\left(t\right)\sigma_1 dW_1(t)+\frac{1}{2}\tilde{\pi}_L\left(t\right)\sigma_2 dW_2(t)\right) \\
&-\frac{\frac{1}{5}\tilde{\pi}_S\left(t\right)\gamma_1}{A\left(t\right)+\frac{1}{5}\tilde{\pi}_S\left(t\right)\gamma_1}dN_2\Bigg] \\
&+\Gamma(t)\left[\left(r\left(t\right)+\tilde{\pi}\left(t\right)'\mathit{\Phi}\mathit{\Upsilon}\right)dt+\tilde{\pi}\left(t\right)'\mathit{\Phi}d\hat{W}\left(t\right)\right]-\beta\frac{X\left(t\right)}{A\left(t\right)}dt.
\end{aligned}
\tag{4.36}
$$

4.5 Conclusion

This chapter analyses the optimal risk control asset portfolio of a bank under CARA preferences when the bank is allowed to invest in treasuries, stock index fund and loans in the presence of jumps. The risk process of bank is modeled by Cramer-Lundberg model. We considered the simultaneous jumps in the dynamics of stock index fund and risk process to consider the unexpected losses whereas an additional jump process is used to model the expected losses which are diminished by the loan loss provision. We derived an optimal investment policy and dynamics of the CAR, which mandates that banks are subject to certain limitations and banks must hold sufficient amount of capital to provide for unexpected losses. The model presented here can be adopted by banking industry as an internal model to make an assessment of CAR. The main thrust of future research may involve simulation of CAR using various models applied in the literature.

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